Jacobi triple product and the like via the exclusion process and the like

Joint with
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University of Bristol

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Jacobi triple product

Theorem

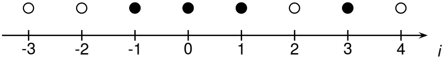
Let $|x| < 1$ and $y \neq 0$ be complex numbers. Then

$$\prod_{i=1}^{\infty}(1 - x^{2i})\Big(1 + \frac{x^{2i-1}}{y^2}\Big)(1 + x^{2i-1}y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in number theory and combinatorics of partitions.

We'll prove it using interacting particles (for real x, y only).

Asymmetric simple exclusion



$$\eta_i \in \{0, 1\}.$$

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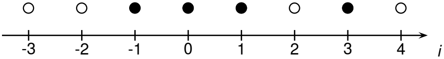

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to the right with rate p,
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The jump is suppressed if the destination site is occupied by another particle.

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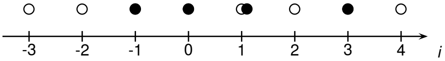

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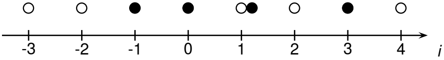

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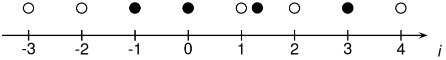

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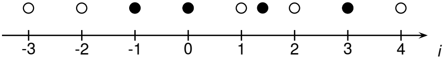

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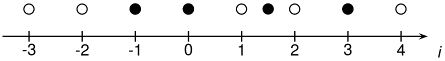

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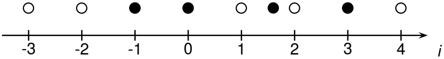

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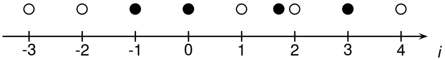

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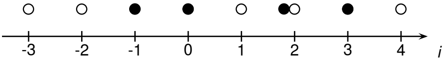

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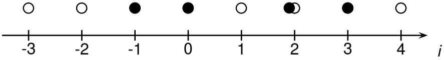

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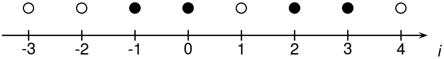

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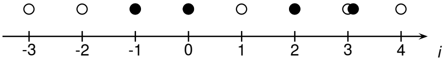

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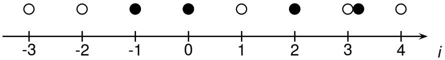

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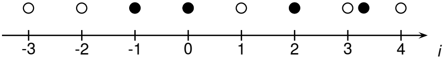

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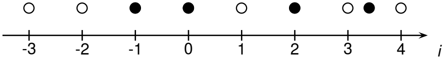

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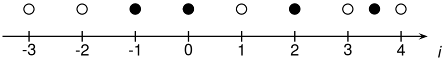

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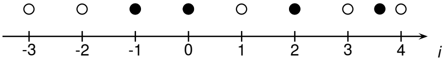

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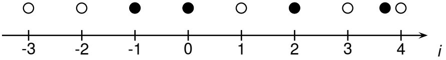

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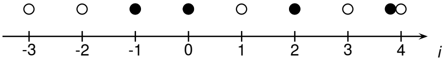

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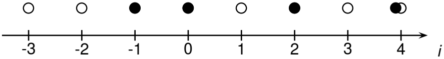

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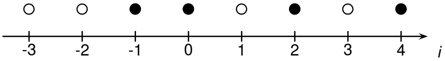

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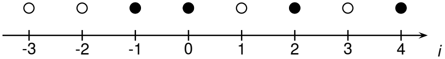

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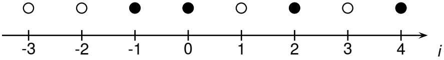

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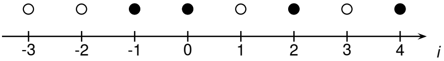

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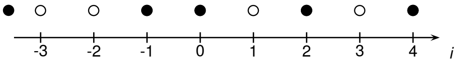

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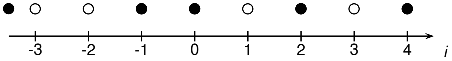

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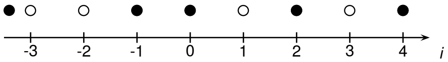

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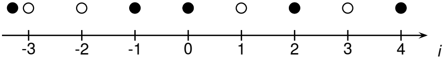

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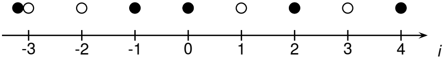

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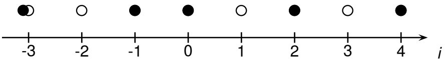

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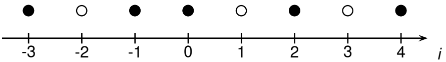

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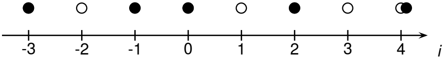

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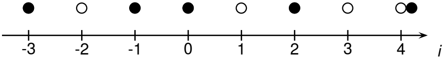

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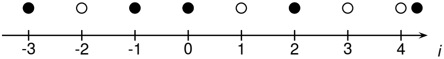

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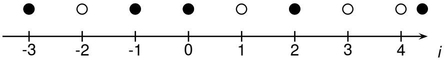

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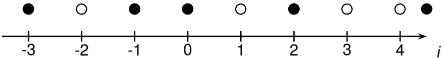

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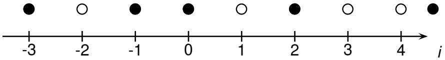

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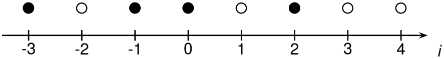

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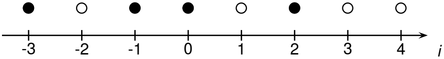

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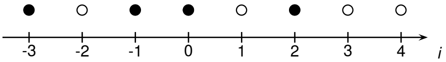

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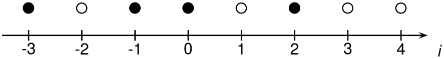

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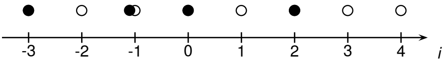

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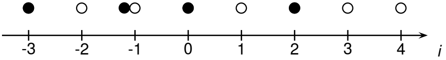

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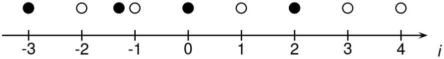

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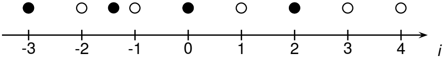

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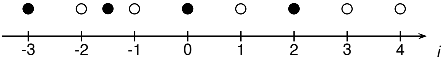

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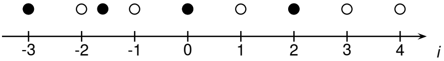

$$\eta_i \in \{0, 1\}.$$

Particles try to jump

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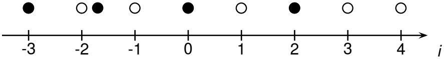

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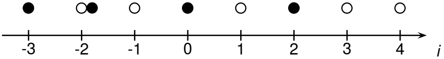

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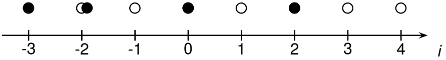

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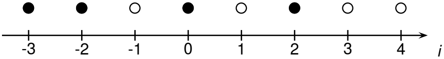

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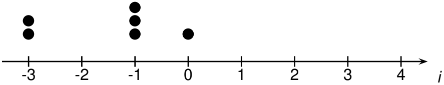
The asymmetric zero range process



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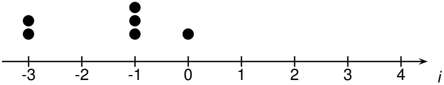

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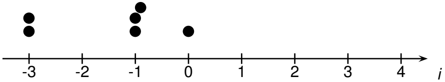

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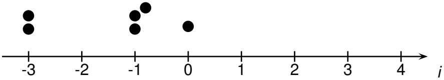

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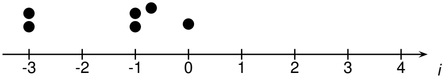

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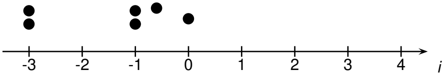

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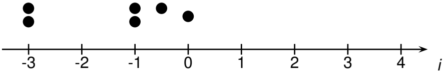

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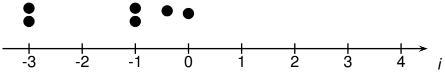

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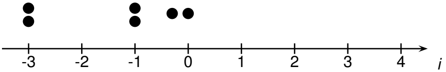

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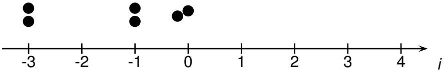

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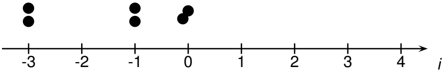

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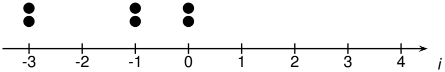

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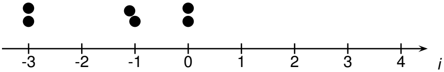

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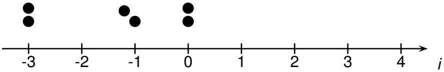

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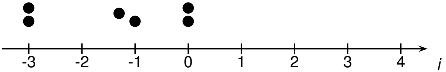

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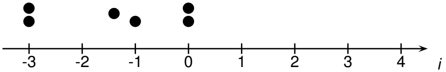

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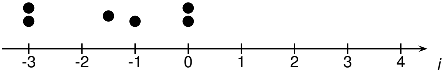

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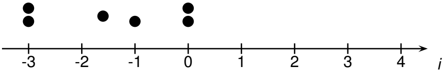

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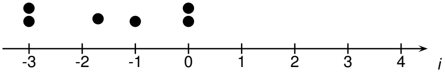

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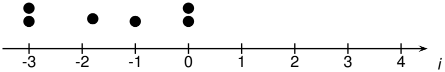

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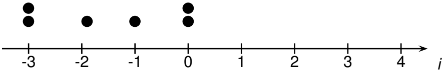

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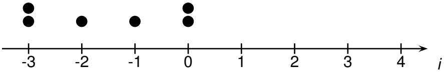

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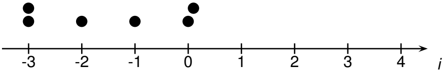

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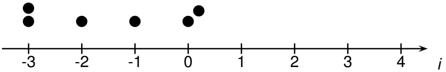

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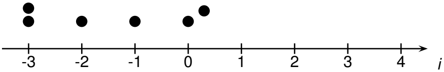

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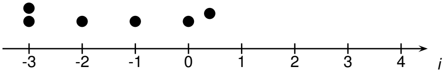

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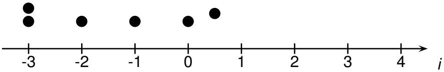

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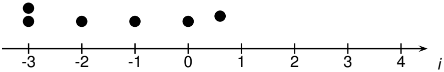

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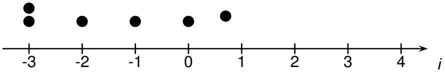

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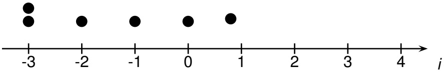

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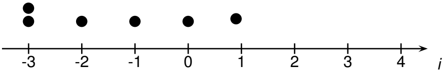

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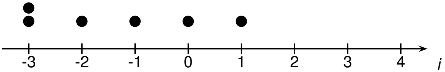

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The asymmetric zero range process

We need r non-decreasing and assume, as before, $q = 1 - p < p$.

Examples:

- 'Classical' ZRP: $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$.
- Independent walkers: $r(\omega_i) = \omega_i$.

Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i\curvearrowright i+1}\big) = \underline{\mu}\big(\underline{\omega}^{i\curvearrowright i+1}\big) \cdot \text{rate}\big(\underline{\omega}^{i\curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

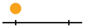
Here

$$\underline{\omega}^{i\curvearrowright i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}\big(\underline{\eta} \to \underline{\eta}^{i\curvearrowright i+1}\big) = \underline{\mu}(\underline{\eta}^{i\curvearrowright i+1}) \cdot \text{rate}\big(\underline{\eta}^{i\curvearrowright i+1} \to \underline{\eta}\big)$$

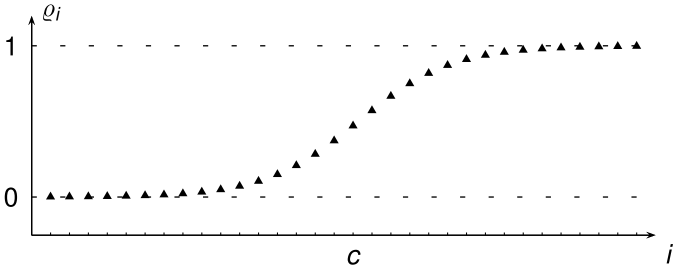
ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;



$\underline{\eta}$

$$\varrho_i(1-\varrho_{i+1}) \cdot p = (1-\varrho_i)\varrho_{i+1} \cdot q$$

Solution: $$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c}+1}$$



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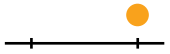

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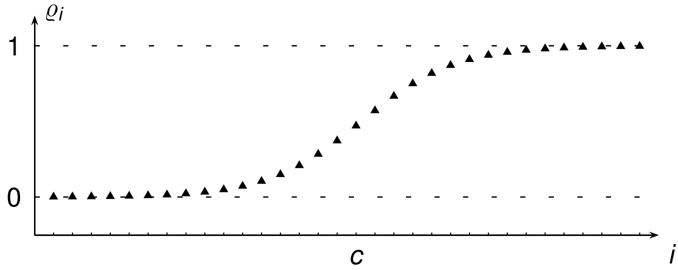


Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}\big(\underline{\omega} \to \underline{\omega}^{i\curvearrowright i+1}\big) = \underline{\mu}(\underline{\omega}^{i\curvearrowright i+1}) \cdot \text{rate}\big(\underline{\omega}^{i\curvearrowright i+1} \to \underline{\omega}\big) \quad ?$$

AZRP:

$$\mu_i(\omega_i)\mu_{i+1}(\omega_{i+1}) \cdot p\mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1)\mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

Solution: $\mu_i \sim \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-\text{const}}\Big).$

State space: ASEP

Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + (\frac{p}{q})^{i-c}} \qquad \text{as } i \to \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{(\frac{q}{p})^{i-c} + 1} \qquad \text{as } i \to -\infty$$

are both summable. Hence by Borel-Cantelli there is $\underline{\mu}$-a.s. a rightmost hole and a leftmost particle,

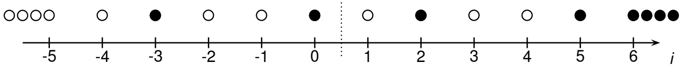
$$N := \sum_{i=1}^{\infty}(1 - \eta_i) - \sum_{i=-\infty}^{0} \eta_i$$

is $\underline{\mu}$-a.s. finite.

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty}(1-\eta_i) - \sum_{i=-\infty}^{0} \eta_i$$

$$1 = 3 - 2$$



Also: it is conserved. Define

$$\Omega^n := \{0,\,1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

the irreducible components of the state space.

$$\underline{\mu}\Big(\bigcup_{n=-\infty}^{\infty} \Omega^n\Big) = 1.$$

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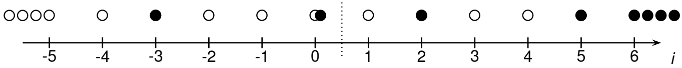

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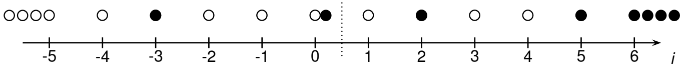

Also: it is conserved. Define

$$\Omega^n := \{0,\,1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

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$$\underline{\mu}\Big(\bigcup_{n=-\infty}^{\infty}\Omega^n\Big) = 1.$$

State space: ASEP

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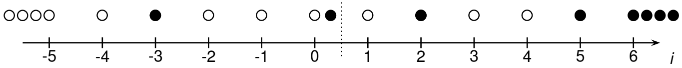

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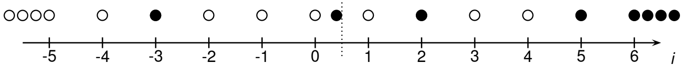

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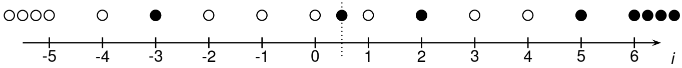

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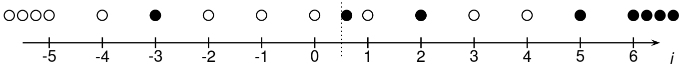

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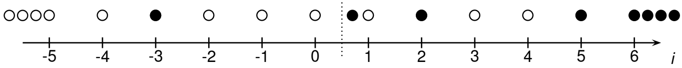

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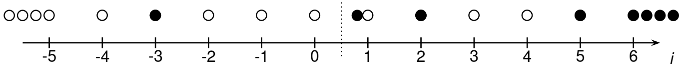

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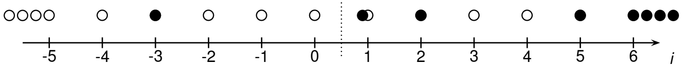

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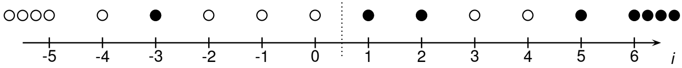

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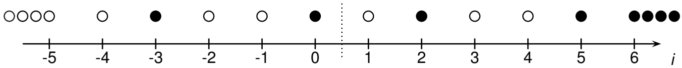

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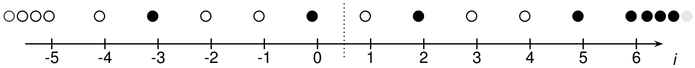

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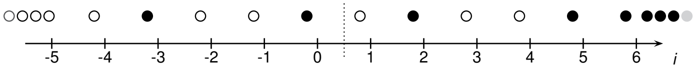

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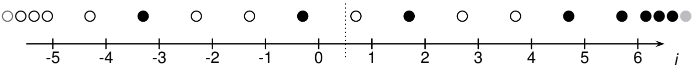

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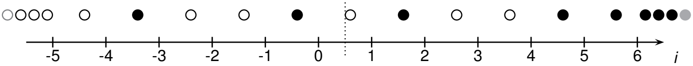

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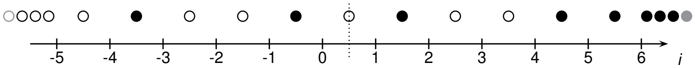

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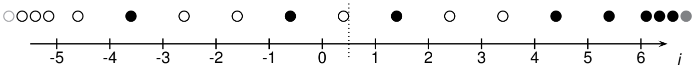

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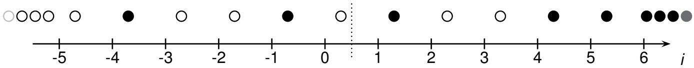

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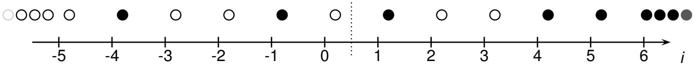

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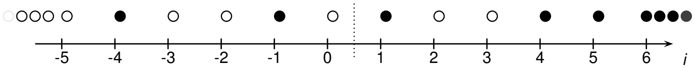

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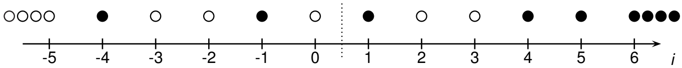

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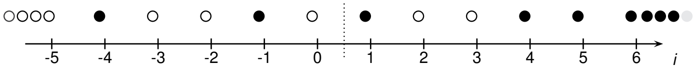

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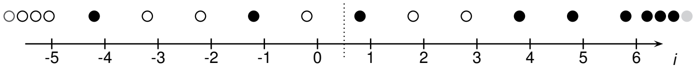

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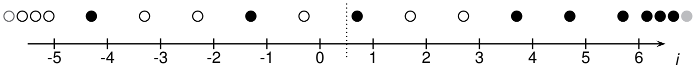

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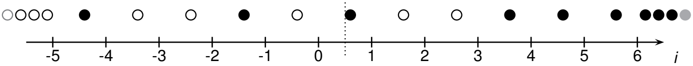

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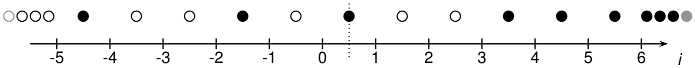

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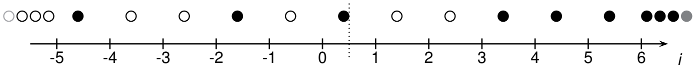

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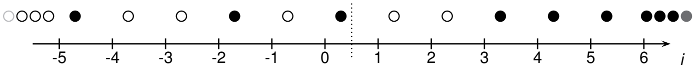

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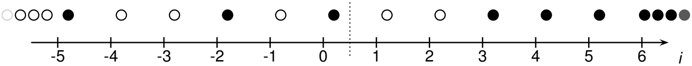

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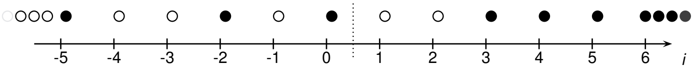

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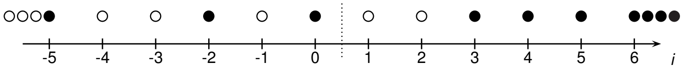
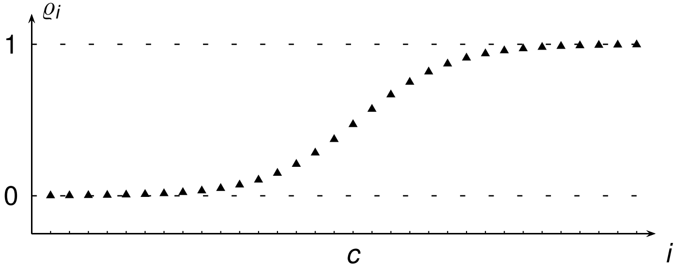


$$\tau \; : \; \Omega^n \to \Omega^{n-1}$$

State space: ASEP

Recall

$$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}},$$

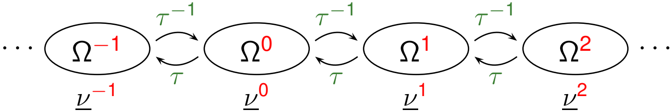


but

$$\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot \mid N(\cdot) = n)$$

doesn't depend on c anymore. Stationary measure on Ω^n.

State space: ASEP



$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot \mid N(\cdot) = n)\underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot)\underline{\mu}(N(\cdot) = n).$$

Ergodic decomposition of $\underline{\mu}$.

Let's find the coefficients $\underline{\mu}(N(\cdot) = n)$!

State space: ASEP

Recall:

$$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1+(\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c}+1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i\le 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1+\left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i>0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c}+1}$$

$$= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}$$

State space: ASEP

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State space: ASEP

$$\underline{\mu}(\tau\underline{\eta}) = \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}$$

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$$= \frac{\prod_{i\le 1}\left(\frac{p}{q}\right)^{(i-c-1)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>1}\left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}$$

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$$\underline{\mu}(\tau\underline{\eta}) = \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}$$

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State space: ASEP

$$
\begin{aligned}
\underline{\mu}(\tau\underline{\eta}) &= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \\
&= \frac{\prod_{i\le 1}\left(\frac{p}{q}\right)^{(i-c-1)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>1}\left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \\
&= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c-1)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \cdot \left(\frac{p}{q}\right)^{-c} \\
&= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}
\end{aligned}
$$

State space: ASEP

$$\begin{aligned}
\underline{\mu}(\tau\underline{\eta}) &= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)}\cdot\frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}\\
&= \frac{\prod_{i\le 1}\left(\frac{p}{q}\right)^{(i-c-1)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)}\cdot\frac{\prod_{i>1}\left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}\\
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&= \frac{\prod_{i\le 0}\left(\frac{p}{q}\right)^{(i-c)\eta_{i}}}{\prod_{i\le 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)}\cdot\frac{\prod_{i>0}\left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i})}}{\prod_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)}\cdot\left(\frac{p}{q}\right)^{N(\underline{\eta})-c}\\
&= \underline{\mu}(\underline{\eta})\cdot\left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.
\end{aligned}$$

State space: ASEP

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta})$$

State space: ASEP

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta})$$

$$= \sum_{\underline{\eta}:N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta})$$

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$$= \sum_{\underline{\eta}:N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta})$$

$$= \sum_{\underline{\eta}:N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}$$

State space: ASEP

So,

$$\begin{aligned}
\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\
&= \sum_{\underline{\eta}:N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\
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\end{aligned}$$

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$$\underline{\mu}(N=n-1) = \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta})$$

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$$= \underline{\mu}(N=n) \cdot \left(\frac{p}{q}\right)^{n-c}.$$

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$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}:N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta})$$
$$= \sum_{\underline{\eta}:N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta})$$
$$= \sum_{\underline{\eta}:N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c}$$
$$= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.$$

Solution:

$$\underline{\mu}(N = n) = \frac{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2}-cn}}{\sum\left(\frac{q}{p}\right)^{\frac{m^2+m}{2}-cm}}$$

discrete Gaussian.

State space: ASEP

and, if $N(\underline{\eta}) = n$,

$$\underline{\nu}^n(\underline{\eta}) = \underline{\mu}(\underline{\eta} \,|\, N(\underline{\eta}) = n) = \frac{\underline{\mu}(\underline{\eta})}{\underline{\mu}(N(\underline{\eta}) = n)}$$

$$= \frac{\prod\limits_{i\le 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod\limits_{i\le 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod\limits_{i>0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod\limits_{i>0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}.$$

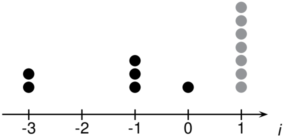
This is the unique stationary distribution on Ω^n.

State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-\text{const}}\Big).$$

$\rightsquigarrow$ we have a problem: cannot do this for all i! We'll pick const $= 1$ and have a *right boundary* instead.



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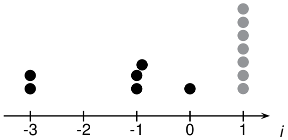

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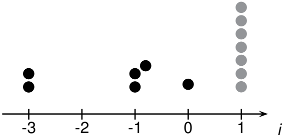

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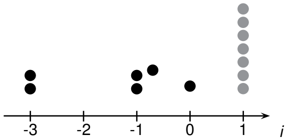

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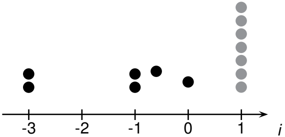

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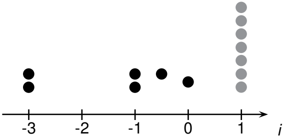

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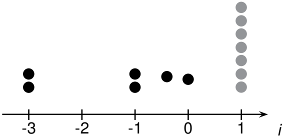

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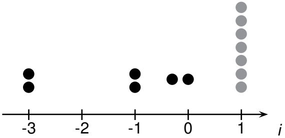

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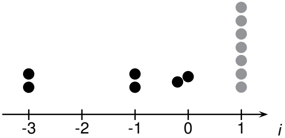

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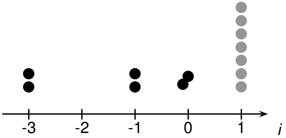

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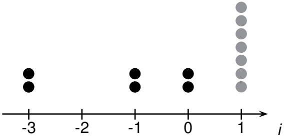

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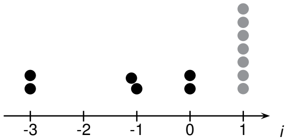

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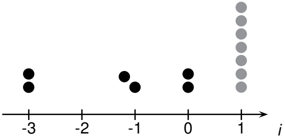

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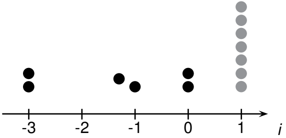

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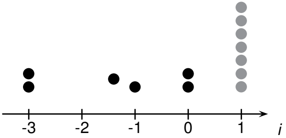

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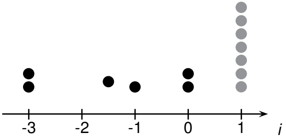

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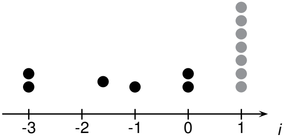

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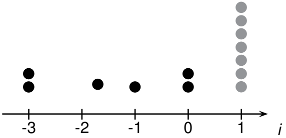

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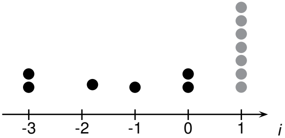

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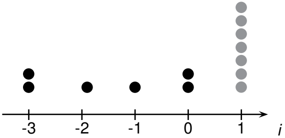

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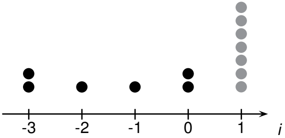

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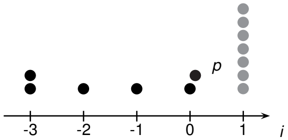

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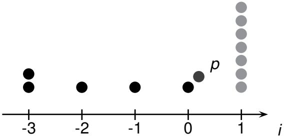

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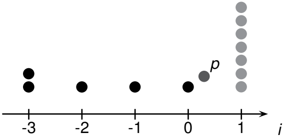

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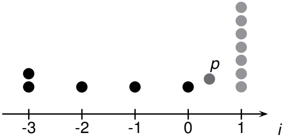

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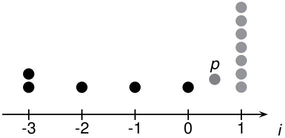

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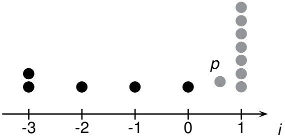

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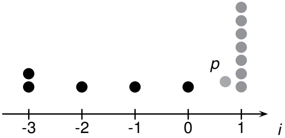
$$\mu_i \sim \text{Geometric}\Big(1 - \Big(\frac{p}{q}\Big)^{i-\text{const}}\Big).$$

$\rightsquigarrow$ we have a problem: cannot do this for all i! We'll pick const $= 1$ and have a *right boundary* instead.



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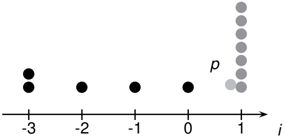

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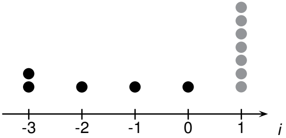

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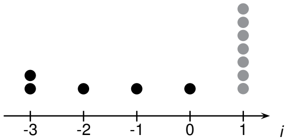

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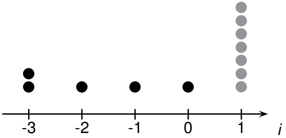

State space: AZRP

Recall: Stationary distribution with marginals

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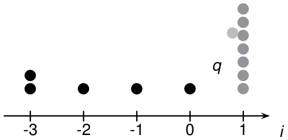

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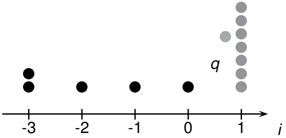

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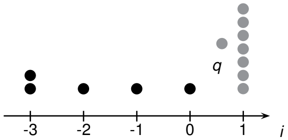

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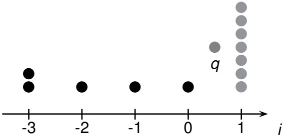

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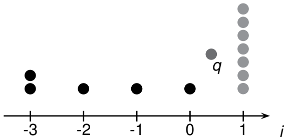

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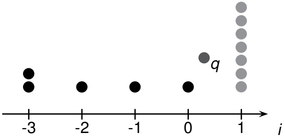

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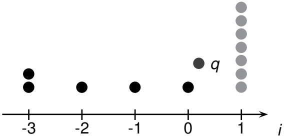

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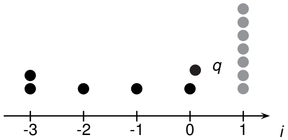

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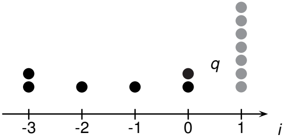
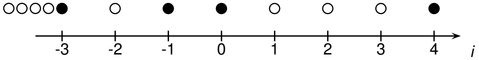


$\rightsquigarrow$ The product measure stays stationary on the half-line.

Lay down / stand up

ASEP



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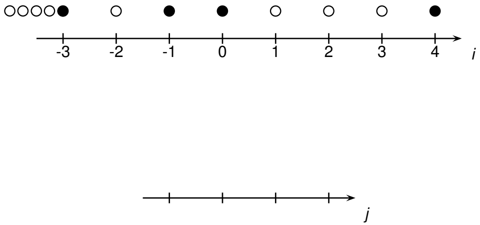

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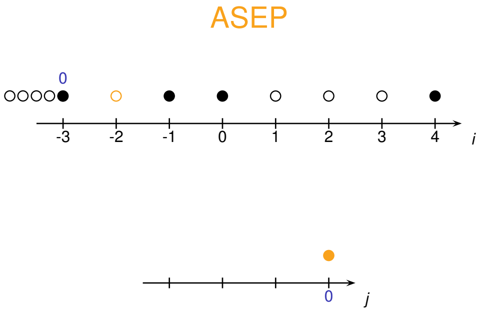

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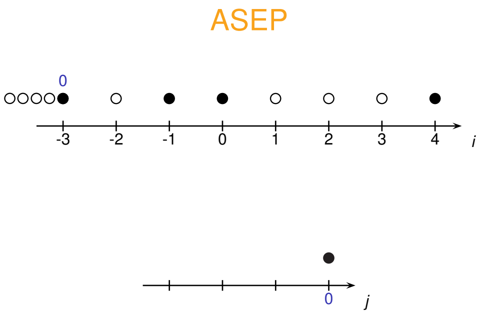

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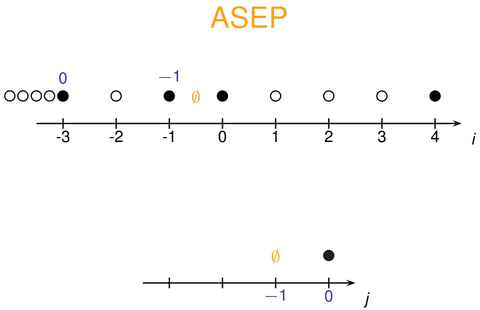

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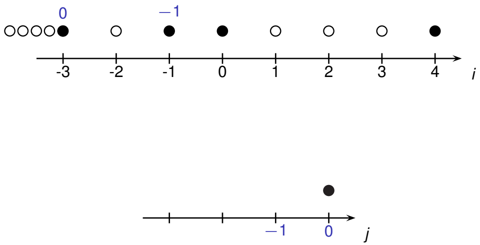

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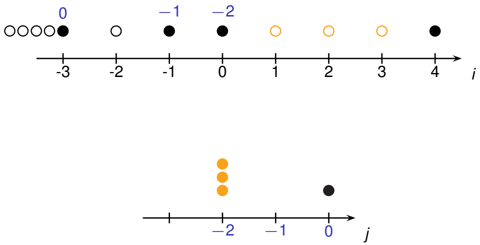

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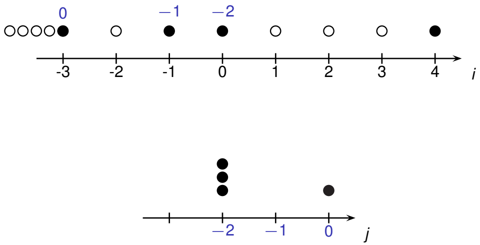

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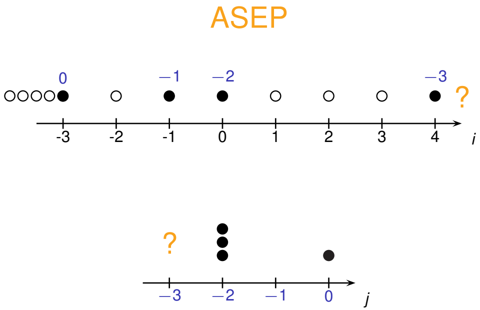

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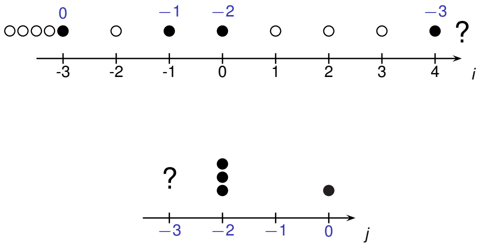

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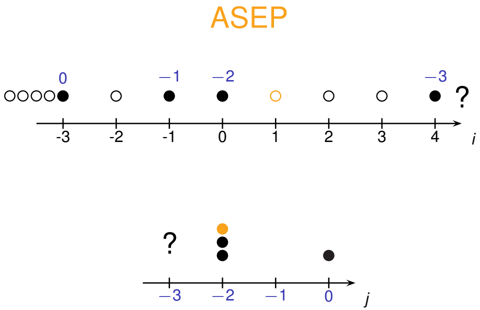

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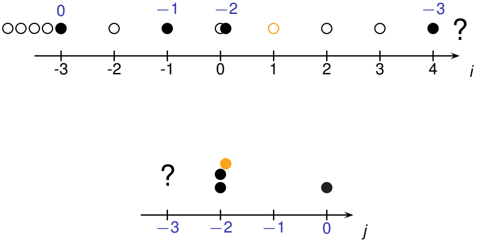

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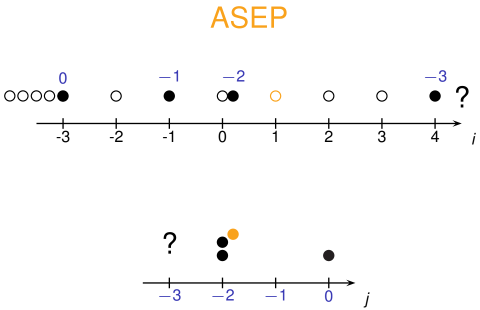

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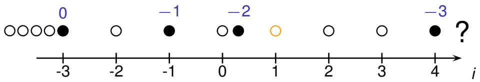

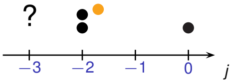

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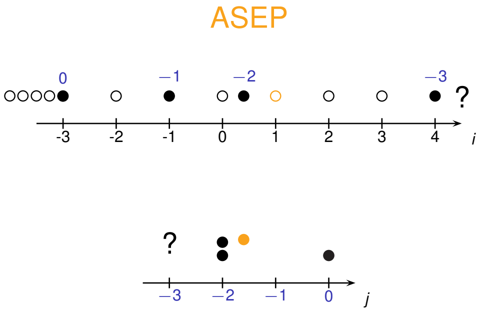

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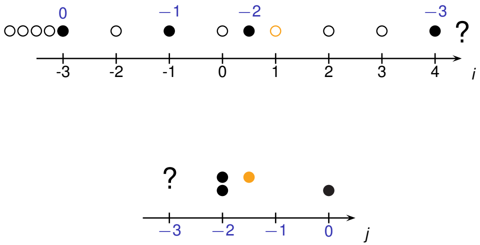

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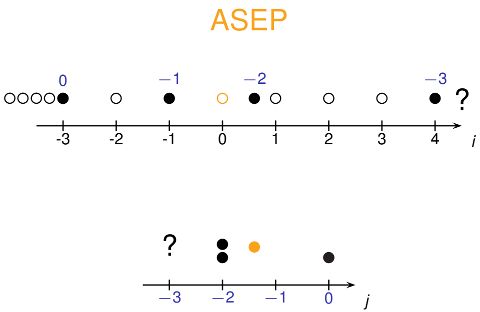

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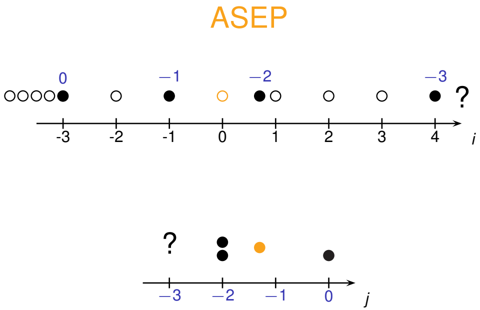

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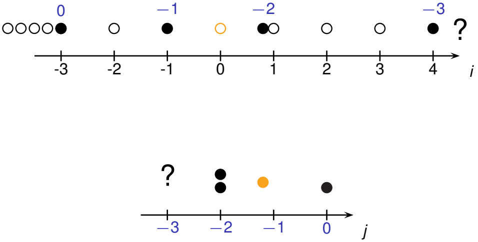

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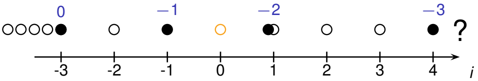

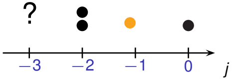

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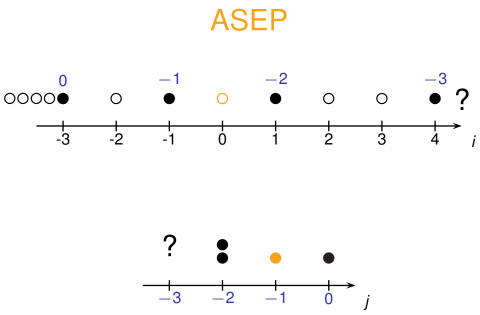

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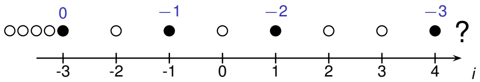
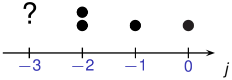


AZRP



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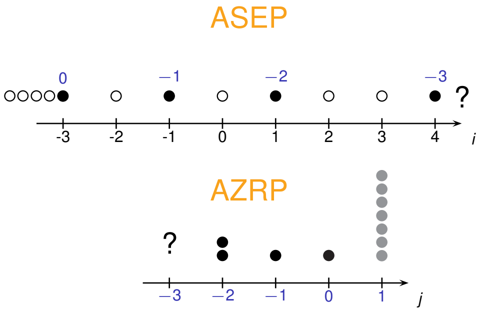

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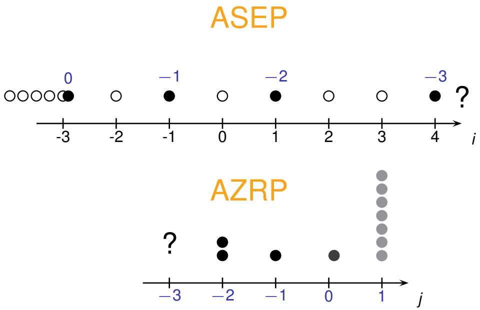

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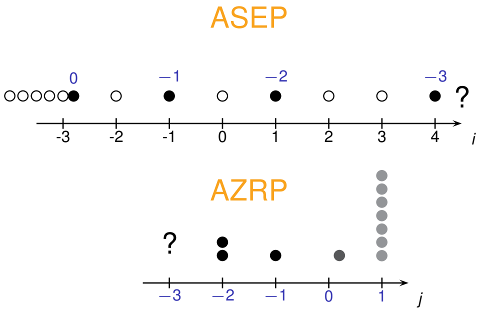

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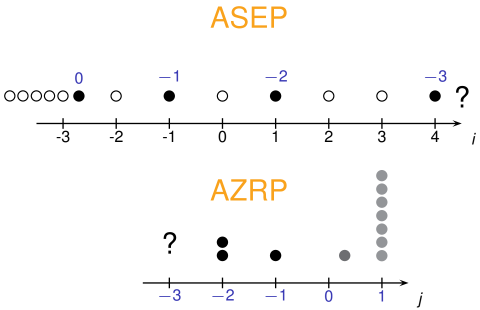

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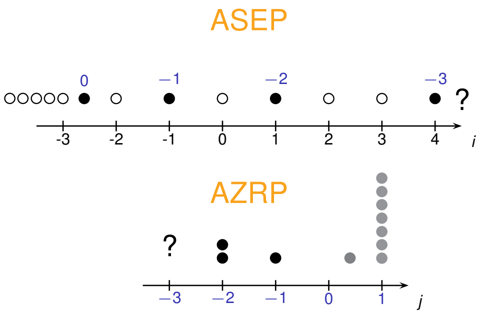

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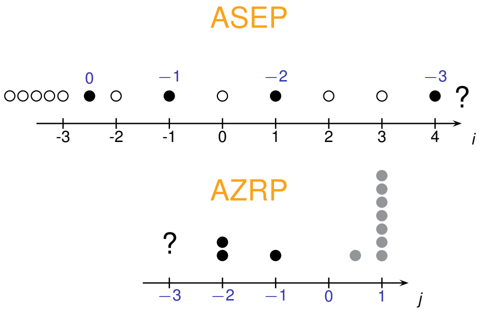

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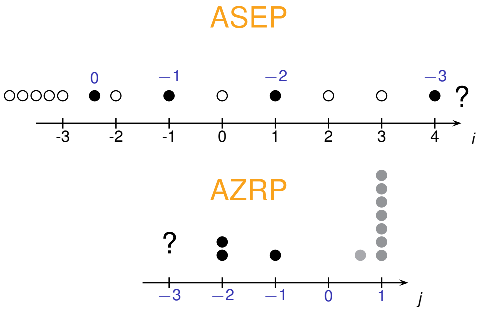

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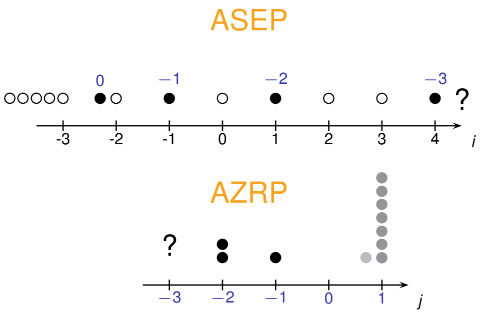

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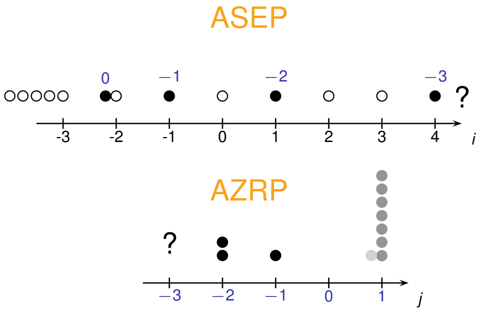

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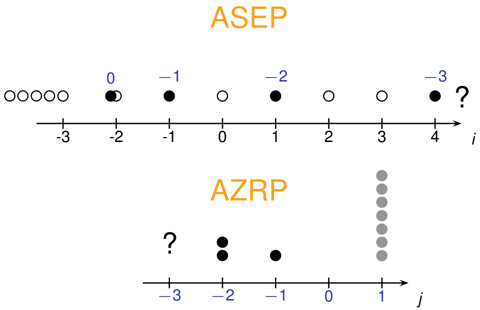

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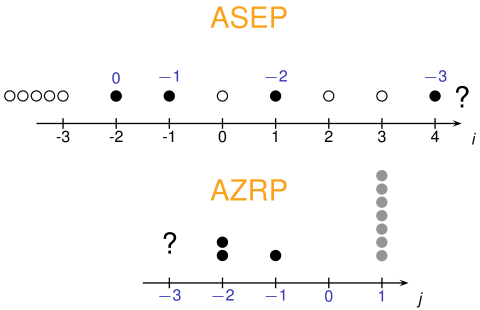


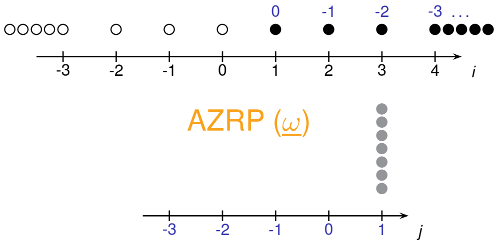
$$\text{ASEP} \stackrel{T^n}{=} \text{AZRP}$$

$$\underline{\nu}^n \stackrel{T^n}{=} \prod_{i\le 0} \text{Geometric}\left(1-\left(\frac{p}{q}\right)^{i-1}\right)$$

since stationary distributions of countable irreducible Markov chains are unique.

Jacobi triple product

ASEP ($\underline{\eta}$)



$$\eta_i = \mathbf{1}\{i \geq 1\}, \qquad N(\underline{\eta}) = 0, \qquad \omega_i \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod\limits_{i\leq 0}\left(\frac{p}{q}\right)^{(i-c)\cdot 0}}{\prod\limits_{i\leq 0}\left(1+\left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod\limits_{i>0}\left(\frac{q}{p}\right)^{(i-c)\cdot(1-1)}}{\prod\limits_{i>0}\left(\left(\frac{q}{p}\right)^{i-c}+1\right)} \cdot \frac{\sum\left(\frac{q}{p}\right)^{\frac{m^2+m}{2}-cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2}-c\cdot 0}}$$

$$\underline{\mu}(\underline{\omega}) = \prod_{i\leq 0}\left(1-\left(\frac{p}{q}\right)^{i-1}\right)$$

Jacobi triple product

$$\prod_{i\le 0}(1-(\frac{p}{q})^{i-1})\cdot\prod_{i\le 0}(1+(\frac{p}{q})^{i-c})\cdot\prod_{i>0}((\frac{q}{p})^{i-c}+1)=\sum_{m=-\infty}^{\infty}\left(\frac{q}{p}\right)^{\frac{m^2+m}{2}-cm}$$

LHS:

$$\prod_{i=1}^{\infty}(1-(\frac{q}{p})^{i})\cdot(1+(\frac{q}{p})^{i-1+c})\cdot((\frac{q}{p})^{i-c}+1)$$

$$=\prod_{i=1}^{\infty}(1-x^{2i})\left(1+\frac{x^{2i-1}}{y^2}\right)(1+x^{2i-1}y^2)$$

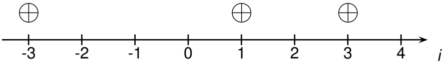
with $x=\left(\frac{q}{p}\right)^{\frac{1}{2}}$, $y=\left(\frac{q}{p}\right)^{\frac{1}{4}-\frac{c}{2}}$.

RHS:

$$\sum_{m=-\infty}^{\infty}\left(\frac{q}{p}\right)^{\frac{m^2}{2}}\left(\frac{q}{p}\right)^{m(\frac{1}{2}-c)}=\sum_{m=-\infty}^{\infty}x^{m^2}y^{2m}.$$

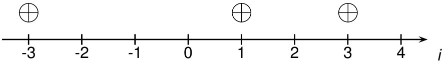
□

$\oplus$ $\ominus$ $\emptyset$ models



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



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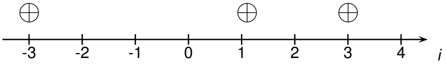

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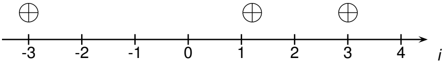
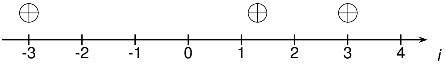


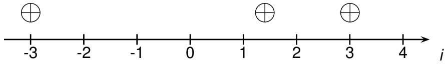
⊕ ⊖ ∅ models

⊕ to the right



$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



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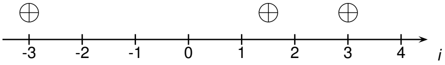
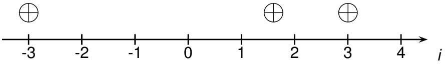


⊕ ⊖ ∅ models

⊕ to the right



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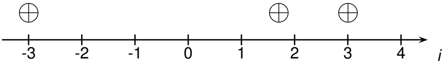
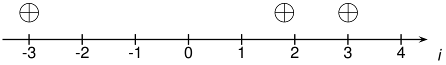


⊕ ⊖ ∅ models

⊕ to the right



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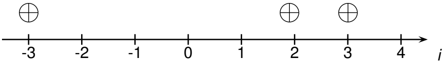
$\oplus$ to the right



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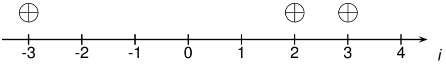
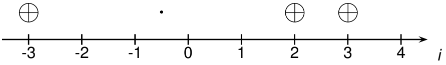


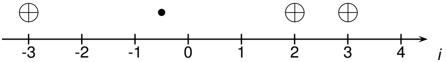
⊕ ⊖ ∅ models

pair creation from vacuum



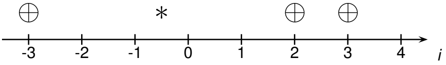
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



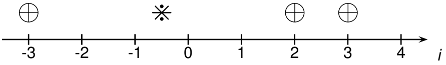
⊕ ⊖ ∅ models

pair creation from vacuum



$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



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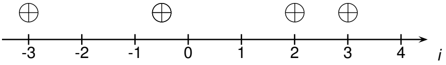
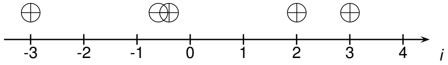
pair creation from vacuum



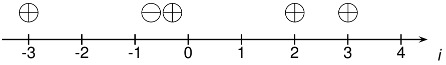
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



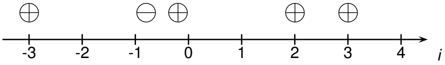
⊕ ⊖ ∅ models

pair creation from vacuum



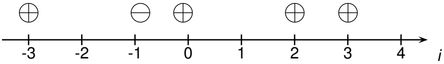
⊕ ⊖ ∅ models

pair creation from vacuum



$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



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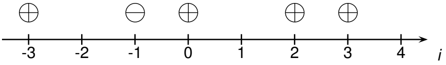
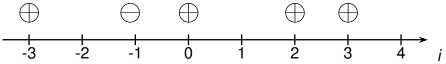
pair creation from vacuum



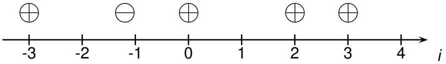
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



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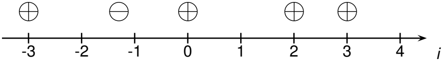

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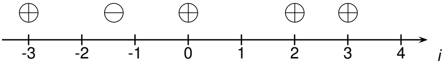

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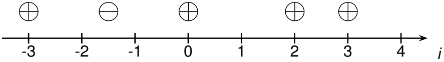

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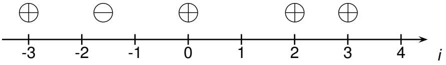

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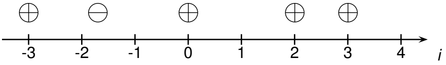
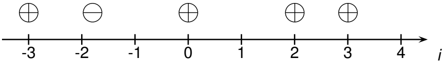


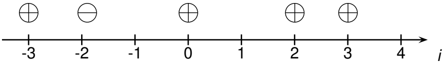
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



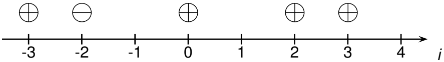
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



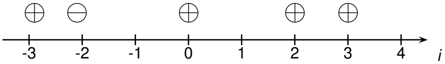
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



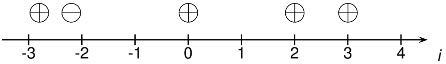
⊕ ⊖ ∅ models

annihilation



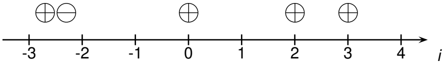
⊕ ⊖ ∅ models

annihilation



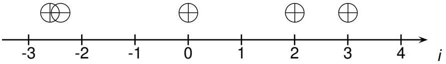
⊕ ⊖ ∅ models

annihilation



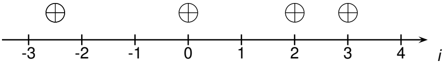
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



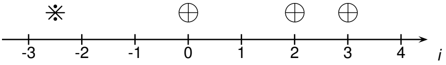
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



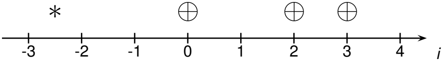
⊕ ⊖ ∅ models

annihilation



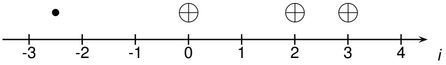
⊕ ⊖ ∅ models

annihilation



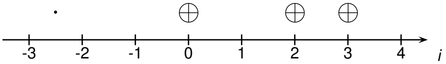
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



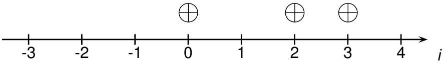
$\oplus$ $\ominus$ $\emptyset$ models

annihilation

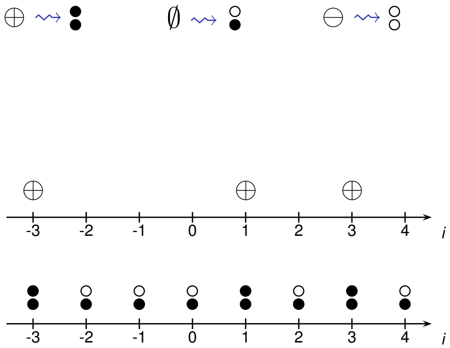


$\oplus$ $\ominus$ $\emptyset$ models

annihilation

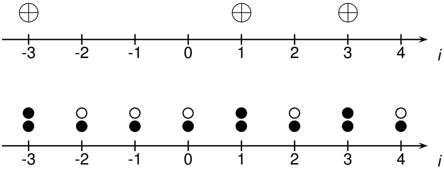


⊕ ⊖ ∅ models



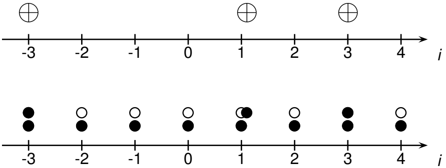
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



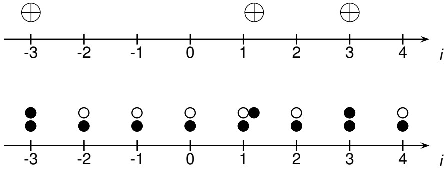
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



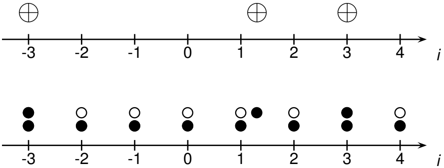
⊕ ⊖ ∅ models

⊕ to the right



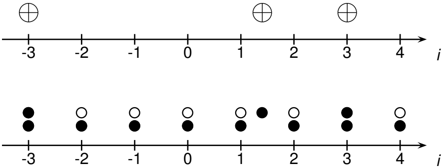
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



⊕ ⊖ ∅ models

⊕ to the right



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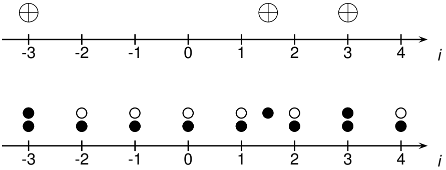
$\oplus$ to the right



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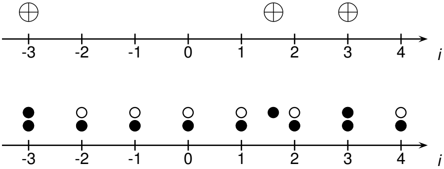
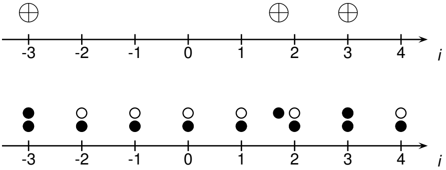
$\oplus$ to the right



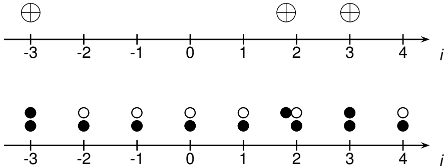
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



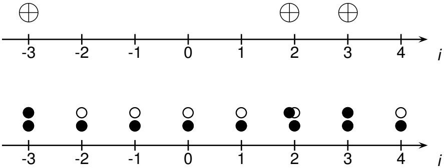
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



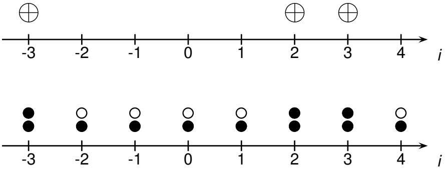
⊕ ⊖ ∅ models

⊕ to the right



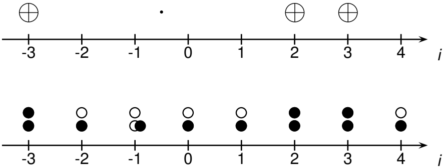
$\oplus$ $\ominus$ $\emptyset$ models

$\oplus$ to the right



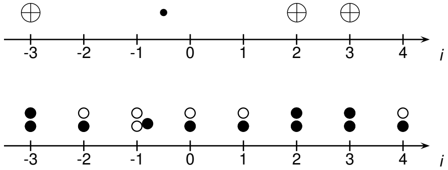
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



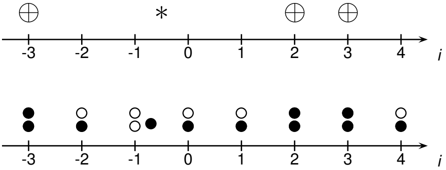
⊕ ⊖ ∅ models

pair creation from vacuum



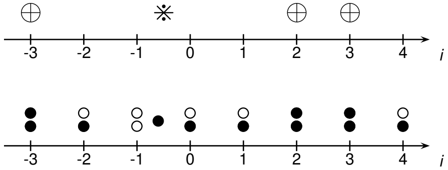
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



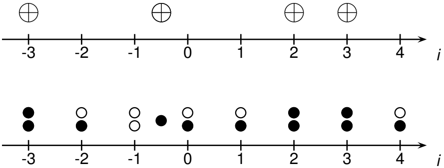
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



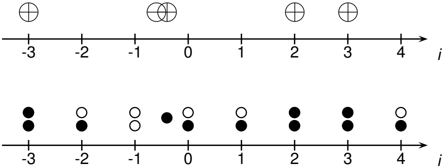
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



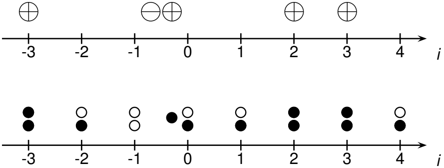
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



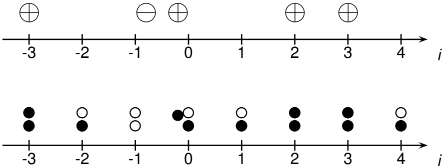
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



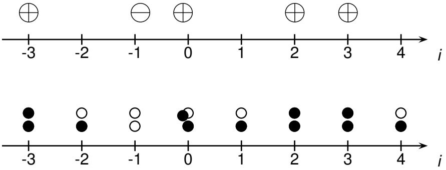
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



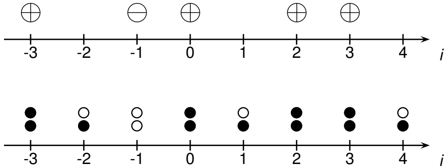
$\oplus$ $\ominus$ $\emptyset$ models

pair creation from vacuum



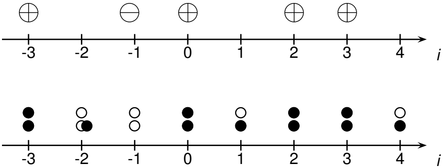
⊕ ⊖ ∅ models

pair creation from vacuum



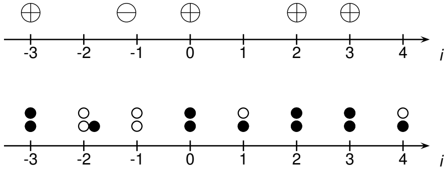
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



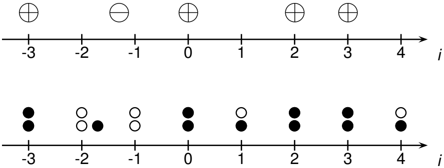
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



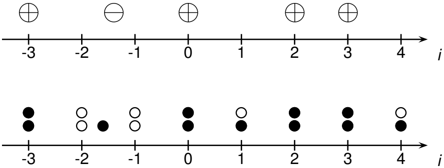
⊕ ⊖ ∅ models

⊖ to the left



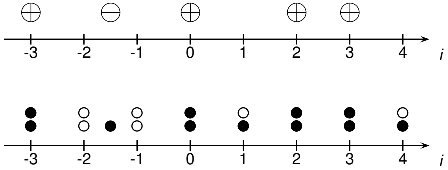
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



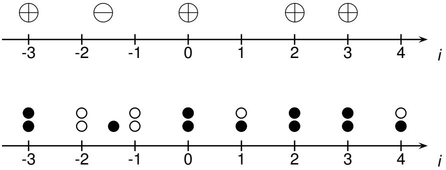
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



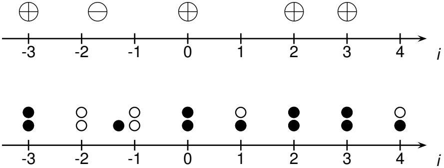
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



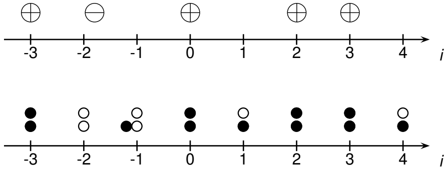
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



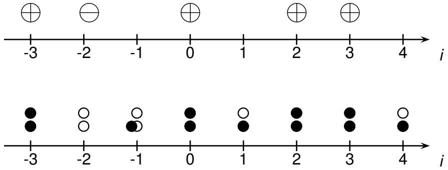
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



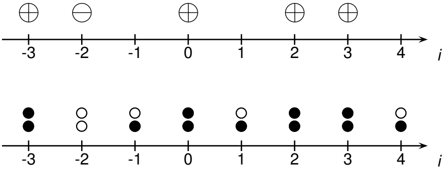
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



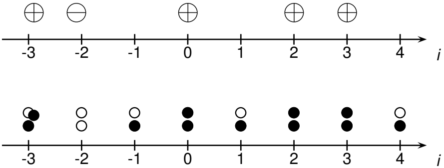
$\oplus$ $\ominus$ $\emptyset$ models

$\ominus$ to the left



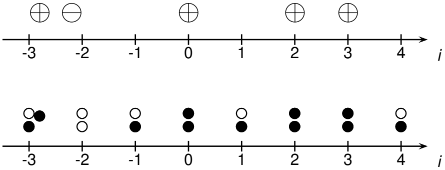
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



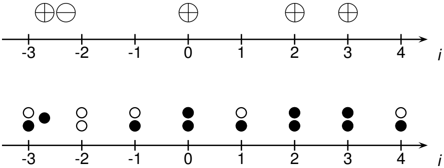
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



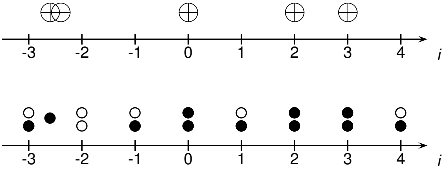
⊕ ⊖ ∅ models

annihilation



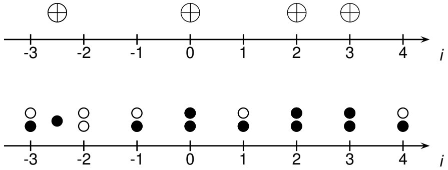
$\oplus$ $\ominus$ $\emptyset$ models

annihilation



$\oplus$ $\ominus$ $\emptyset$ models

annihilation



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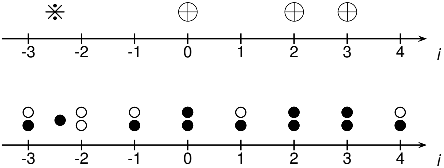
annihilation



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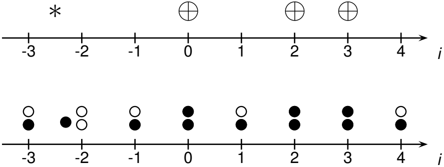
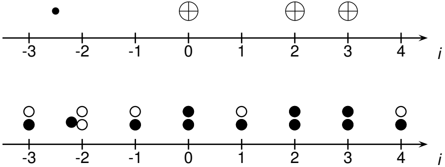
annihilation



$\oplus$ $\ominus$ $\emptyset$ models

annihilation



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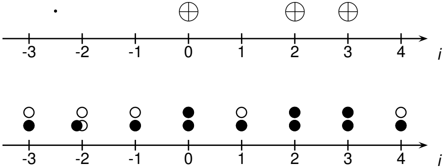
annihilation



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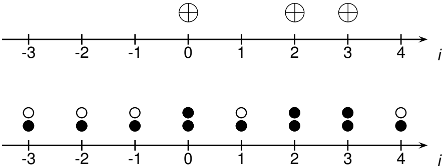

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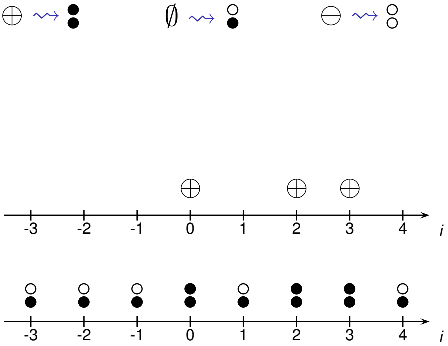
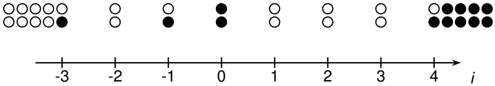


…and the same steps to the left.

Lay down / stand up a bit differently



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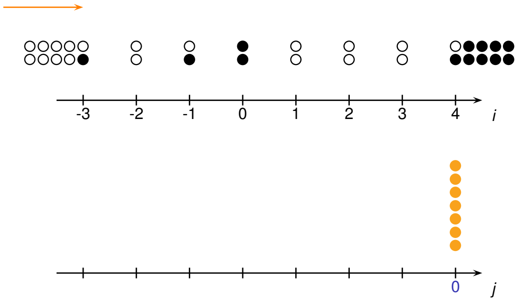

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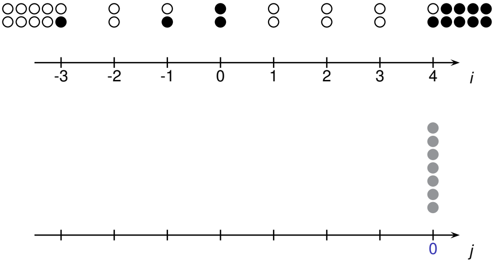

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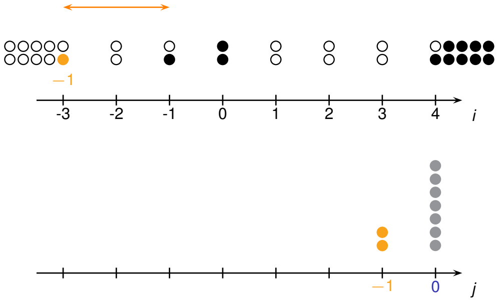

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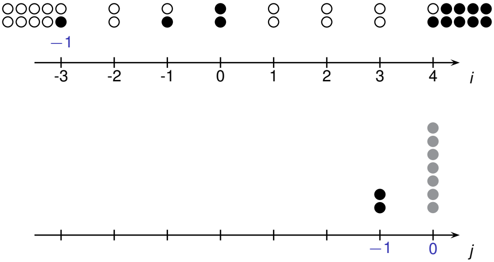

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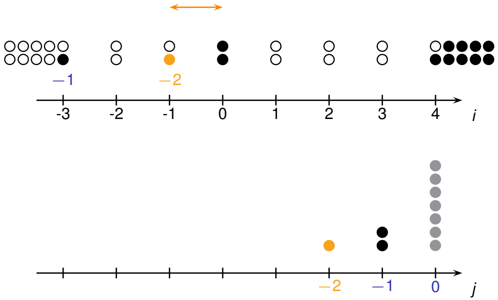

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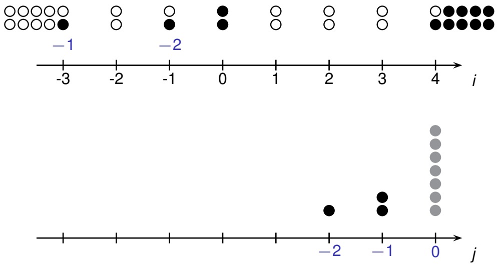

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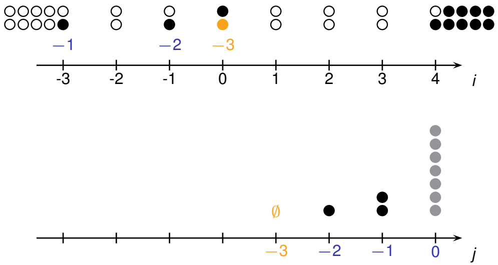

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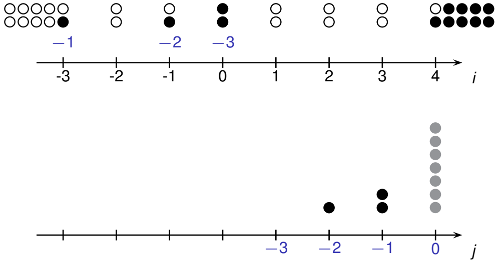

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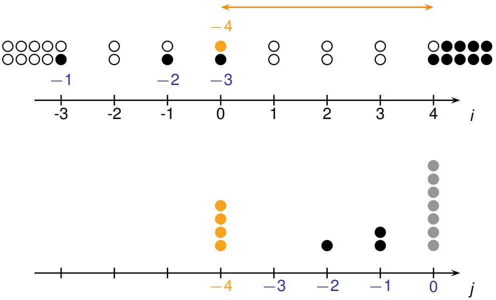

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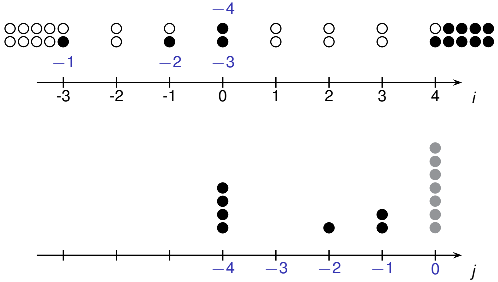

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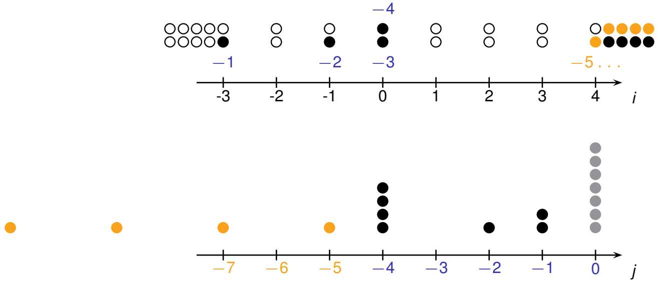

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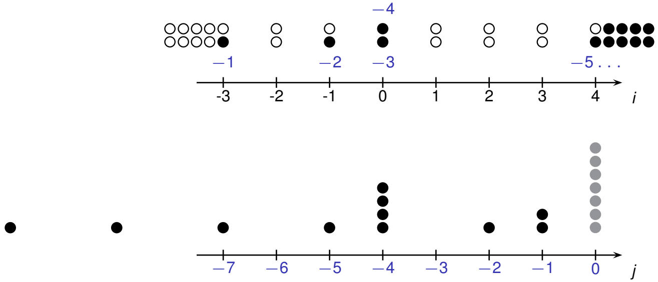

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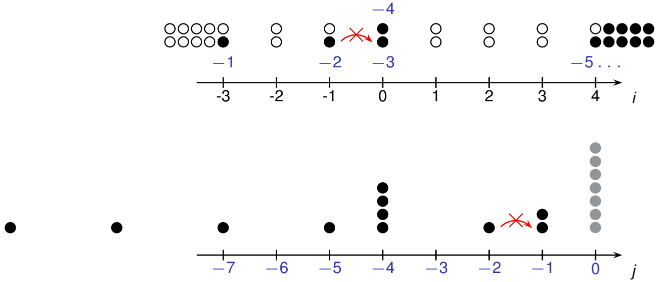
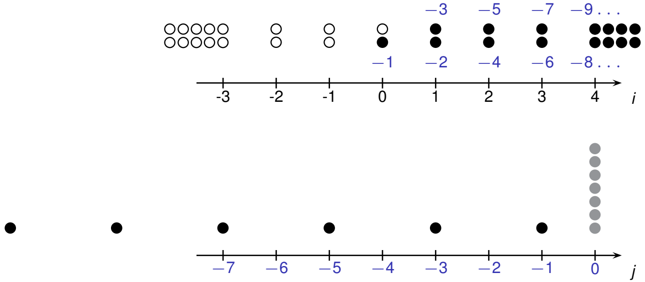


No two consecutive 0's!

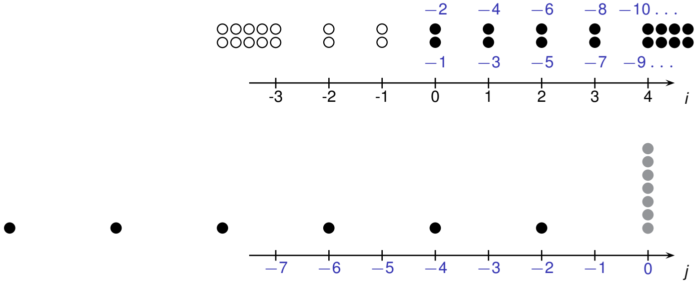
Lay down / stand up a bit differently

Odd ground state:



Lay down / stand up a bit differently

Even ground state:



Lay down / stand up a bit differently

- *No two consecutive 0's!* is a nonlocal constraint.

Lay down / stand up a bit differently

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Lay down / stand up a bit differently

- *No two consecutive 0's!* is a nonlocal constraint.
- The stood up model is nice otherwise. It has reversible product blocking measures.
- *Reversible* measures survive forbidden jumps. Yay!

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

ASEP(q, 1) Carinci, Giardiná, Redig, Sasamoto:

Jump	Rate	Jump	Rate
$\oplus \rightarrow$	$q^{-1} + q^{-3}$	$\leftarrow \oplus$	$q + q^3$
$\leftarrow \ominus$	$q^{-1} + q^{-3}$	$\ominus \rightarrow$	$q + q^3$
$\emptyset \rightsquigarrow \ominus\oplus$	q^{-3}	$\emptyset \rightsquigarrow \oplus\ominus$	q^3
$\oplus\ominus \rightsquigarrow \emptyset$	$(1 + q^2)(q^{-1} + q^{-3})$	$\ominus\oplus \rightsquigarrow \emptyset$	$(1 + q^{-2})(q + q^3)$

Identity: *Has to do with the odd and even terms of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

A nice three-state model:

Jump	**Rate**	**Jump**	**Rate**
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus\oplus$	c	$\emptyset \rightsquigarrow \oplus\ominus$	qc
$\oplus\ominus \rightsquigarrow \emptyset$	2	$\ominus\oplus \rightsquigarrow \emptyset$	$2q$

Identity: *Has to do with the square of Jacobi's triple product.*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

2-exclusion:

Jump	**Rate**	**Jump**	**Rate**
$\oplus \rightarrow$	1	$\leftarrow \oplus$	q
$\leftarrow \ominus$	1	$\ominus \rightarrow$	q
$\emptyset \rightsquigarrow \ominus\oplus$	1	$\emptyset \rightsquigarrow \oplus\ominus$	q
$\oplus\ominus \rightsquigarrow \emptyset$	1	$\ominus\oplus \rightsquigarrow \emptyset$	q

Identity: *Looks new and interesting. . .*

New identities

The identities we get have to do with *generalised Frobenius partitions* and *generalised Young diagrams*.

<u>K-exclusion:</u>

Identity: *Rather nice generalisation using the K^{th} roots of unity.*

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Thank you.