

# Jacobi triple product via the exclusion process

Joint with  
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# Jacobi triple product

## Theorem

Let  $|x| < 1$  and  $y \neq 0$  be complex numbers. Then

$$\prod_{i=1}^{\infty} (1 - x^{2i}) \left(1 + \frac{x^{2i-1}}{y^2}\right) (1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in **number theory** and **combinatorics of partitions**.

We'll prove it using interacting particles (for real  $x, y$  only).

## Models

- Asymmetric simple exclusion
- Zero range

## Blocking measures

## State space

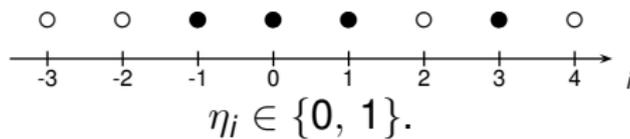
- No boundaries
- Boundaries

## Lay down - stand up

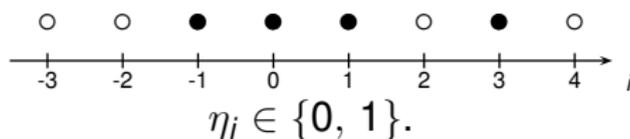
## Jacobi triple product

## More models

# Asymmetric simple exclusion



# Asymmetric simple exclusion



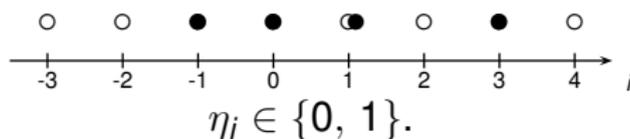
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to the left with rate  $q = 1 - p < p$ .

The jump is suppressed if the destination site is occupied by another particle.

# Asymmetric simple exclusion



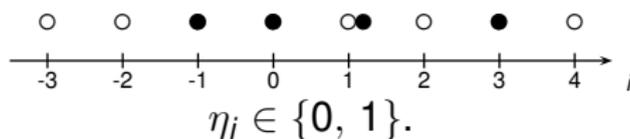
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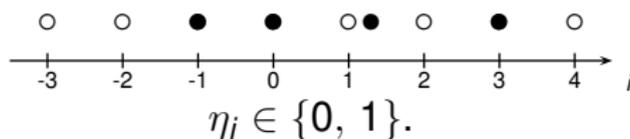
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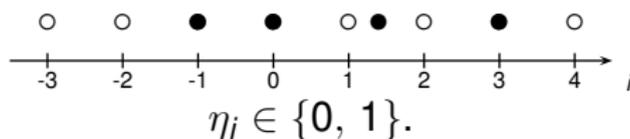
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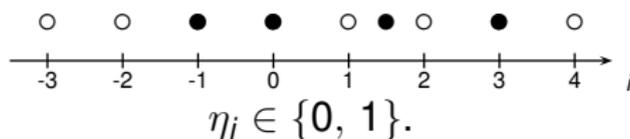
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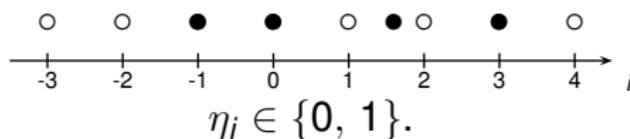
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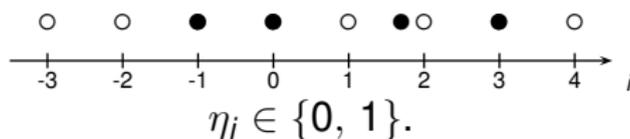
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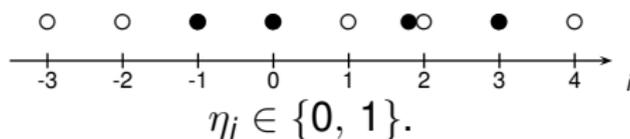
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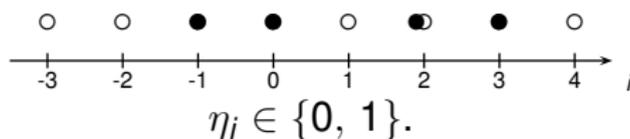
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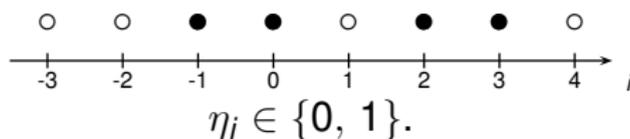
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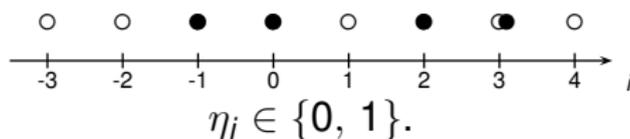
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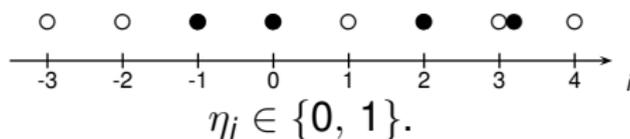
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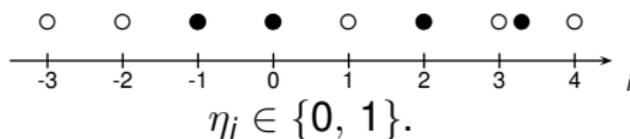
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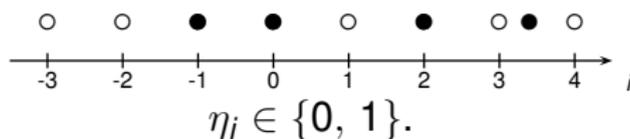
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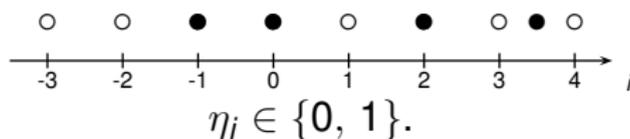
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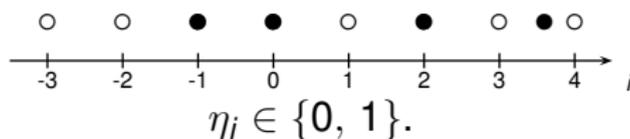
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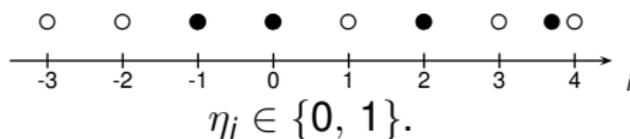
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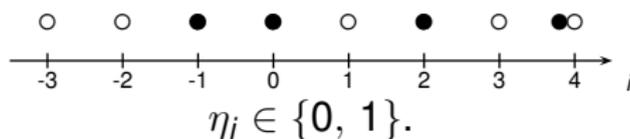
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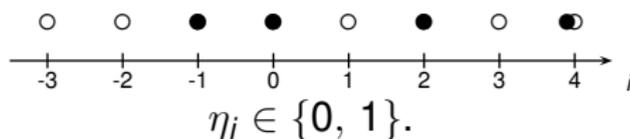
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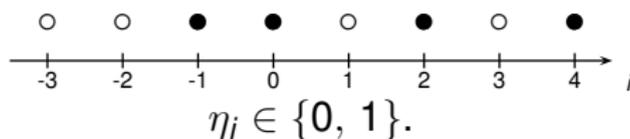
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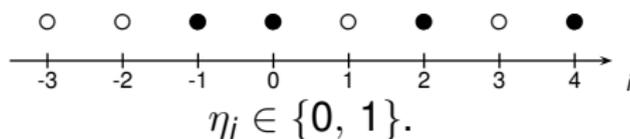
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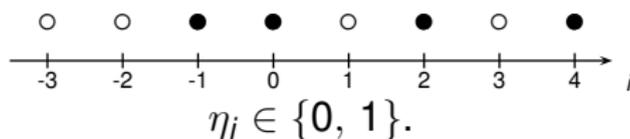
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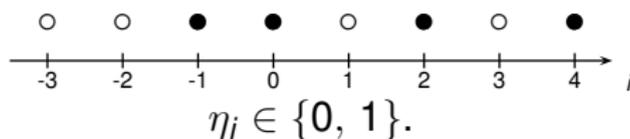
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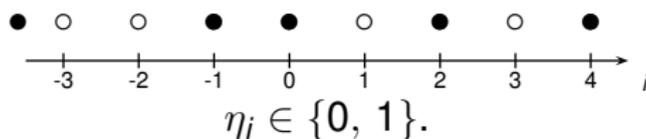
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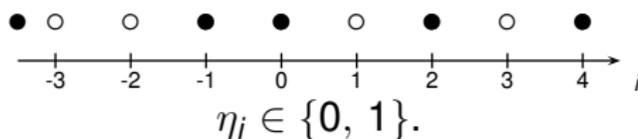
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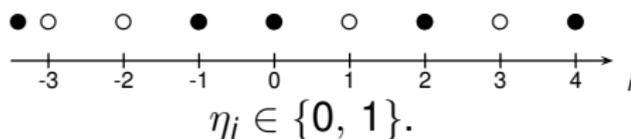
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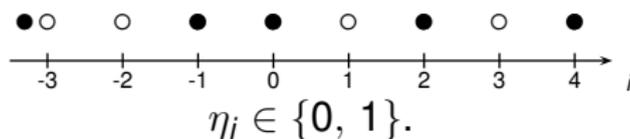
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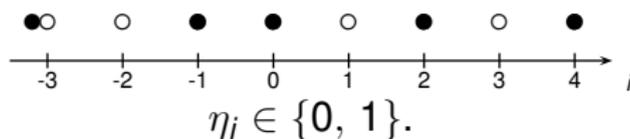
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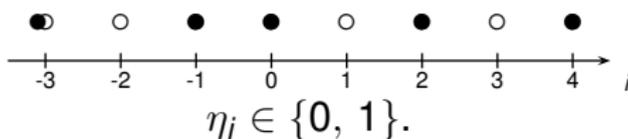
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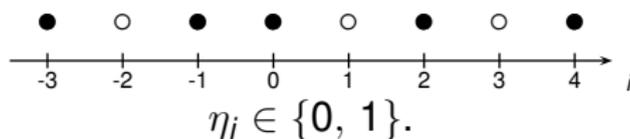
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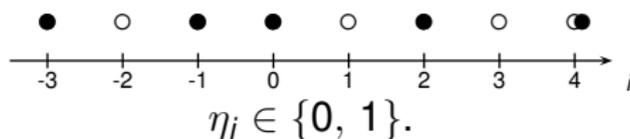
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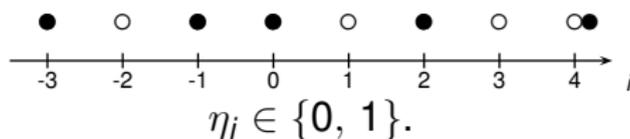
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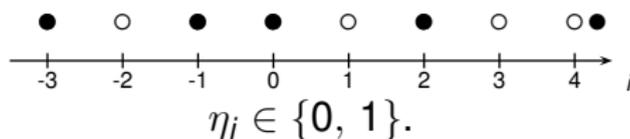
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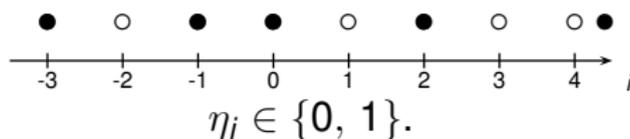
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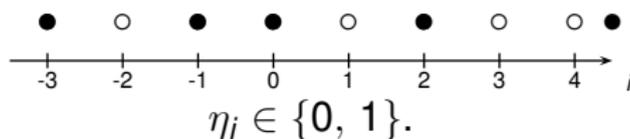
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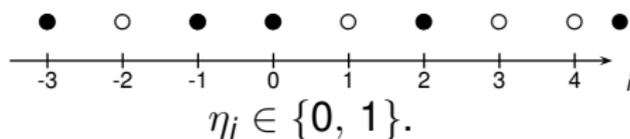
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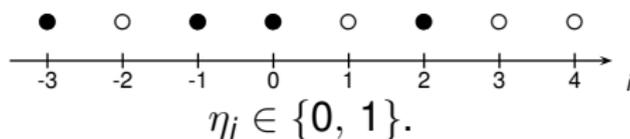
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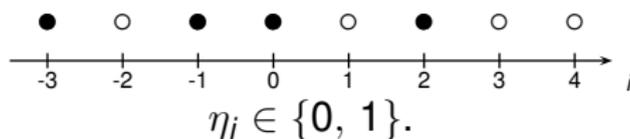
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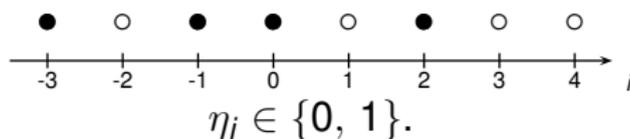
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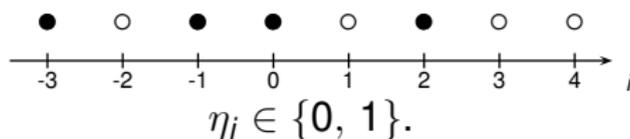
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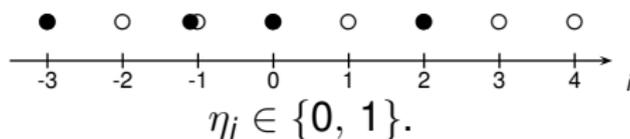
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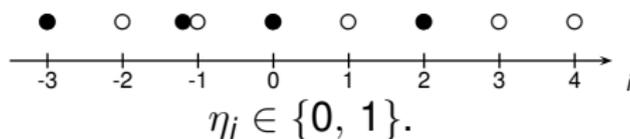
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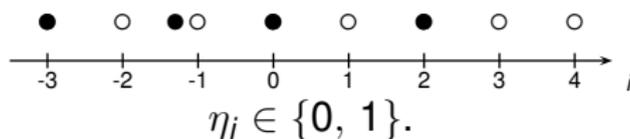
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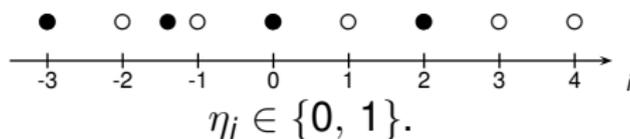
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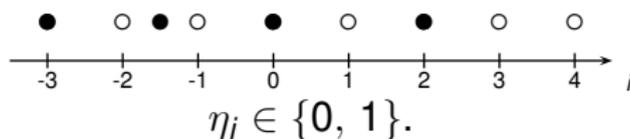
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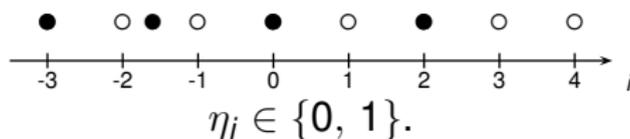
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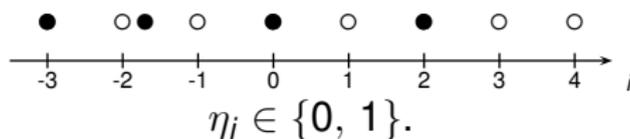
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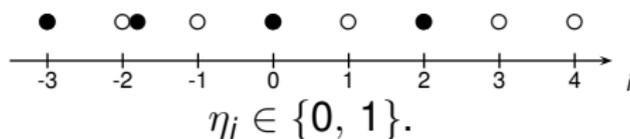
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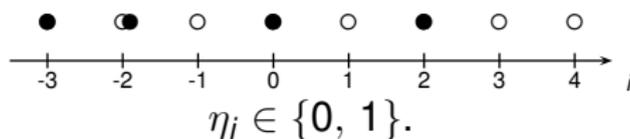
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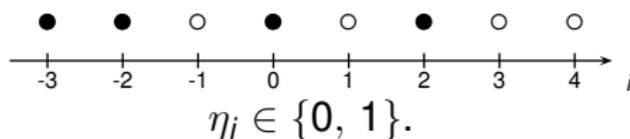
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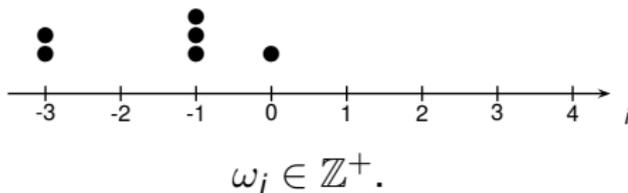
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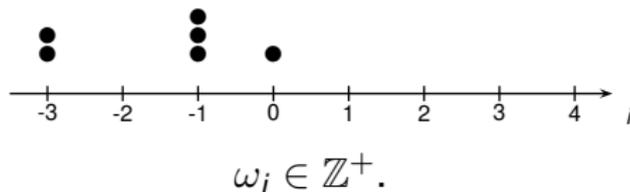
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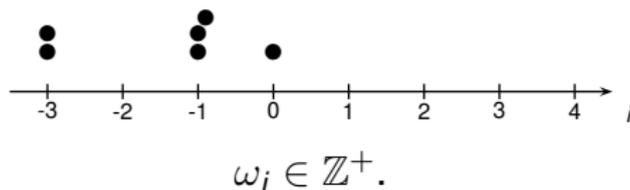


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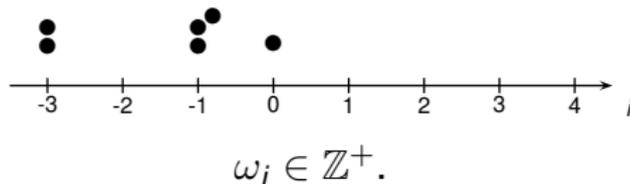
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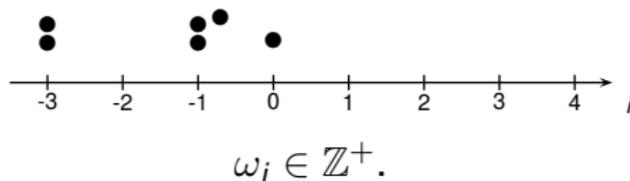
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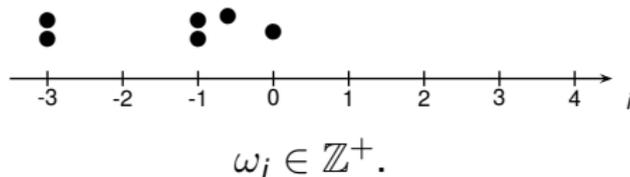
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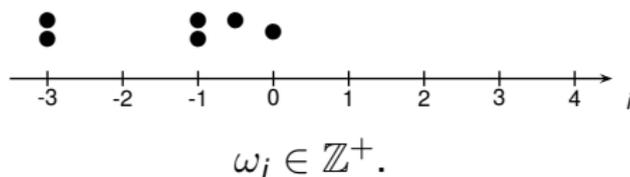
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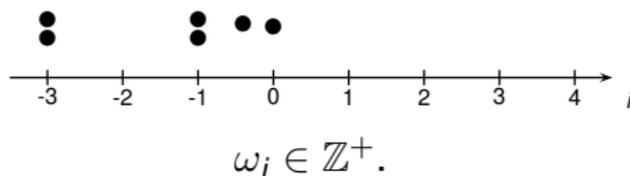
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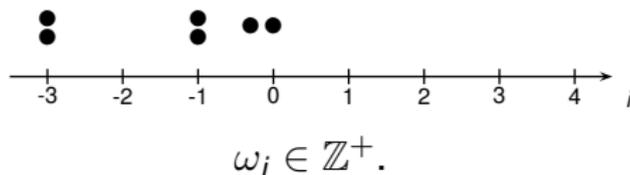
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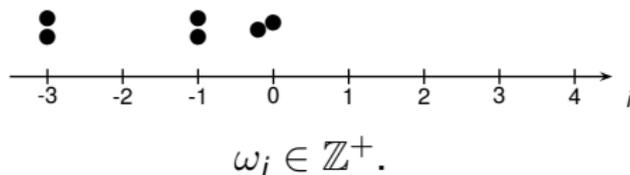
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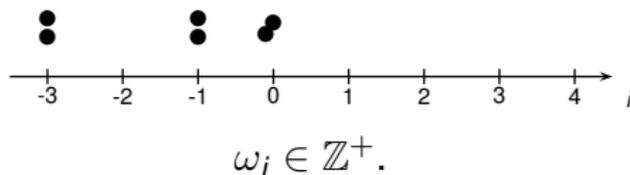
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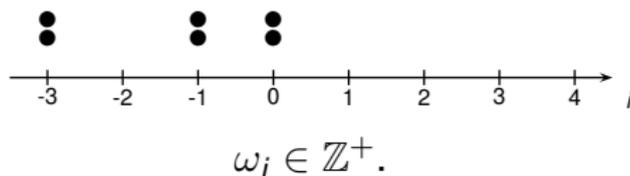
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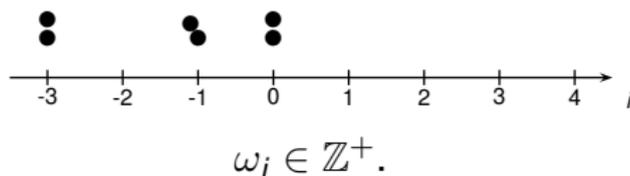
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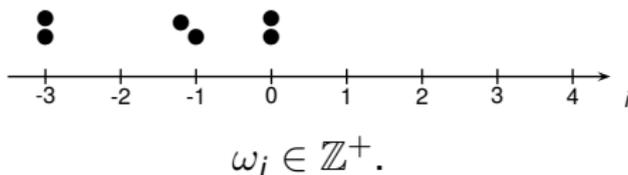
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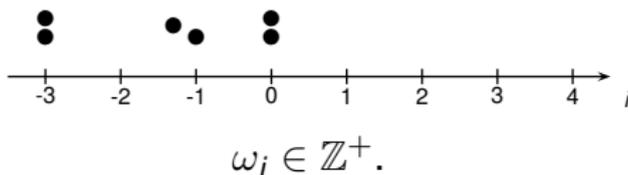
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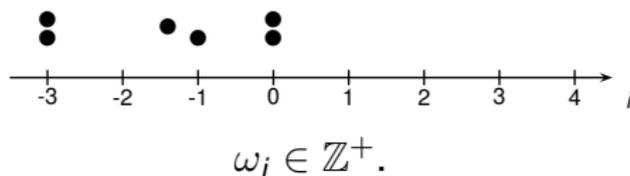
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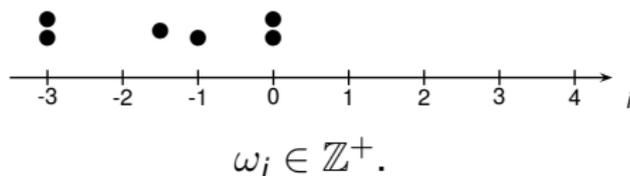
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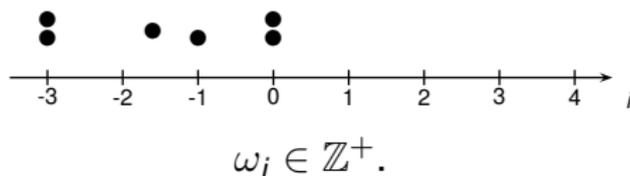
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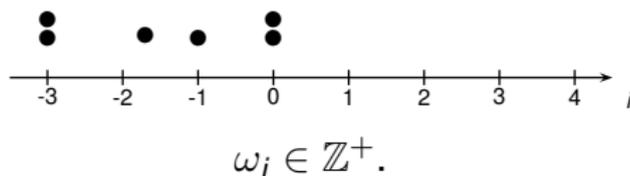
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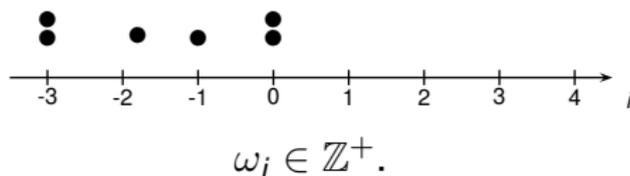
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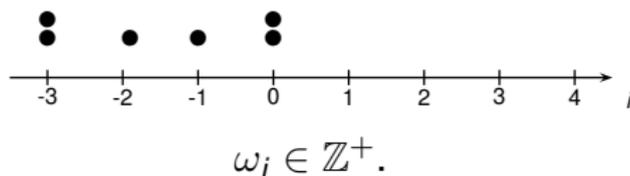
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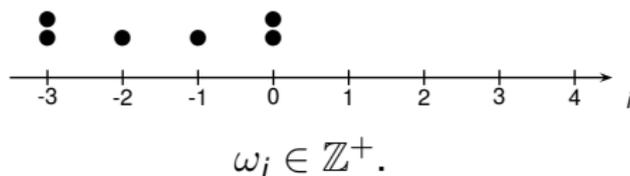
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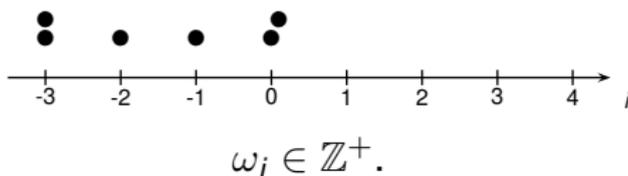
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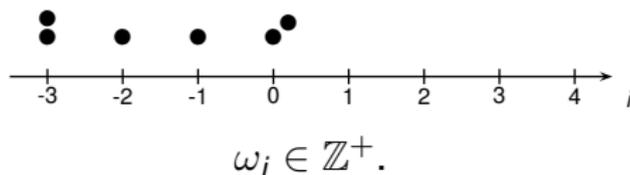
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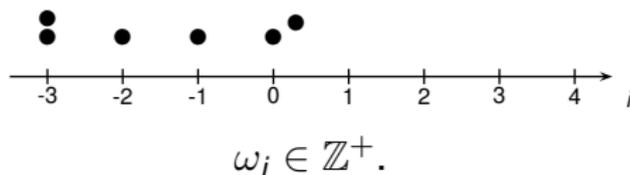
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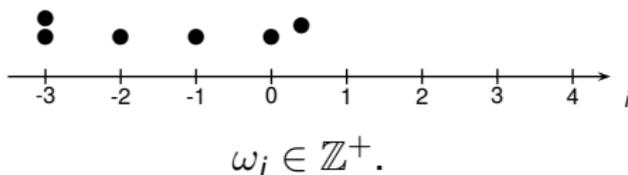
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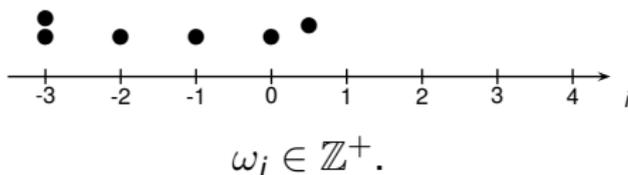
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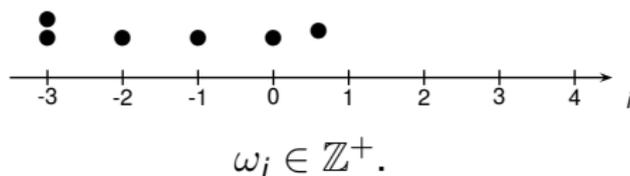
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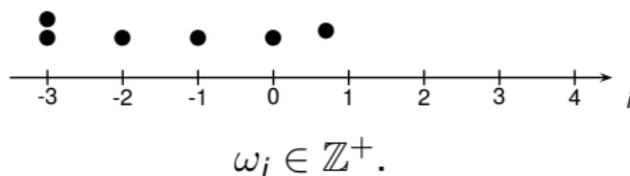
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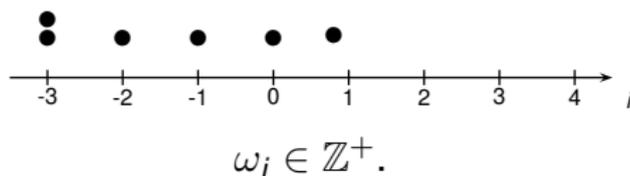
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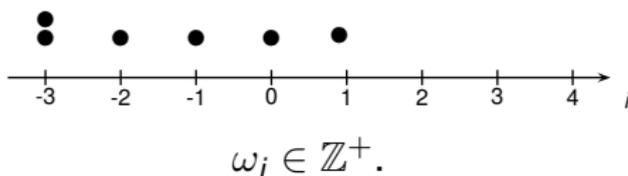
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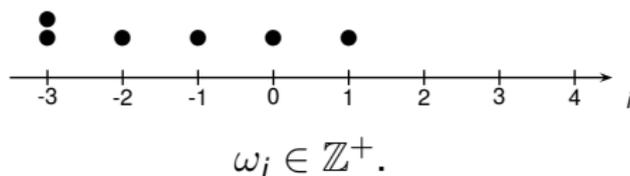
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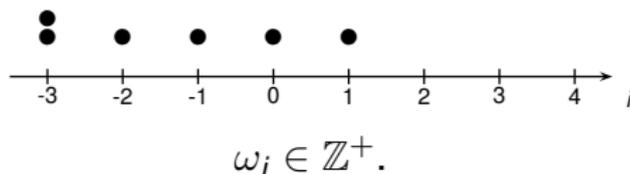
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Examples:

- ▶ 'Classical' ZRP:  $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$ .
- ▶ Independent walkers:  $r(\omega_i) = \omega_i$ .

## Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \rightsquigarrow i+1}) = \underline{\mu}(\underline{\omega}^{i \rightsquigarrow i+1}) \cdot \text{rate}(\underline{\omega}^{i \rightsquigarrow i+1} \rightarrow \underline{\omega}) \quad ?$$

Here

$$\underline{\omega}^{i \rightsquigarrow i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

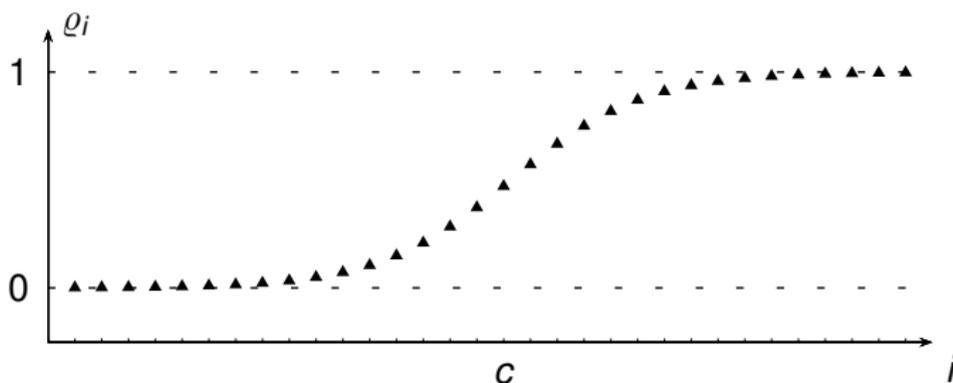
# Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

**ASEP:**  $\mu_i \sim \text{Bernoulli}(\varrho_i)$ ;   $\eta$

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

**Solution:** 
$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$



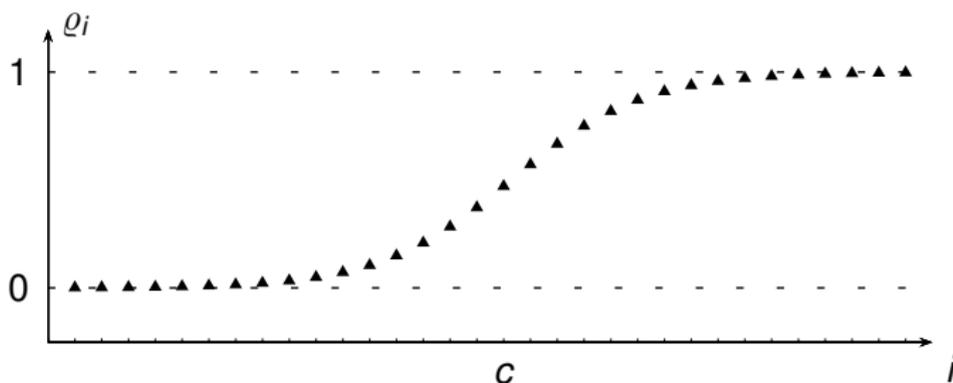
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## Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\omega}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\omega}^{i \curvearrowright i+1} \rightarrow \underline{\omega}) \quad ?$$

AZRP:

$$\mu_i(\omega_i) \mu_{i+1}(\omega_{i+1}) \cdot p \mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1) \mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

**Solution:**  $\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i - \text{const}}\right).$

# State space: ASEP

## Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + \left(\frac{\rho}{q}\right)^{i-c}} \quad \text{as } i \rightarrow \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{\left(\frac{q}{\rho}\right)^{i-c} + 1} \quad \text{as } i \rightarrow -\infty$$

are both summable. Hence by Borel-Cantelli there is  $\underline{\mu}$ -a.s. a rightmost hole and a leftmost particle,

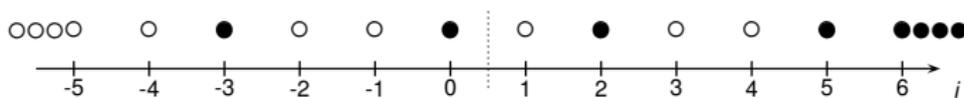
$$N := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

is  $\underline{\mu}$ -a.s. finite.

# State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

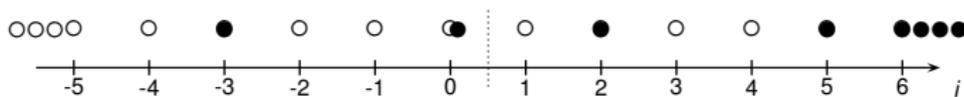
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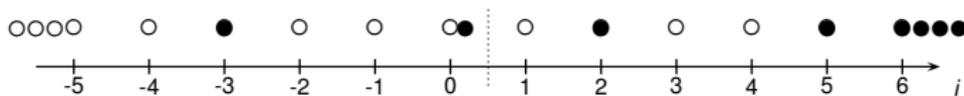
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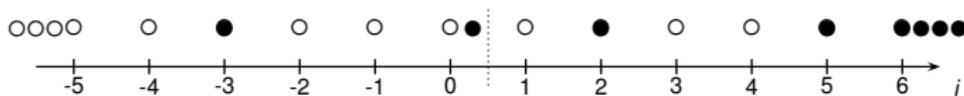
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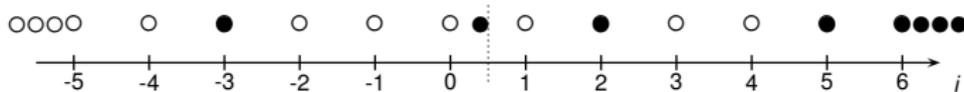
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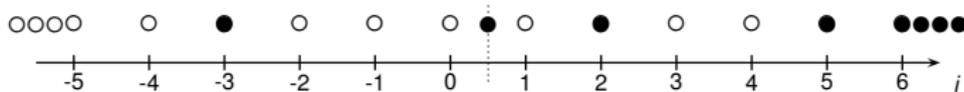
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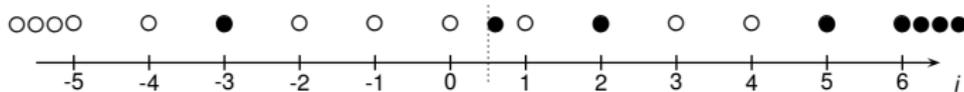
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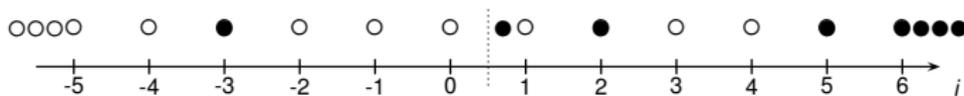
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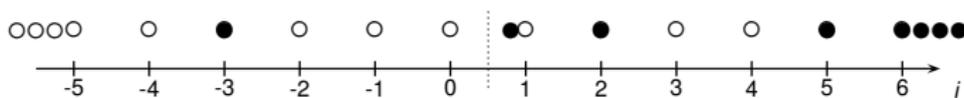
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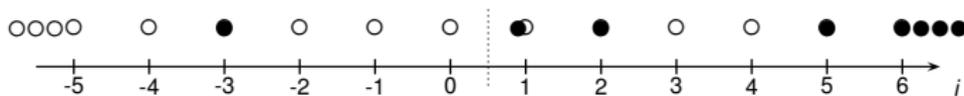
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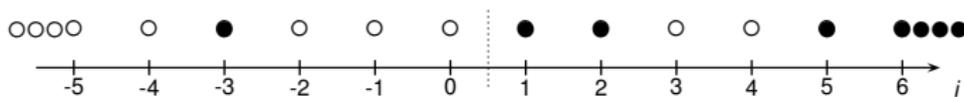
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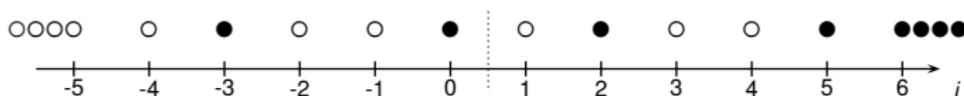
# State space: ASEP

Left shift:  $(\tau\underline{\eta})_i = \eta_{i+1}$ .

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

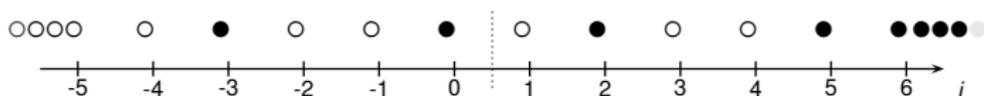
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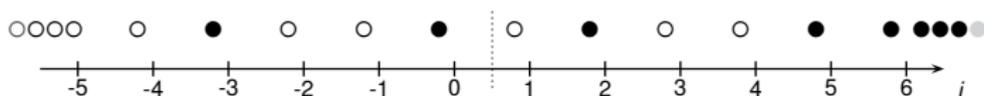
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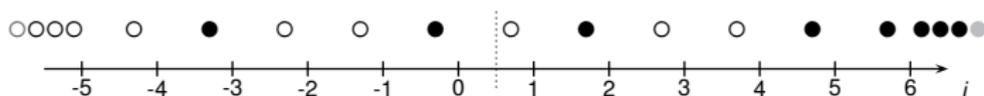
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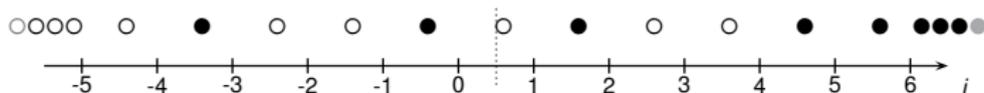
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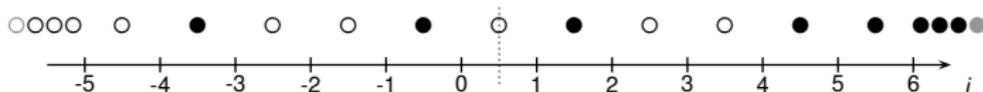
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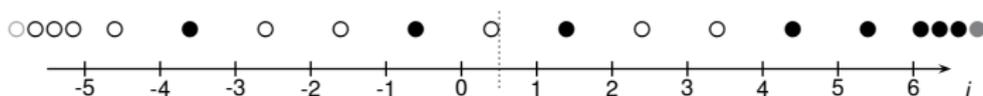
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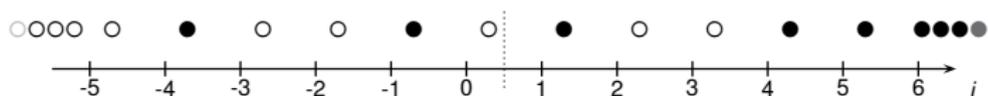
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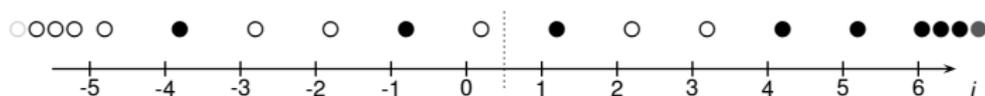
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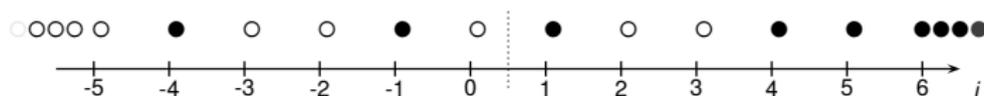
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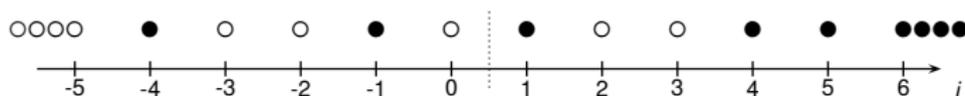
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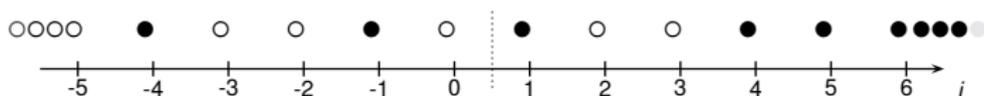
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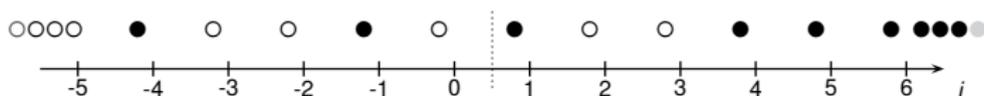
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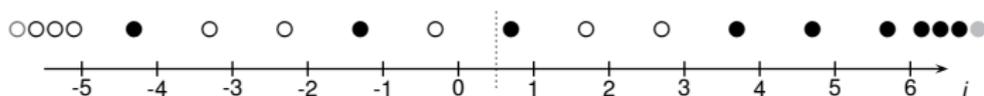
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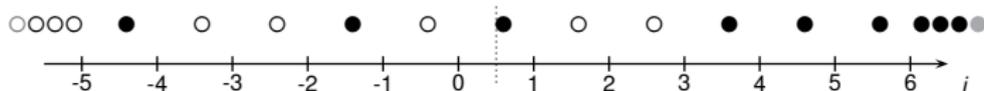
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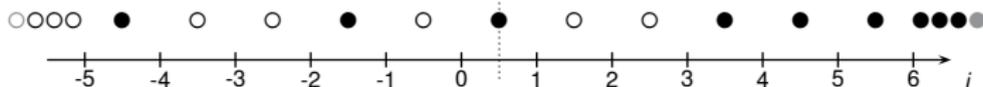
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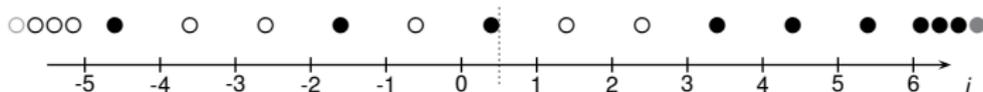
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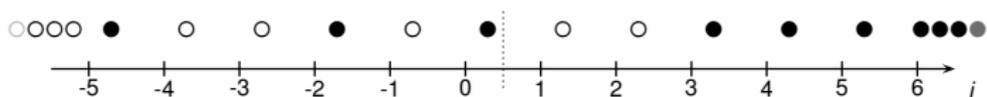
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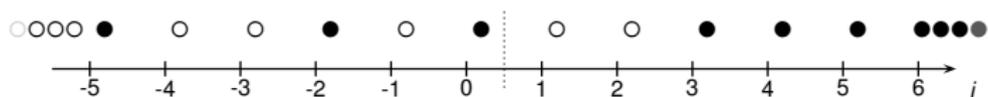
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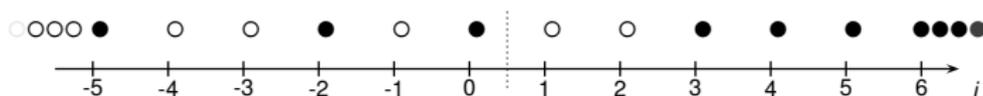
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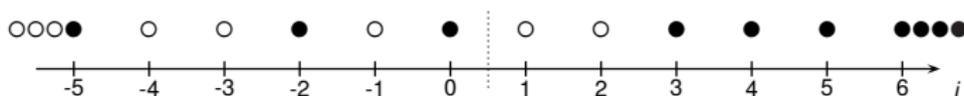
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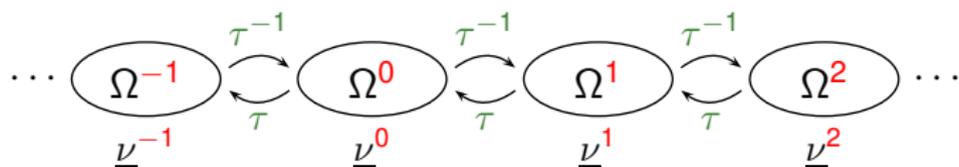
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# State space: ASEP



$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot | N(\cdot) = n) \underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot) \underline{\mu}(N(\cdot) = n).$$

with

$$\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot | N(\cdot) = n)$$

the **stationary measure on  $\Omega^n$** .

*Ergodic decomposition of  $\underline{\mu}$ .*

Let's find the coefficients  $\underline{\mu}(N(\cdot) = n)$ !

## State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\tau\eta}) = \dots = \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.$$

## State space: ASEP

Recall:

$$q_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

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So,

$$\underline{\mu}(N = n - 1) = \dots = \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.$$

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So,

$$\underline{\mu}(N = n - 1) = \dots = \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.$$

Solution:

$$\underline{\mu}(N = n) = \frac{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2}-cn}}{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2}-cm}}$$

*discrete Gaussian.*

## State space: ASEP

and, if  $N(\underline{\eta}) = n$ ,

$$\begin{aligned} \underline{\nu}^n(\underline{\eta}) &= \underline{\mu}(\underline{\eta} \mid N(\underline{\eta}) = n) = \frac{\underline{\mu}(\underline{\eta})}{\mu(N(\underline{\eta}) = n)} \\ &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}. \end{aligned}$$

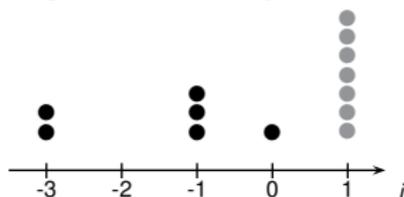
This is the unique stationary distribution on  $\Omega^n$ .

# State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

$\rightsquigarrow$  we have a problem: cannot do this for all  $i$ ! We'll pick  $\text{const} = 1$  and have a *right boundary* instead.

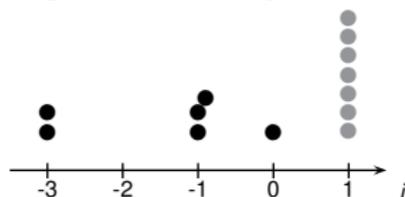


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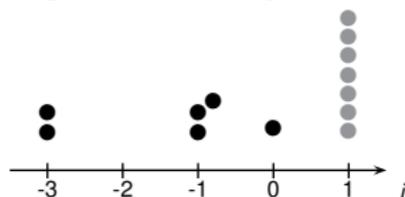


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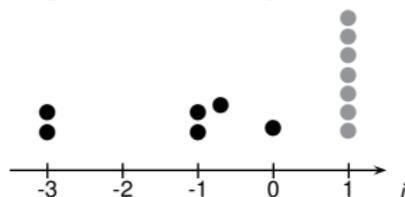


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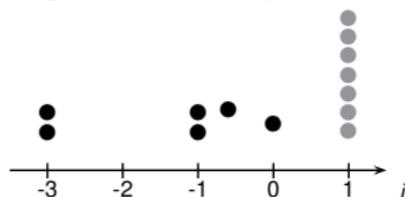


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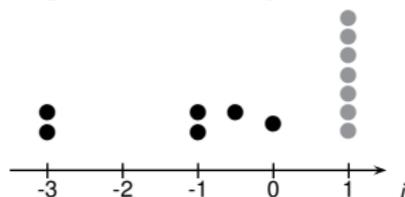


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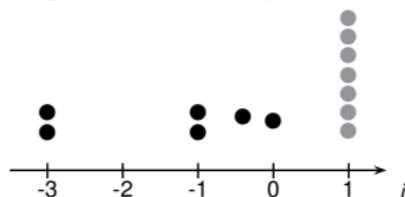


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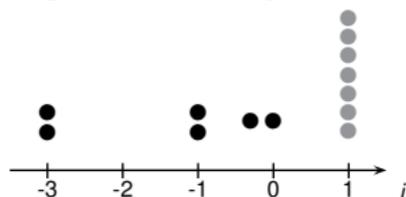


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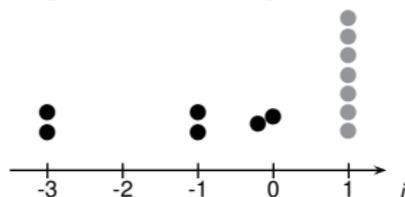


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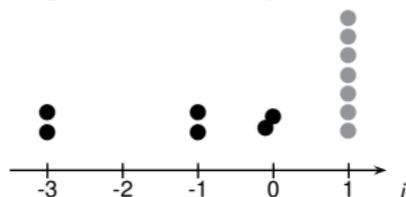


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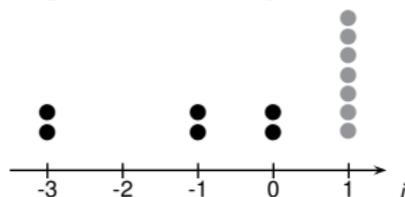


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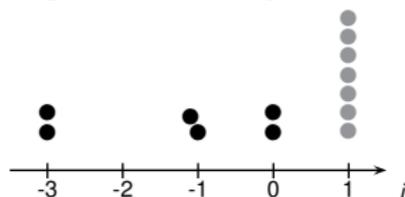


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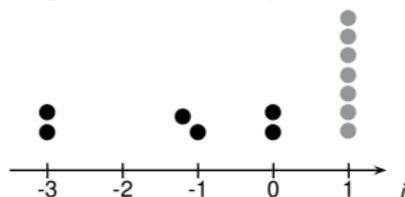


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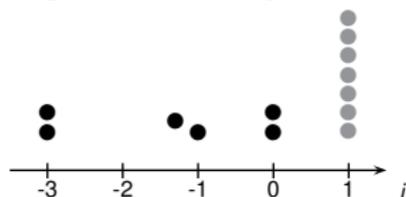


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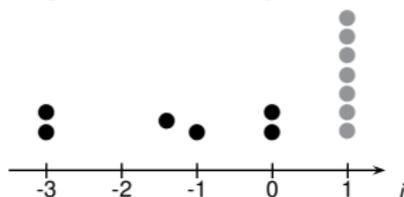


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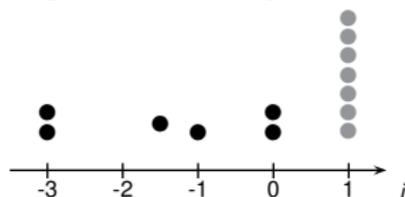


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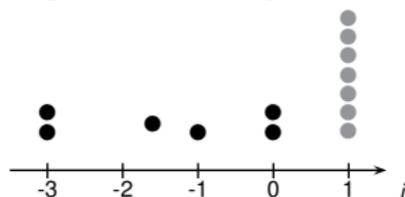


# State space: AZRP

Recall: Stationary distribution with marginals

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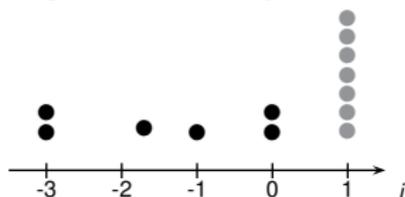


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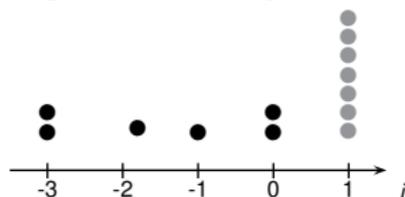


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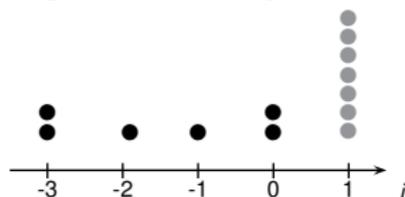


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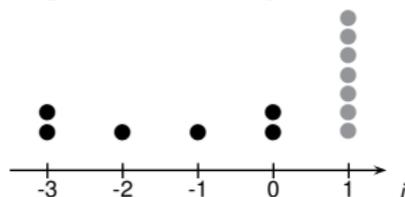


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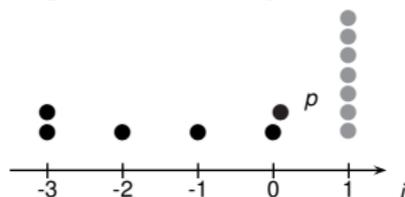


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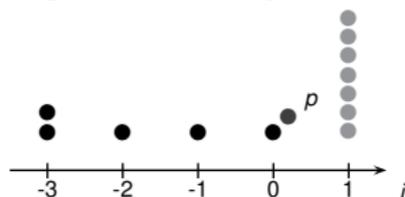


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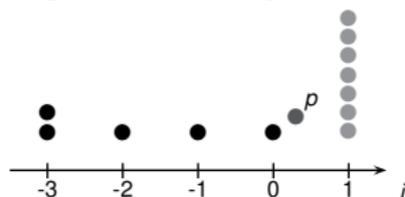


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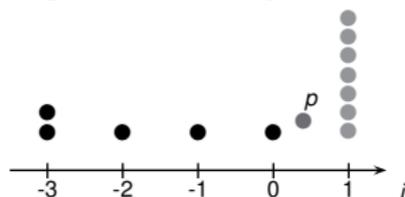


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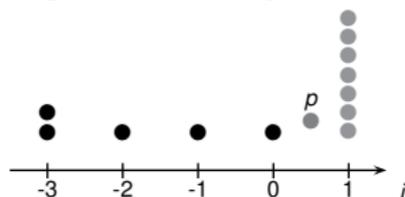


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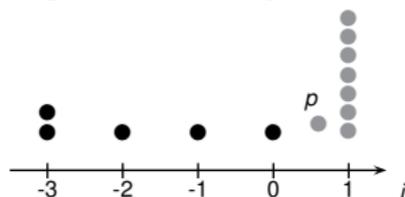


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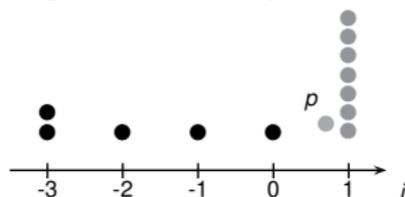


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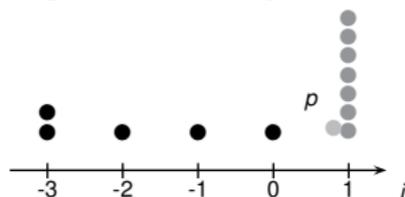


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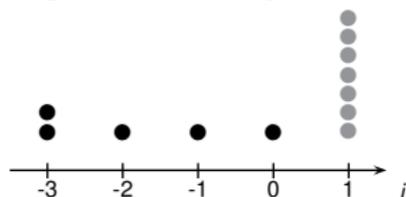


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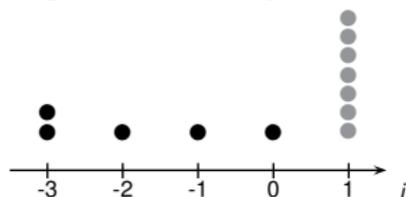


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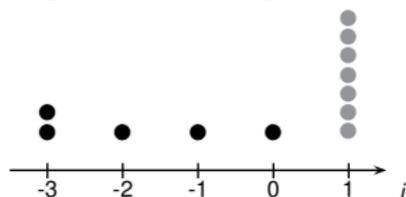


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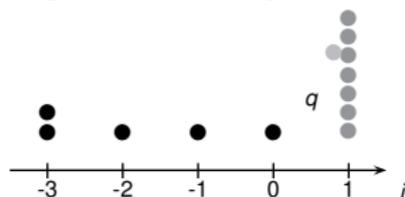


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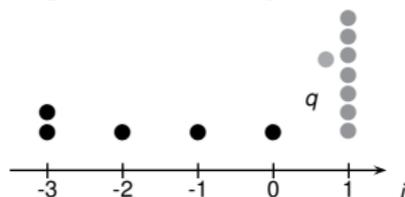


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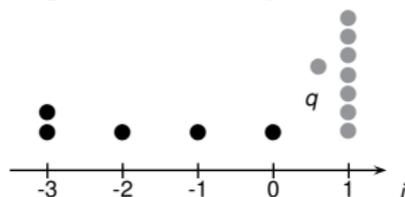


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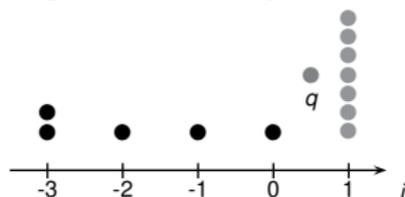


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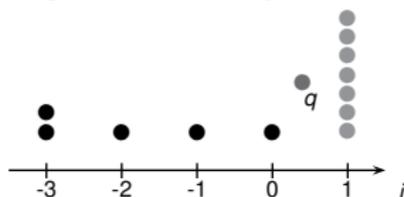


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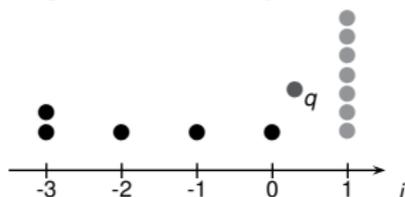


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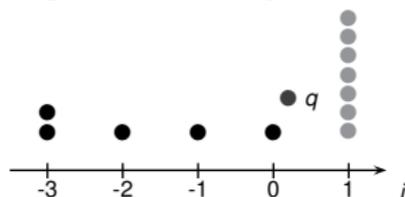


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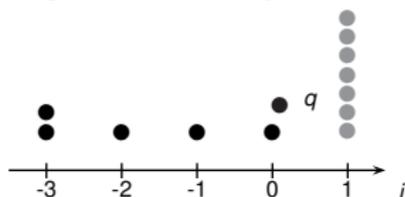


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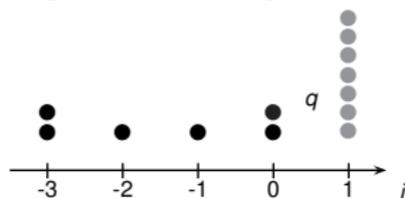


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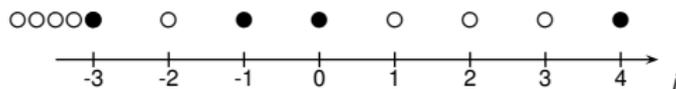
$\rightsquigarrow$  we have a problem: cannot do this for all  $i$ ! We'll pick  $\text{const} = 1$  and have a *right boundary* instead.



$\rightsquigarrow$  The product measure stays stationary on the half-line.

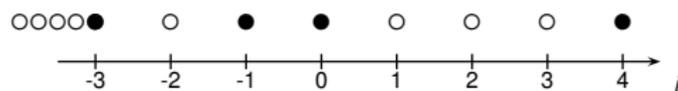
# Lay down / stand up

ASEP



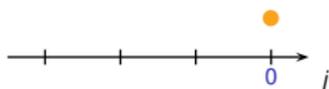
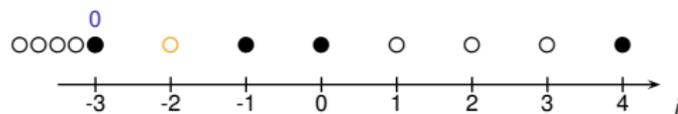
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ASEP



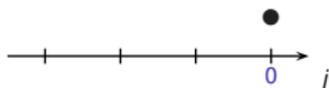
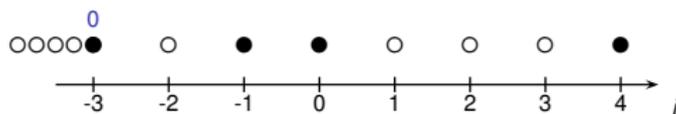
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## ASEP



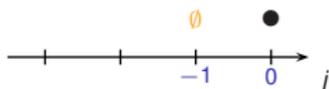
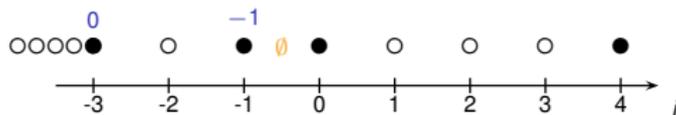
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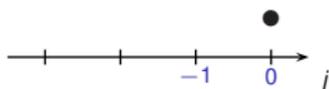
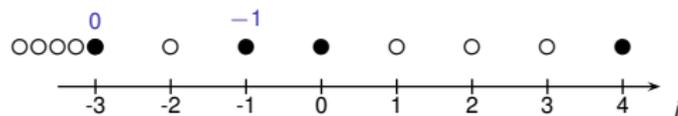
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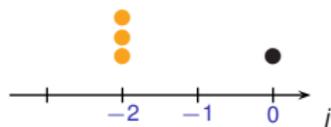
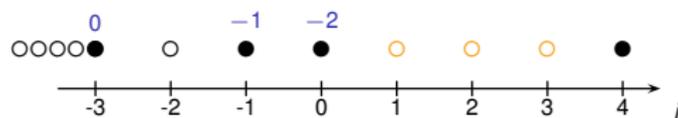
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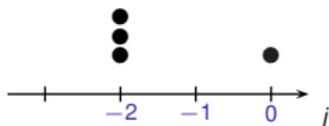
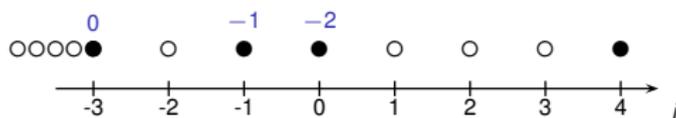
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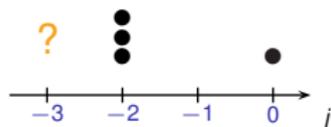
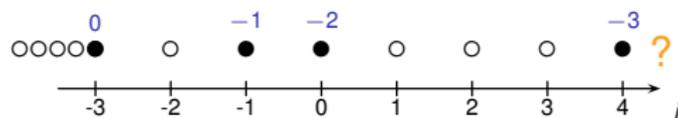
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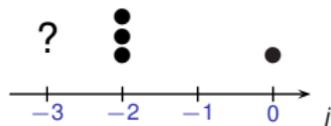
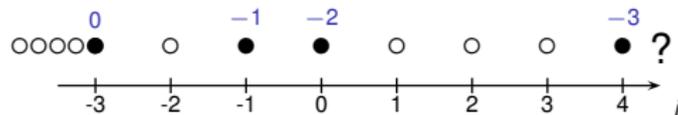
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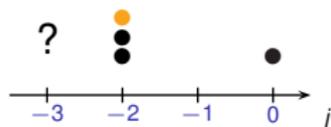
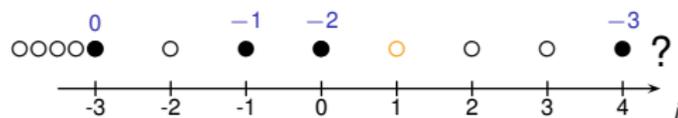
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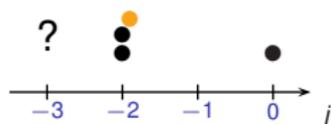
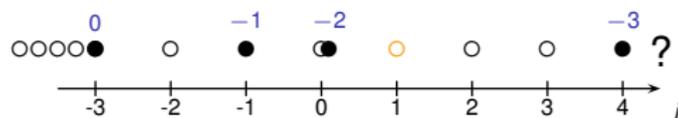
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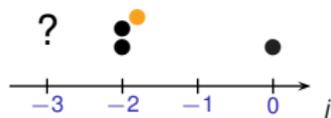
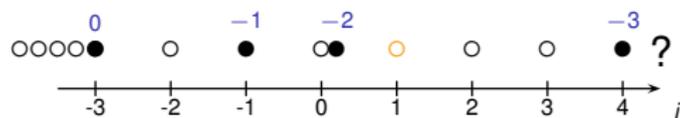
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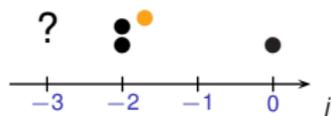
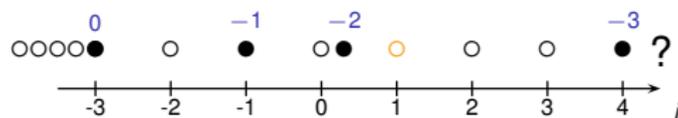
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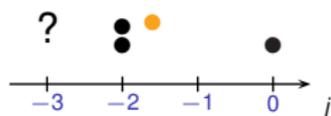
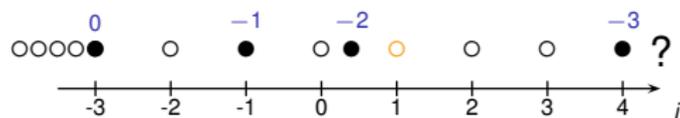
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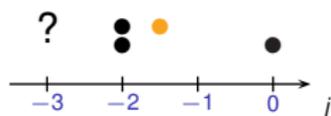
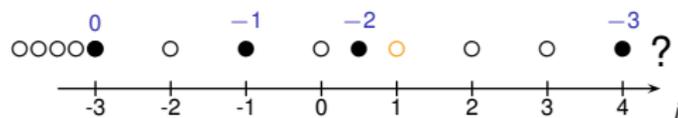
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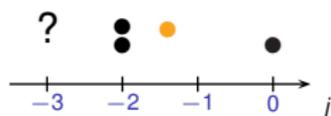
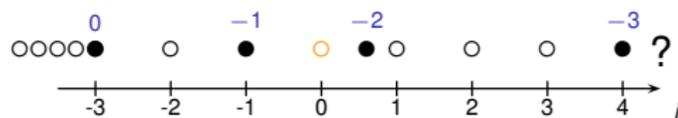
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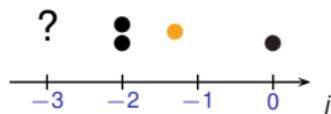
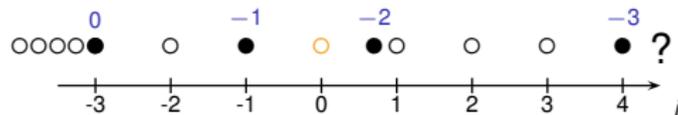
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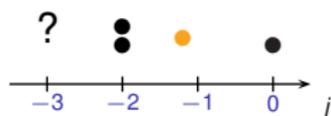
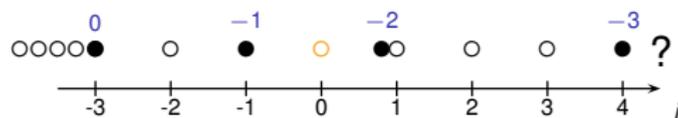
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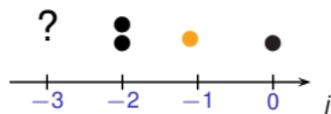
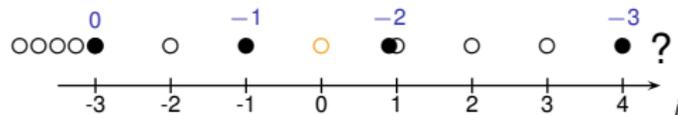
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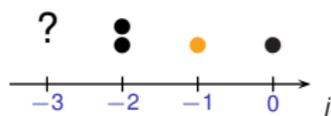
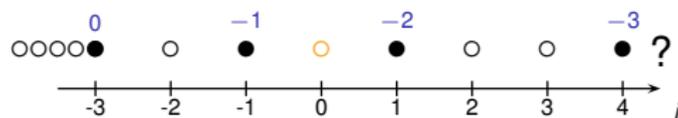
## Lay down / stand up

## ASEP



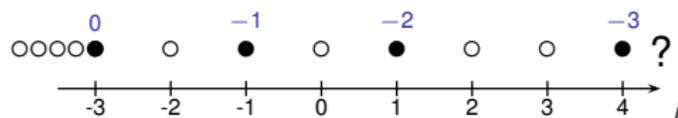
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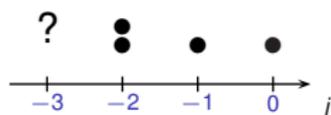


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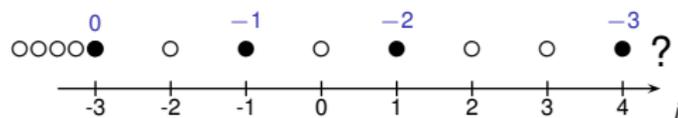


AZRP



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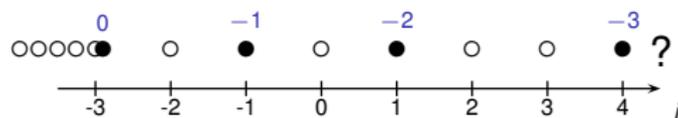


AZRP

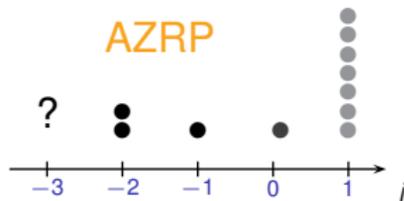


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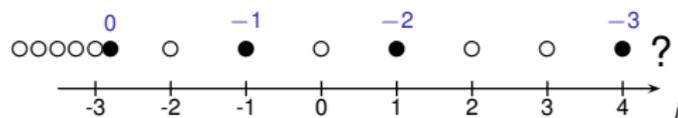


AZRP

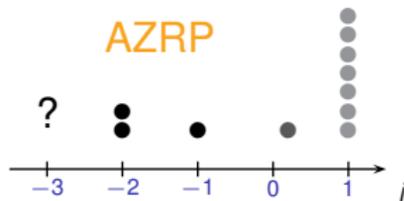


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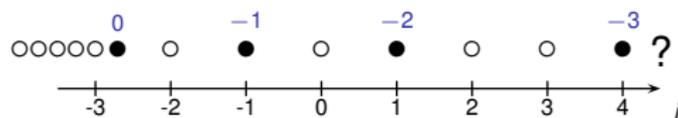


AZRP

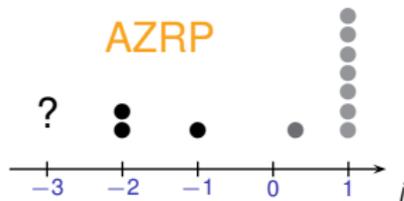


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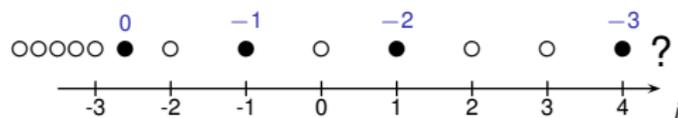


AZRP

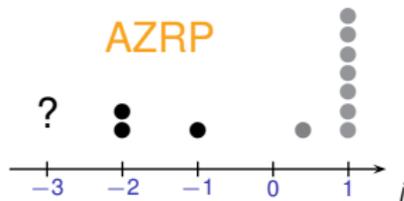


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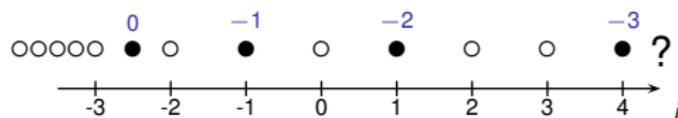


AZRP

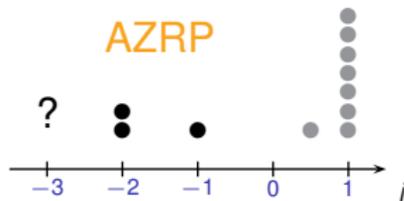


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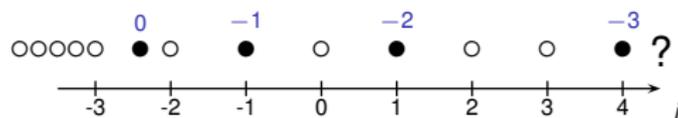


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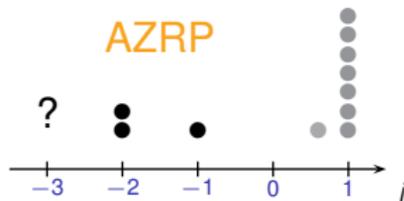


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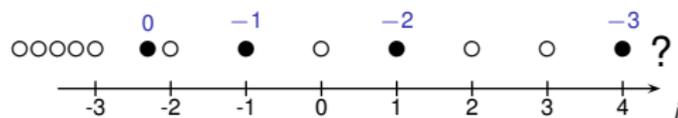


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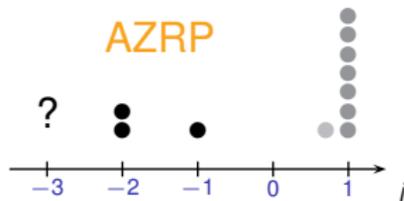


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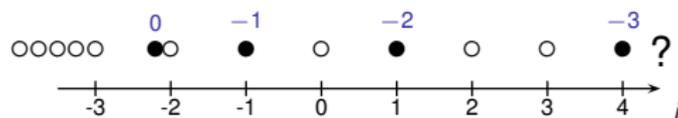


AZRP

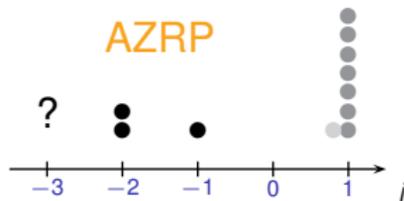


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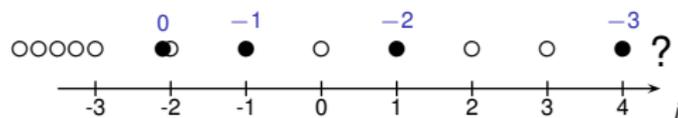


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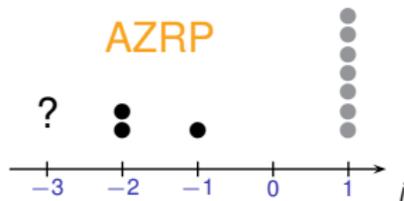


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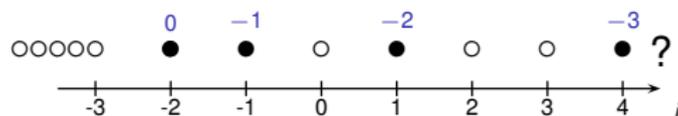


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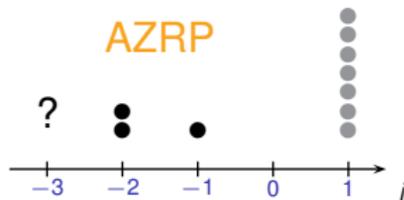


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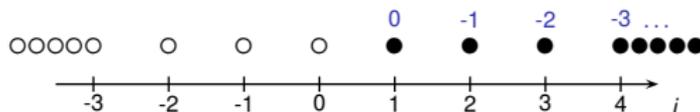
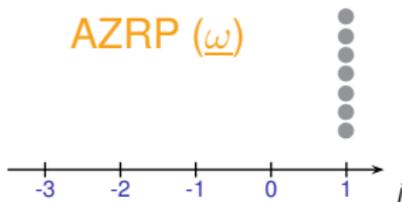


$$\text{ASEP} \stackrel{T^n}{=} \text{AZRP}$$

$$\underline{\nu}^n \stackrel{T^n}{=} \prod_{i \leq 0} \text{Geometric} \left( 1 - \left( \frac{p}{q} \right)^{i-1} \right)$$

since stationary distributions of countable irreducible Markov chains are unique.

## Jacobi triple product

ASEP ( $\underline{\eta}$ )AZRP ( $\underline{\omega}$ )

$$\eta_i = \mathbf{1}\{i \geq 1\}, \quad N(\underline{\eta}) = 0, \quad \omega_i \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c) \cdot 0}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c) \cdot (1-1)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2} - c \cdot 0}}$$

$$\underline{\mu}(\underline{\omega}) = \prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right)$$

# Jacobi triple product

$$\prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right) \cdot \prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right) \cdot \prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) = \sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}$$

LHS:

$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 - \left(\frac{q}{p}\right)^i\right) \cdot \left(1 + \left(\frac{q}{p}\right)^{i-1+c}\right) \cdot \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) \\ = \prod_{i=1}^{\infty} \left(1 - x^{2i}\right) \left(1 + \frac{x^{2i-1}}{y^2}\right) \left(1 + x^{2i-1}y^2\right) \end{aligned}$$

with  $x = \left(\frac{q}{p}\right)^{\frac{1}{2}}$ ,  $y = \left(\frac{q}{p}\right)^{\frac{1}{4}-\frac{c}{2}}$ .

RHS:

$$\sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2}{2}} \left(\frac{q}{p}\right)^{m(\frac{1}{2}-c)} = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$



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The point was that ASEP is in both lists.

Thank you.