Nonexistence of bi-infinite geodesics in exponential last passage percolation - a probabilistic way

Joint with
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Statistics Seminars
Durham University
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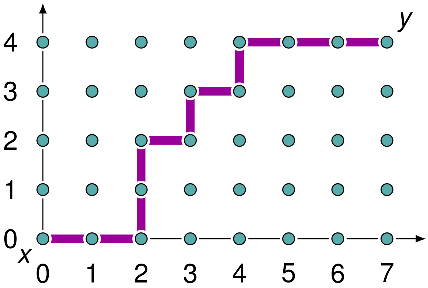
Last passage percolation

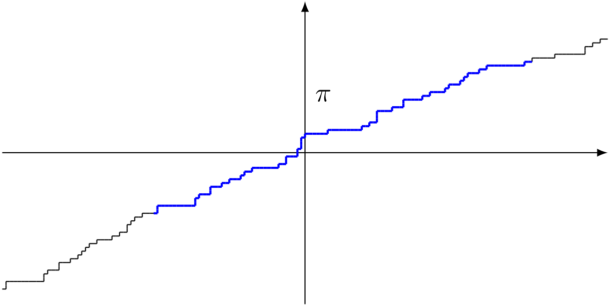
- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{x,y}$ from x to y is the a.s. unique heaviest up-right from x to y.
- $G_{x,y} = \sum_{z \in \pi_{x,y}} \omega_z$ is its weight.



Surface growth, TASEP, queuing…

Bi-infinite geodesics

A bi-infinite up-right path is a *bi-infinite geodesic*, if any of its segments is itself a geodesic between the two endpoints.



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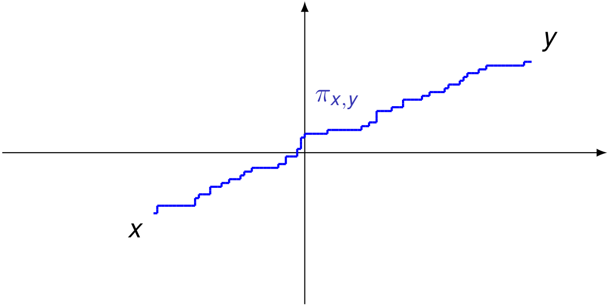

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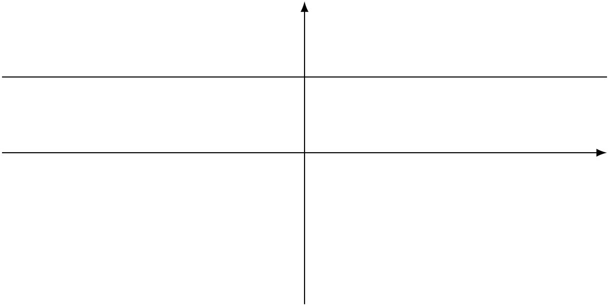

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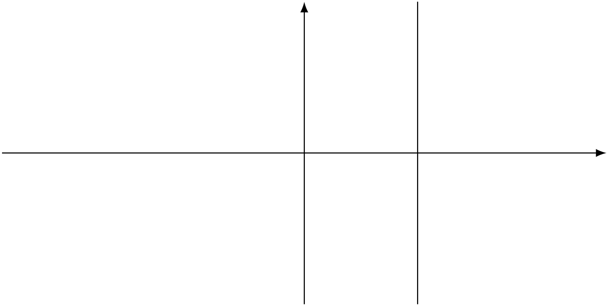

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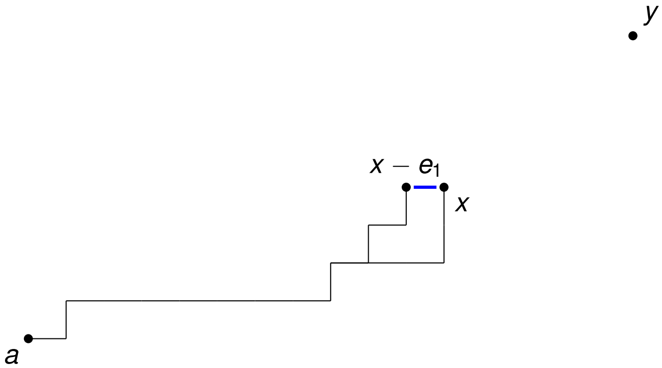
- We only need a bit of random walks, queuing, couplings.

1. Increments as new boundary



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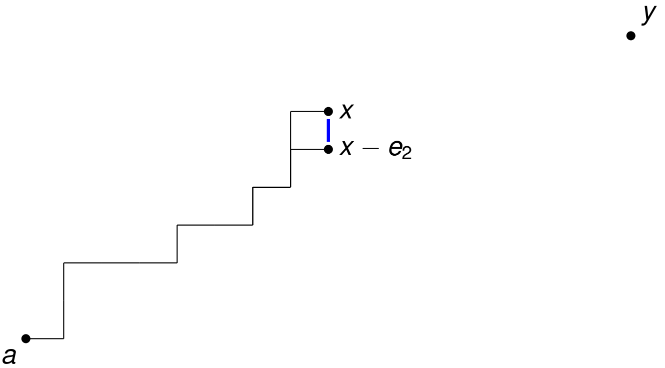

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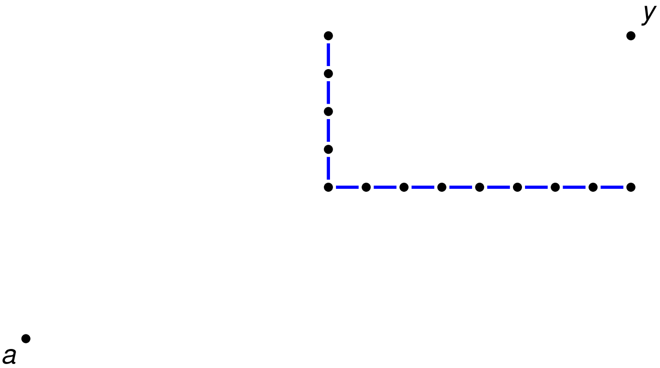

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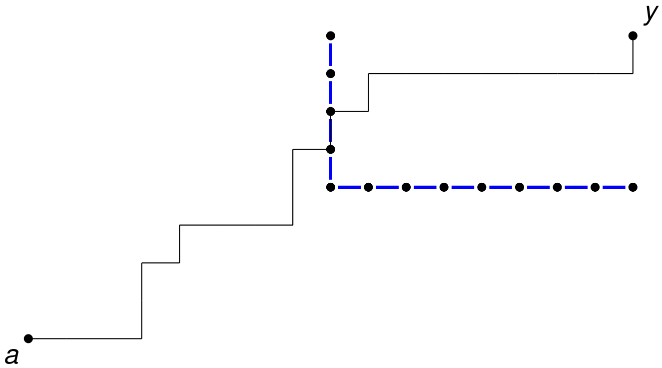

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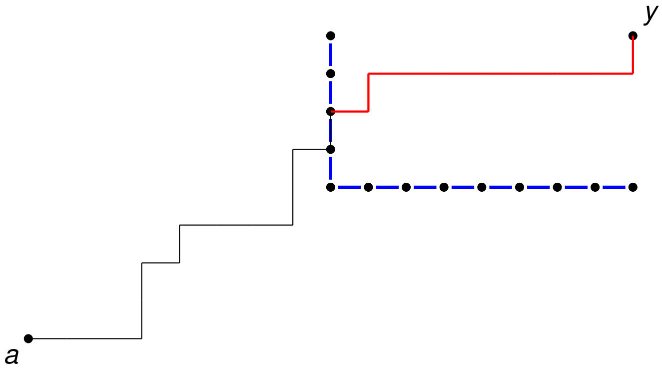

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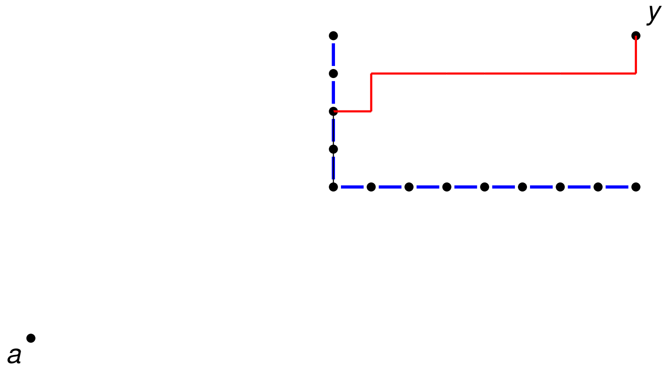

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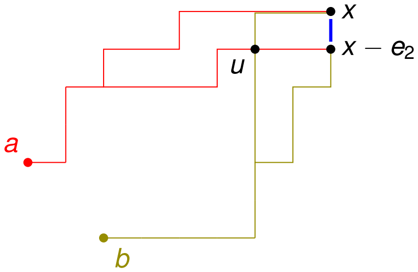
$\rightsquigarrow$ Act as boundary weights for a smaller, embedded model.

2. Crossing lemma

Let a be North-West of b.



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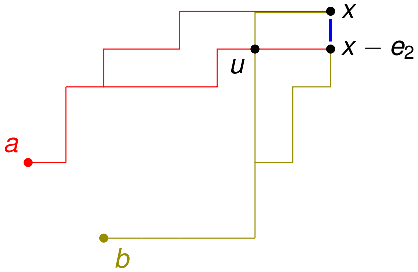

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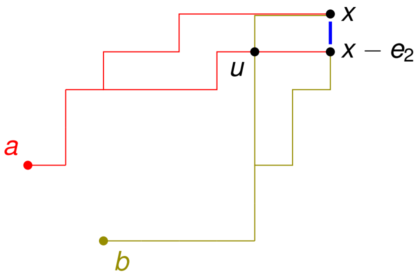

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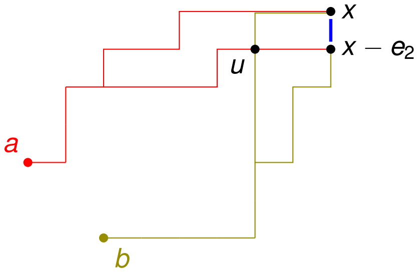

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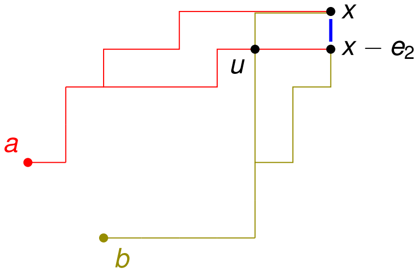

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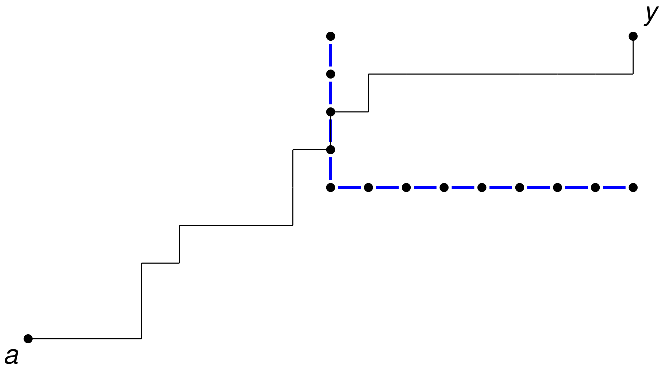
Similarly, $I_x^{(a)} \leq I_x^{(b)}$.

3. Stationary LPP



$$I_x = G_{a,x} - G_{a,x-e_1} \qquad J_x = G_{a,x} - G_{a,x-e_2}$$

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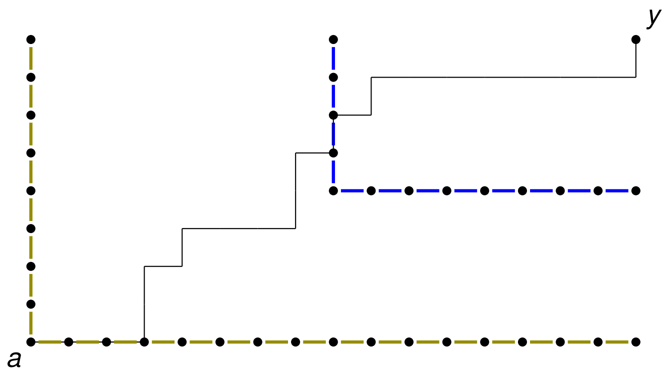
Replace the boundary to | $\sim \mathrm{Exp}(\varrho)$, — $\sim \mathrm{Exp}(1-\varrho)$ independent.



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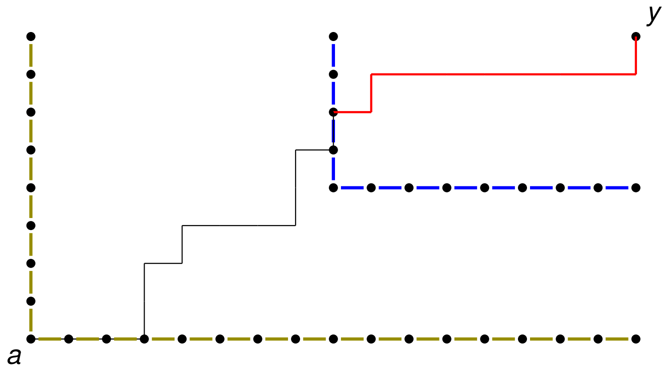

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The embedded model has the same structure.

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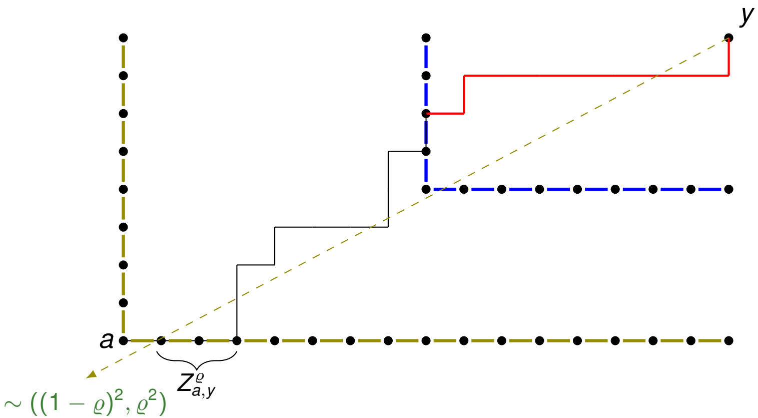
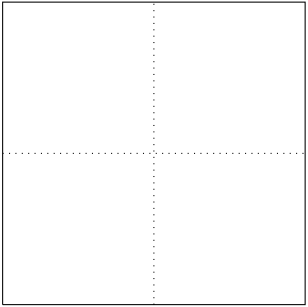


B., Cator, Seppäläinen '06: $\mathbb{P}\{|Z^{\varrho}_{a,y}| \geq \ell\} \leq \mathrm{box}^2/\ell^3$, *good directional control*.

Proof

Take larger and larger boxes and show that geodesics avoid more and more the origin when crossing from one side to the other (Newman '95).



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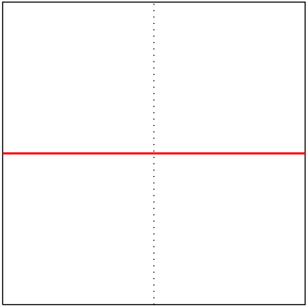

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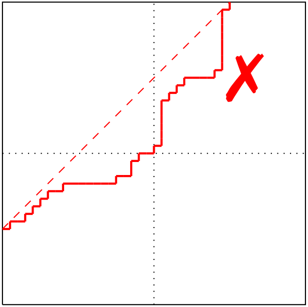
2. Otherwise, geodesics don't like to turn too much.



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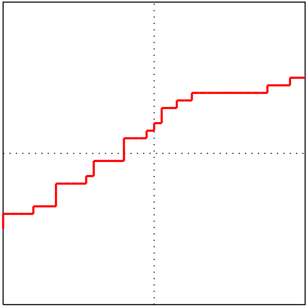
3. We are left with roughly diagonal ones, show that they fluctuate too much.



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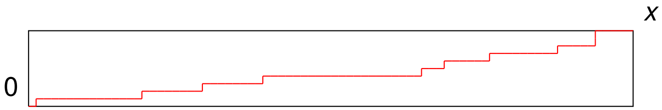

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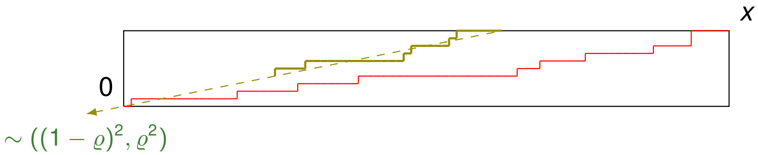

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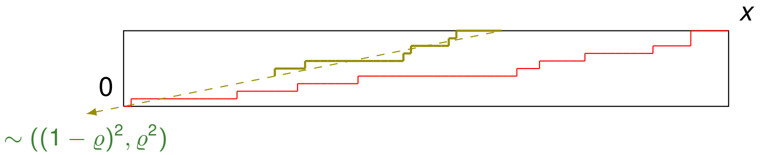

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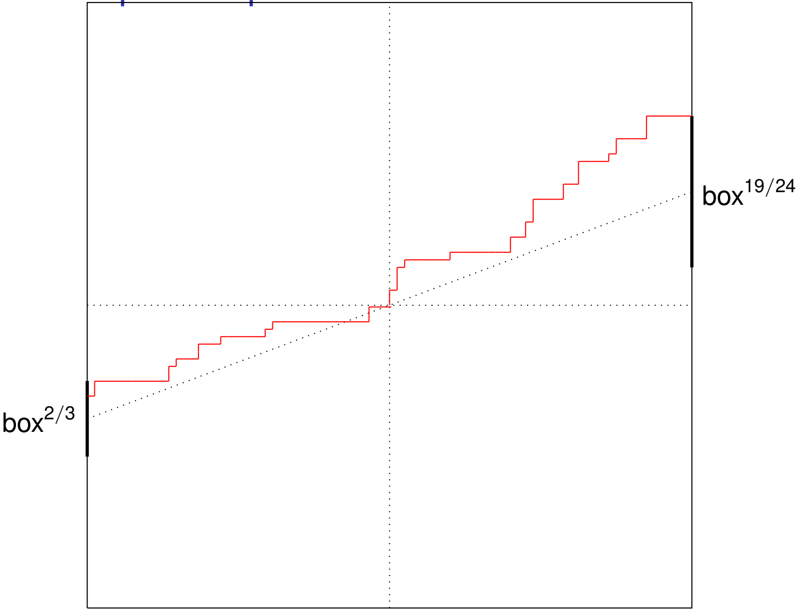
$$G_{0,x} - G_{e_2,x} = \hat{J}_{e_2} \geq \hat{J}^{\varrho}_{e_1} \sim \mathrm{Exp}(\varrho),$$

and can take $\varrho \to 0$ as the box flattens with $x \to \infty$. So, it's never worth leaving from e_2 compared from 0.

2. No sharp turns please



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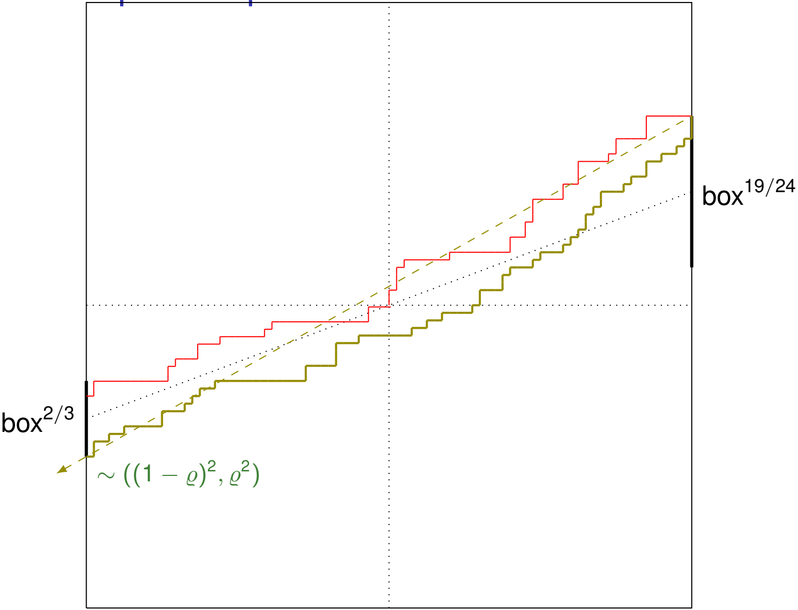

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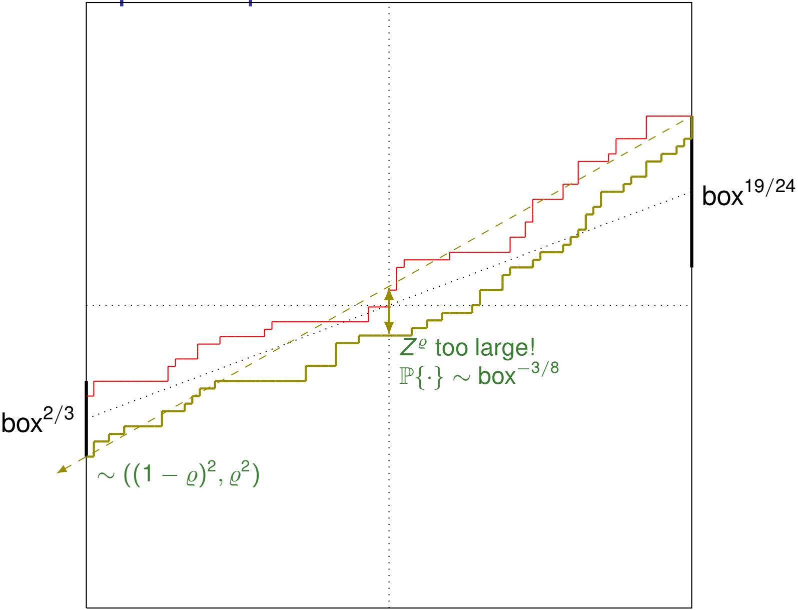
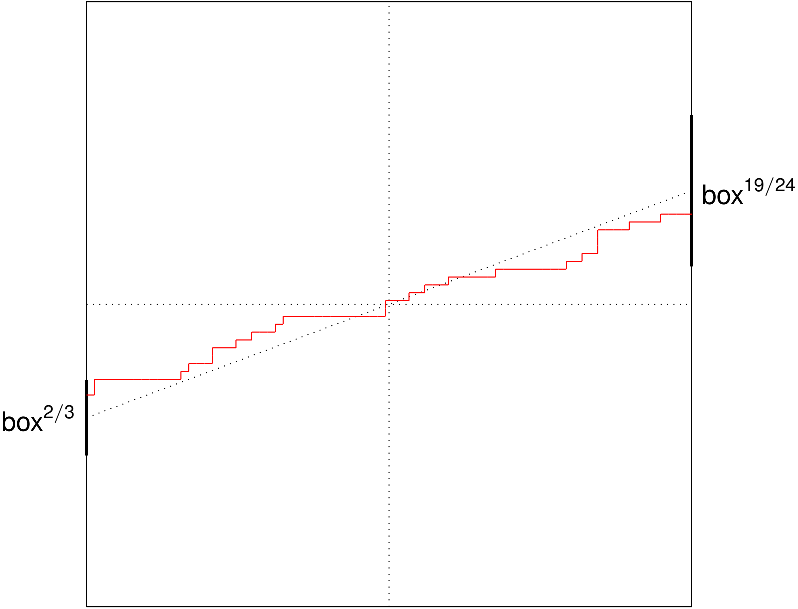


3. The diagonal case: the attack of the geodesics



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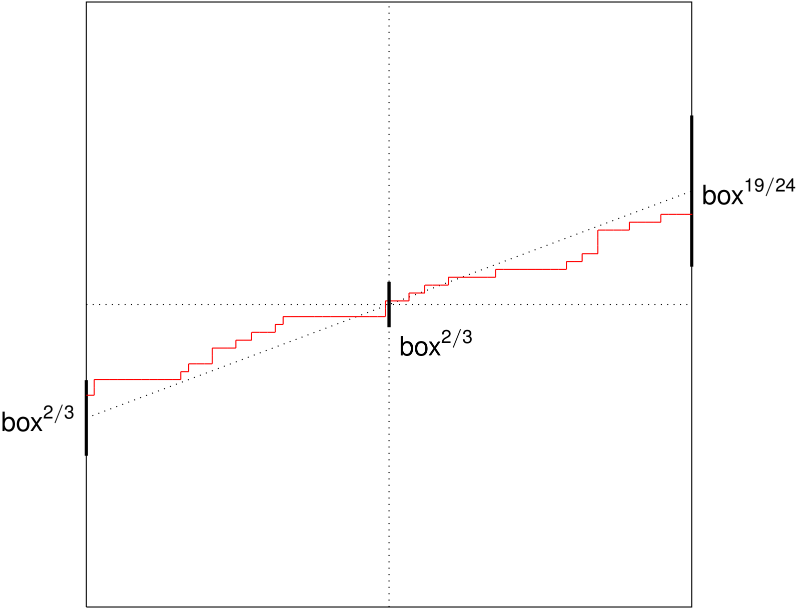

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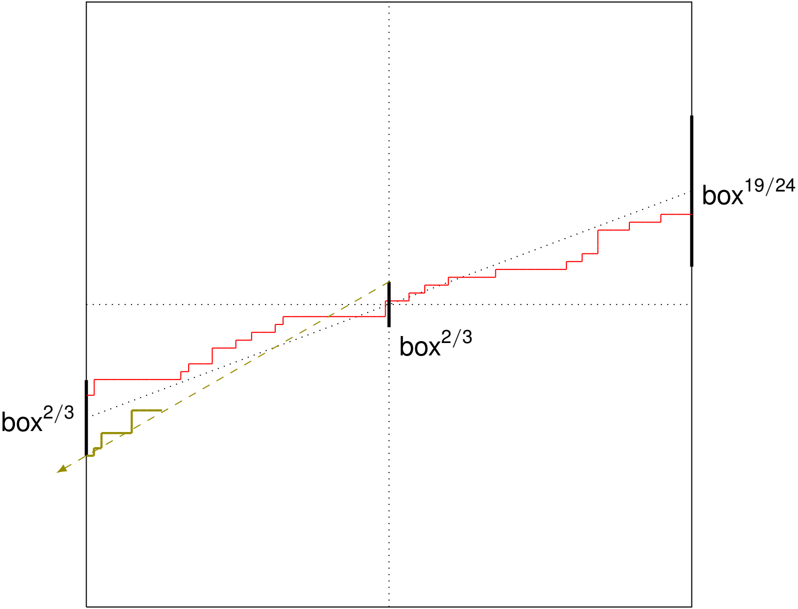

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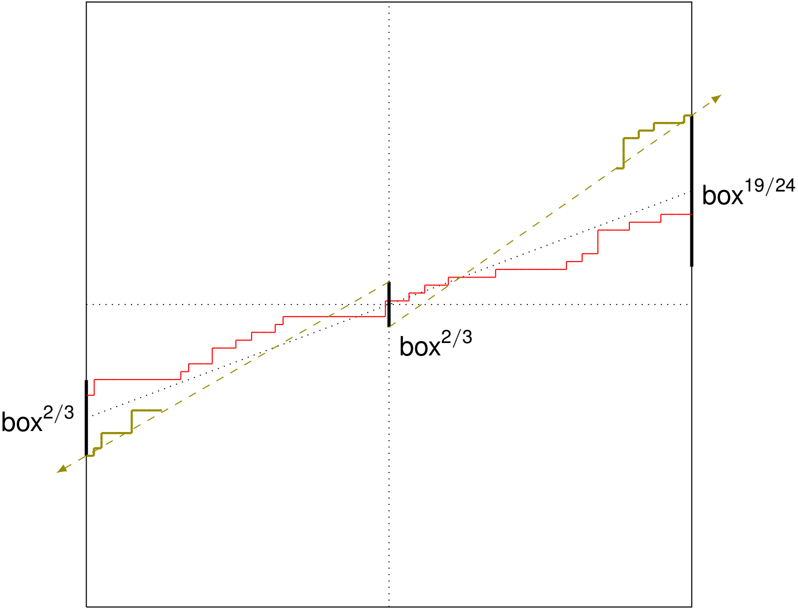

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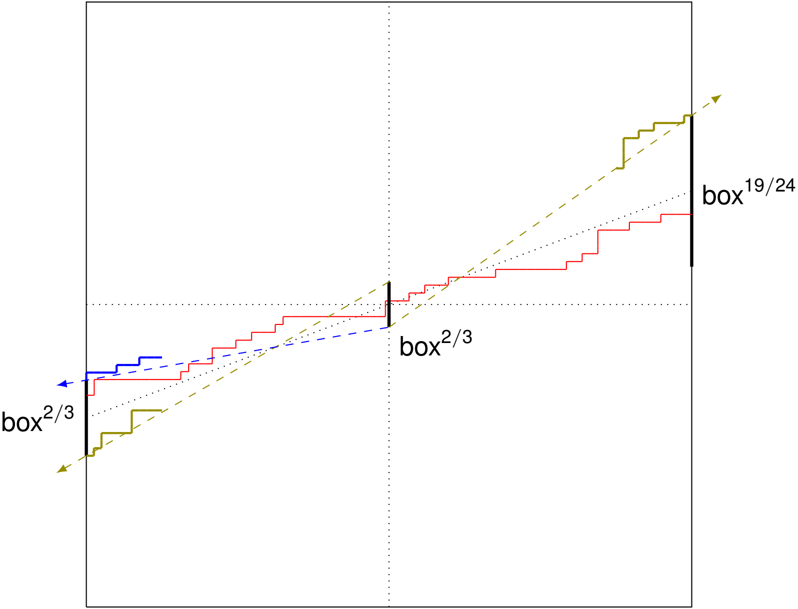

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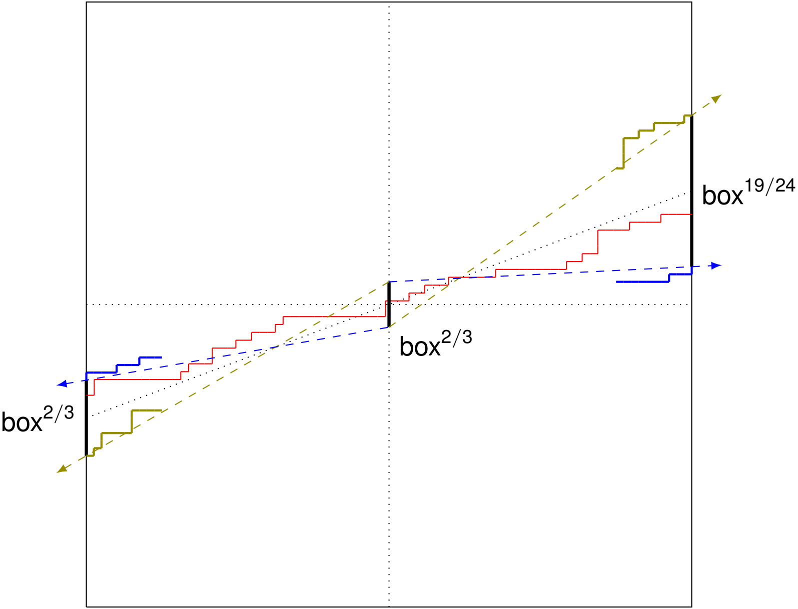

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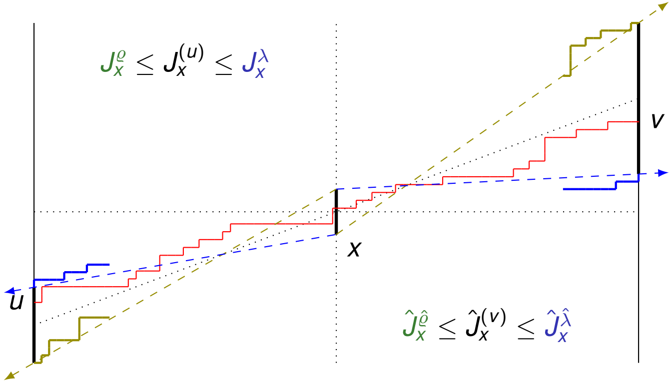

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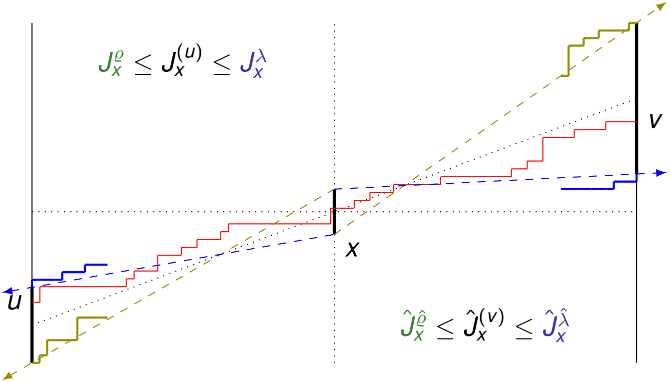

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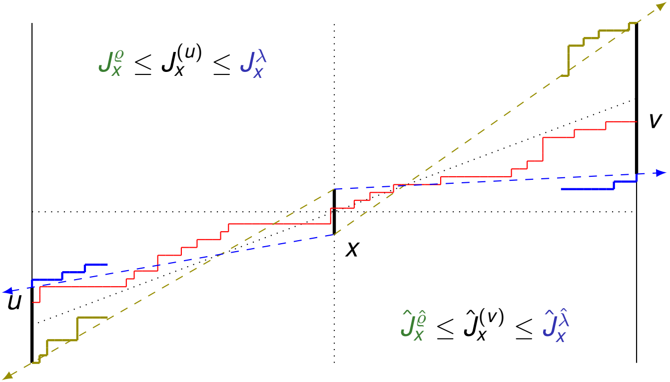

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- The bounds $J_j^{\varrho} - \hat{J}_j^{\hat{\lambda}} \leq J_j^{(u)} - \hat{J}_j^{(v)} \leq J_j^{\lambda} - \hat{J}_j^{\hat{\varrho}}$ are independent and nicely distributed.

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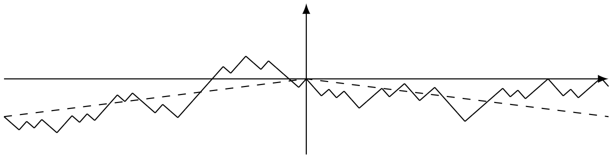
The problem boils down to whether a simple random walk minus drift reaches its maximum at 0. The answer is an asymptotic *no*, the drift is beaten by the fluctuations.



$\mathbb{P}\{\cdot\} \sim \text{box}^{-2/5}$.

So, the counting

- Intervals on the left are of size $\sim \text{box}^{2/3}$.
- Have $\text{box}/\text{box}^{2/3} \sim \text{box}^{1/3}$ many of these.

$\rightsquigarrow$ Union bound:

$$\begin{aligned}\mathbb{P}\{\text{any geodesic crosses } 0\} &\sim \text{box}^{1/3} \cdot \left(\text{box}^{-3/8} + \text{box}^{-2/5}\right) \\ &= \text{box}^{-1/24} \to 0.\end{aligned}$$

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These sharper, probabilistic estimates open up the way to further understanding of geodesics, with rather intuitive arguments.

Thank you.