

# Electric network for non-reversible Markov chains

joint with Áron Folly

Márton Balázs

University of Bristol

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## Reversible chains and resistors

Thomson, Dirichlet principles

Monotonicity, transience, recurrence

## Irreversible chains and electric networks

The part

From network to chain

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## Properties

Effective resistance

What works

Nonmonotonicity

Dirichlet principle

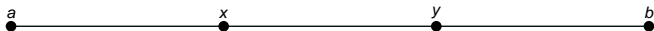
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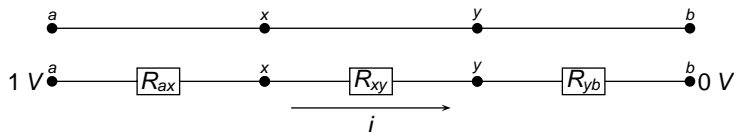
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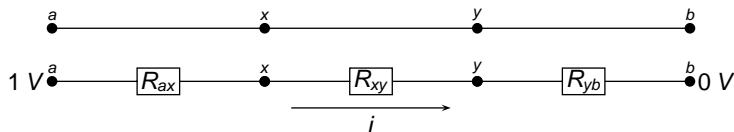
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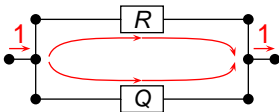
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Notice  $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$ , so **the chain is reversible**.



# Thomson, Dirichlet principles

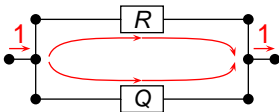
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The physical unit current is the unit flow that minimizes the sum of the ohmic power losses  $\sum i^2 R$ .

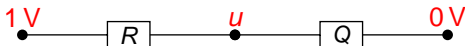
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Dirichlet principle:



The physical voltage is the function that minimizes the ohmic power losses  $\sum (\nabla u)^2 / R$ .

# Monotonicity, transience, recurrence

## The monotonicity property:

Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

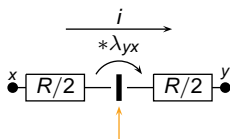
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↪ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

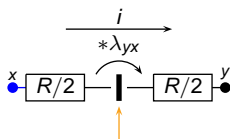
# The part



**Voltage amplifier:** keeps the current, multiplies the potential.

$$\left(u_x - i \cdot \frac{R}{2}\right) \cdot \lambda_{yx} - i \cdot \frac{R}{2} = u_y$$

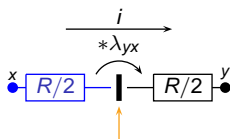
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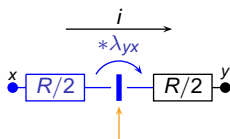
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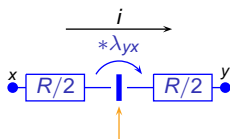


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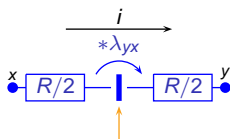
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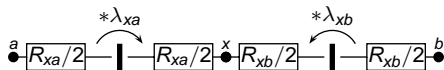


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It is clear that  $\lambda_{yx} = \frac{1}{\lambda_{xy}}$ .

# Harmonicity



$$u_x = \sum_y \frac{D_{xy} \gamma_{xy}}{D_x} \cdot u_y$$

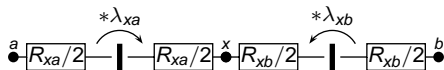
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$$\gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}},$$

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Stationary distribution:  $\rightsquigarrow D_x = \mu_x$ .

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$$\mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2;$$

$$\frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma_{xy}^2 = \lambda_{xy}.$$

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**Reversed chain:** Replace  $P_{xy}$  by  $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$ .

$\rightsquigarrow D_{xy}$  stays,  $\lambda_{xy}$  reverses to  $\lambda_{yx}$ .

$$\gamma_{xy} = \frac{1}{\gamma_{yx}}$$

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# Markovian network

$$u_x = \sum_z P_{xz} u_z; \quad \sum_z P_{xz} = 1$$

$u_x \equiv \text{const.}$  is a solution of the network with no external sources. This is now nontrivial.

## Effective resistance

Suppose  $u_a, u_b$  given, the solution is  $\{u_x\}_{x \in \Omega}$  and  $\{i_{xy}\}_{x \sim y \in \Omega}$ .  
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↪ Then

$$\frac{i_a}{u_a - u_b} = \text{const.} =: C_{ab}^{\text{eff}} = \frac{1}{R_{ab}^{\text{eff}}}.$$

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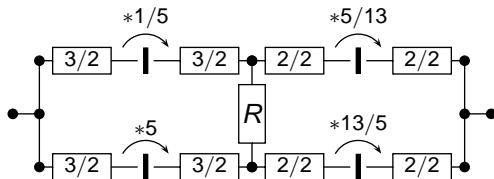
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## Theorem

*Commute time =  $R^{\text{eff}}$  · all conductances.*

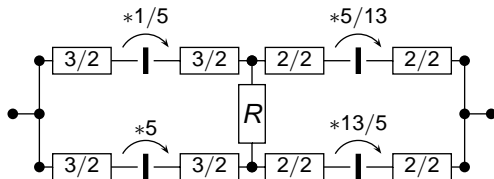
# Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$



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