

Jacobi triple product via the exclusion process

Joint with
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Márton Balázs

University of Bristol

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Jacobi triple product

Theorem

Let $|x| < 1$ and $y \neq 0$ be complex numbers. Then

$$\prod_{i=1}^{\infty} (1 - x^{2i}) \left(1 + \frac{x^{2i-1}}{y^2}\right) (1 + x^{2i-1} y^2) = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$

Mostly appears in **number theory** and **combinatorics of partitions**.

We'll prove it using interacting particles (for real x, y only).

Models

- Asymmetric simple exclusion
- Zero range

Blocking measures

State space

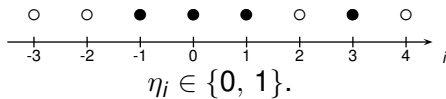
- No boundaries
- Boundaries

Lay down - stand up

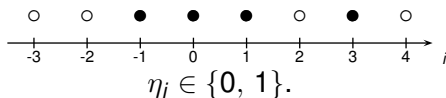
Jacobi triple product

More models

Asymmetric simple exclusion



Asymmetric simple exclusion



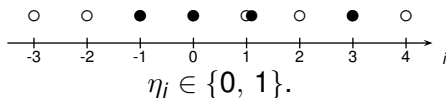
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Asymmetric simple exclusion



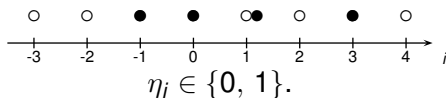
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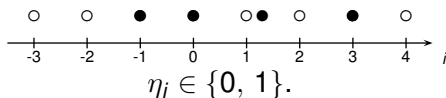
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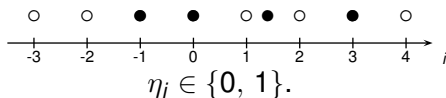
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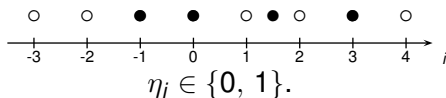
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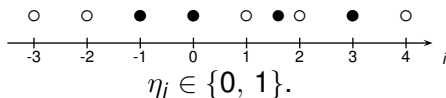
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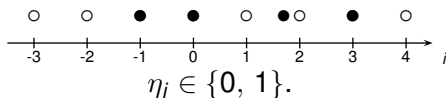
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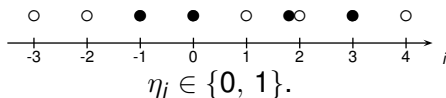
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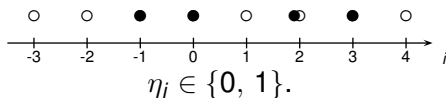
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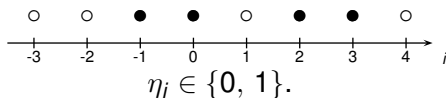
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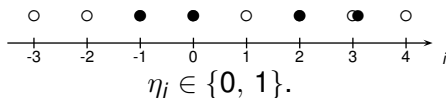
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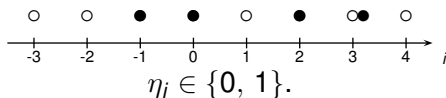
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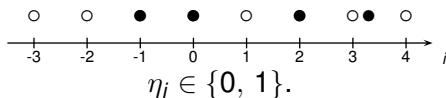
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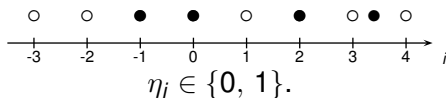
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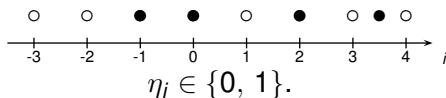
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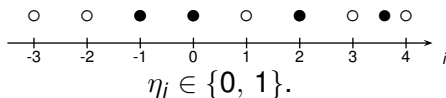
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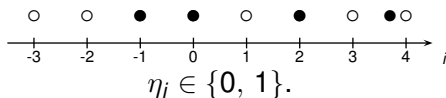
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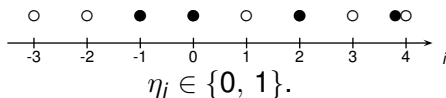
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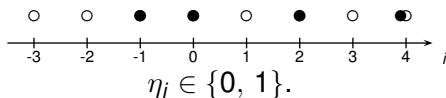
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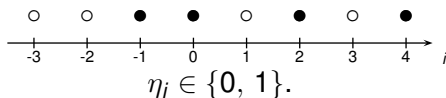
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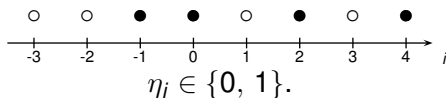
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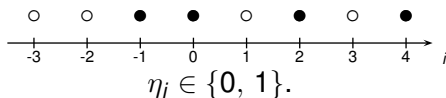
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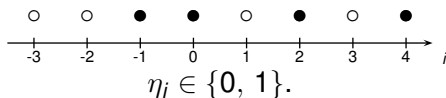
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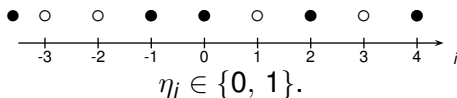
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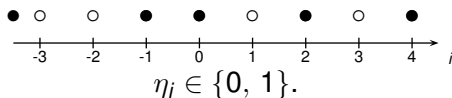
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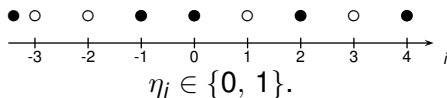
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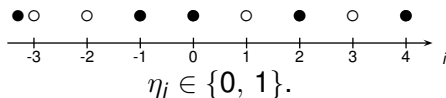
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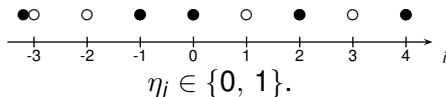
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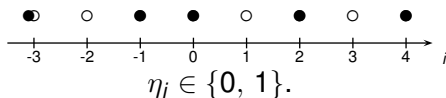
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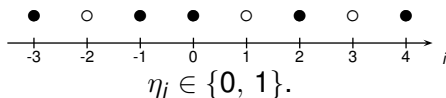
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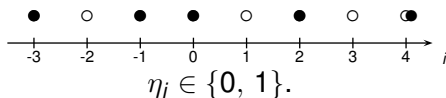
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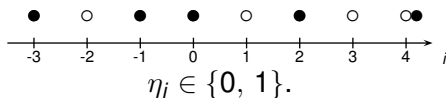
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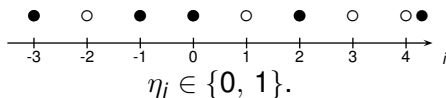
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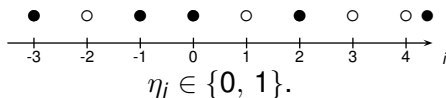
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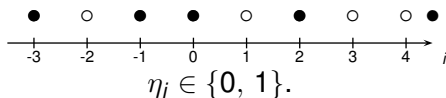
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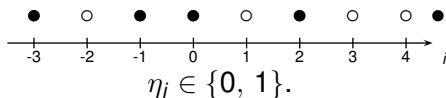
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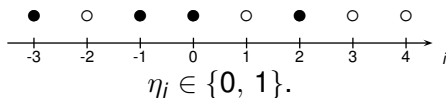
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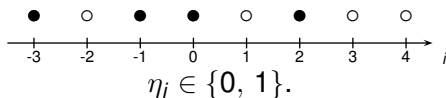
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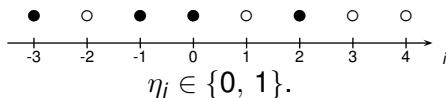
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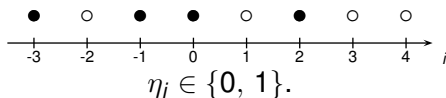
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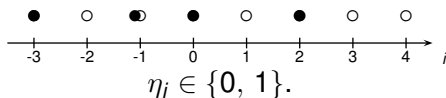
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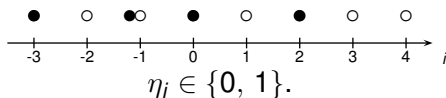
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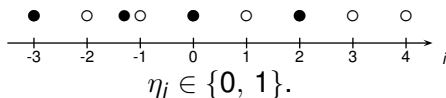
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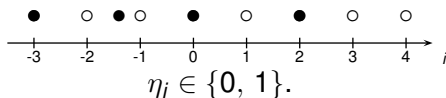
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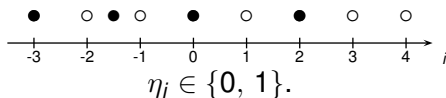
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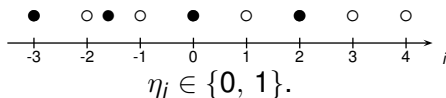
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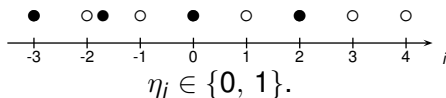
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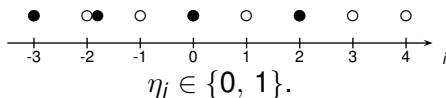
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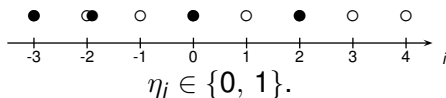
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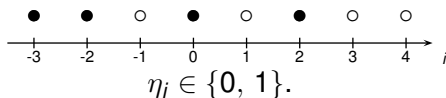
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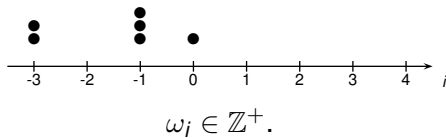
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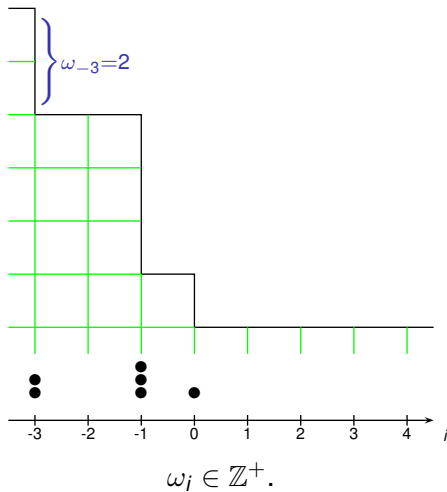
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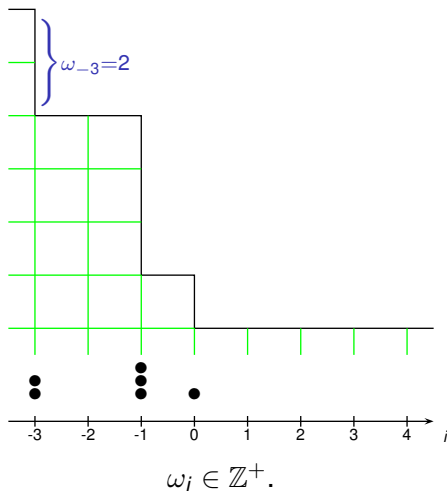
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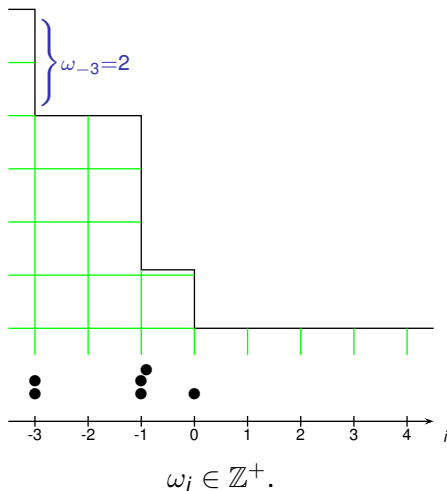


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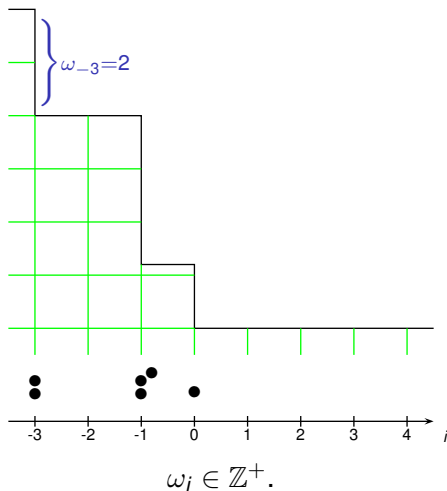
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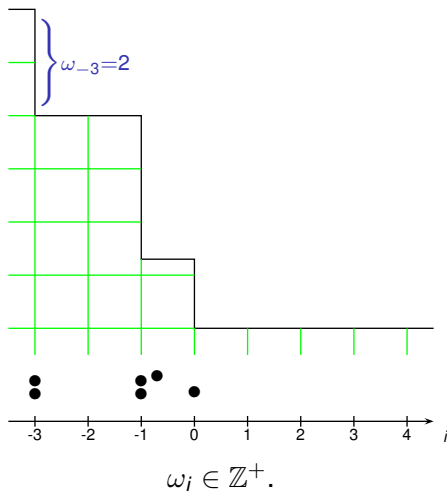
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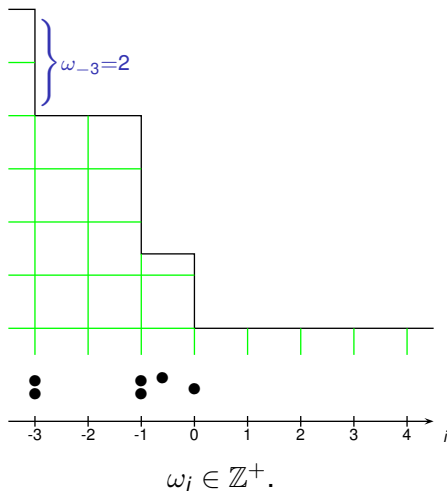
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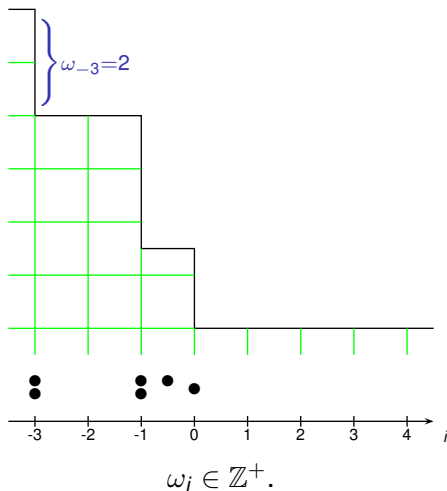
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The asymmetric zero range process



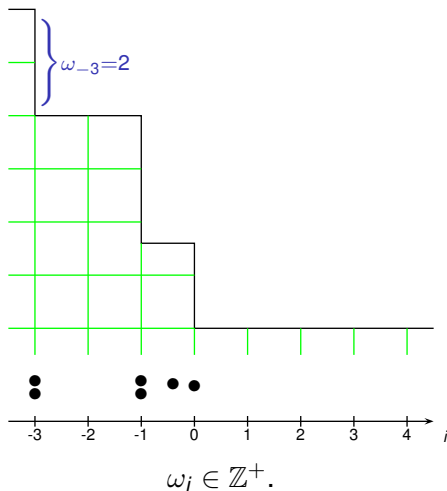
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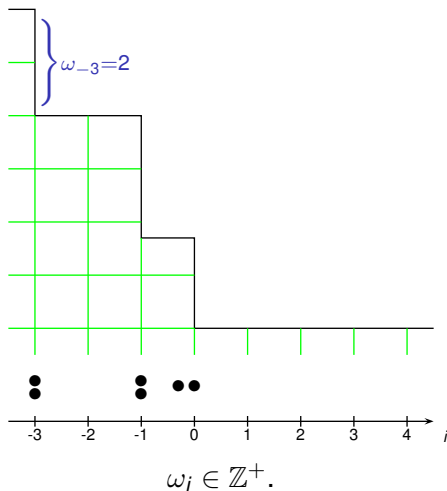
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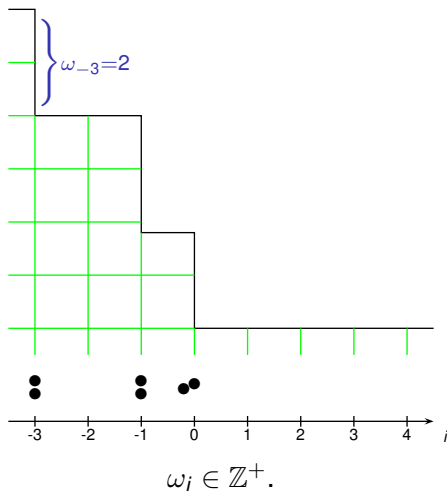
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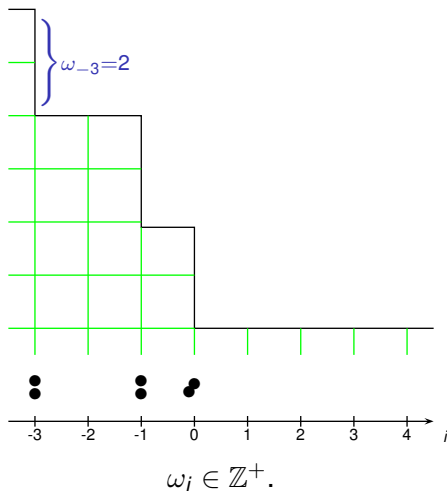
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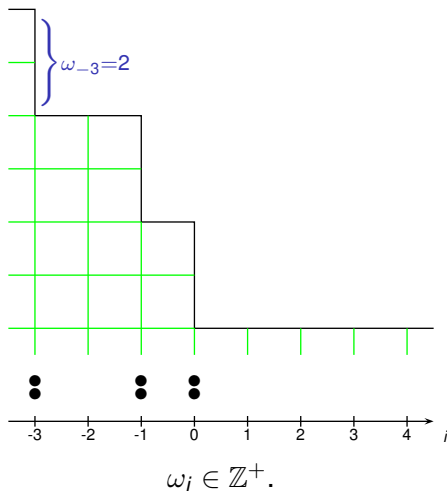
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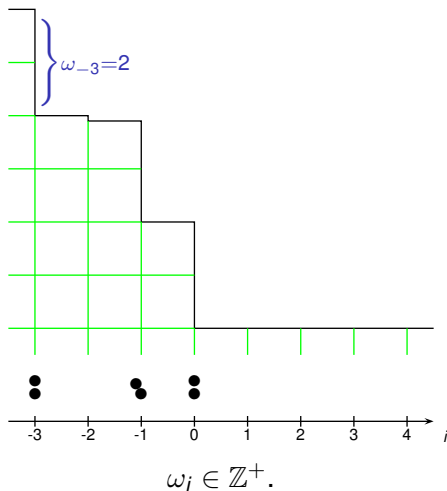
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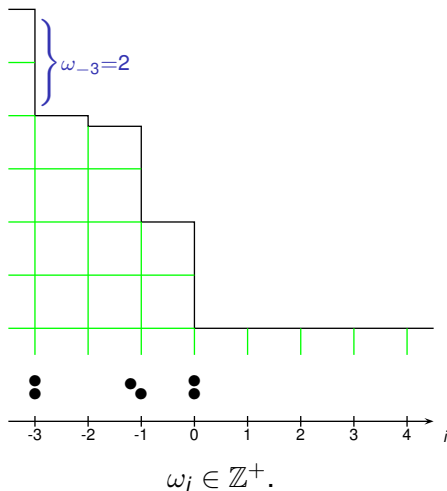
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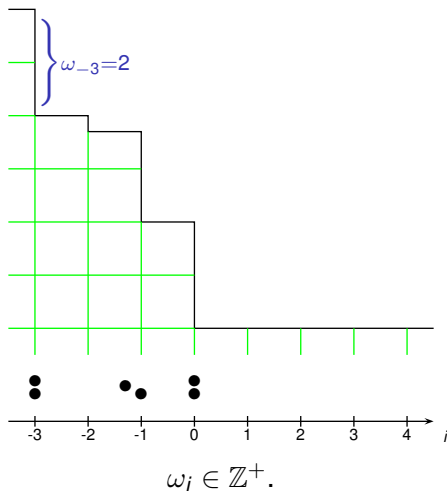
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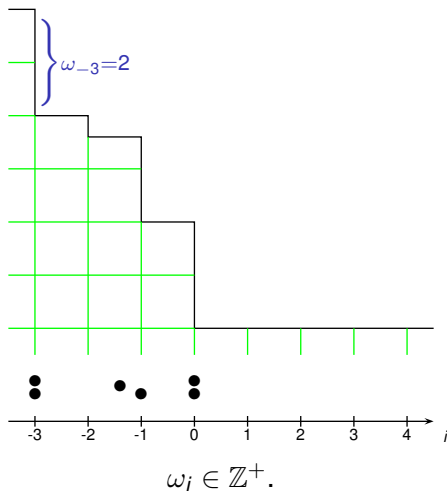
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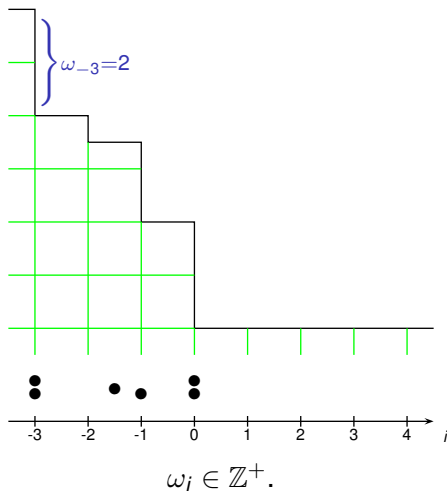
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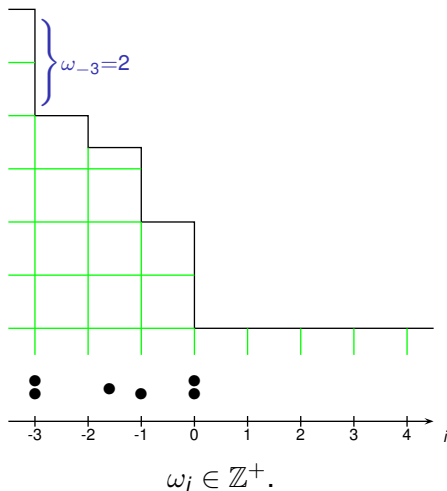
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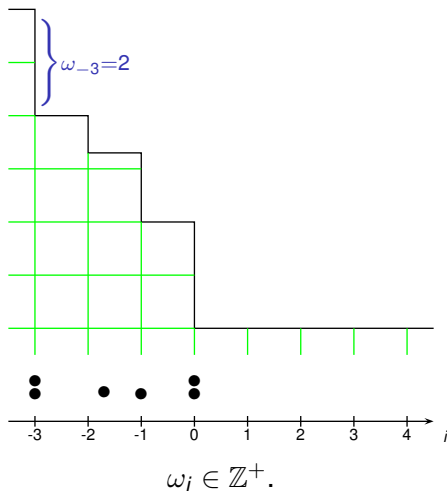
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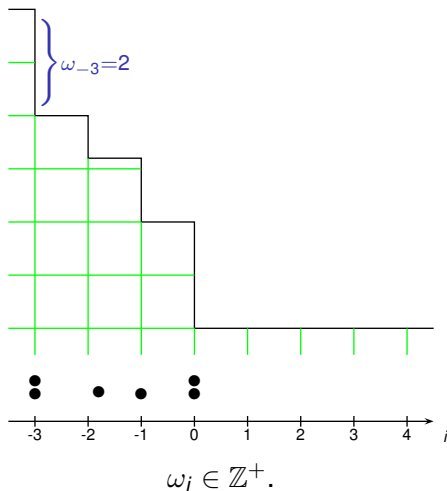
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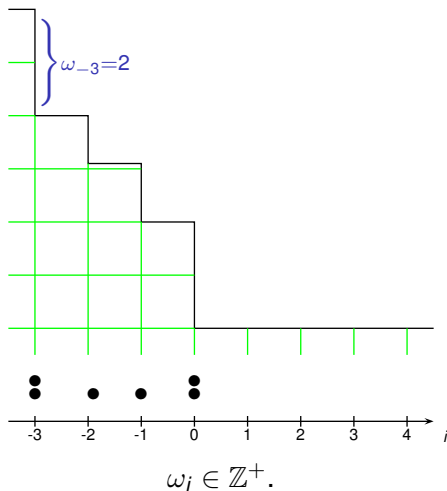
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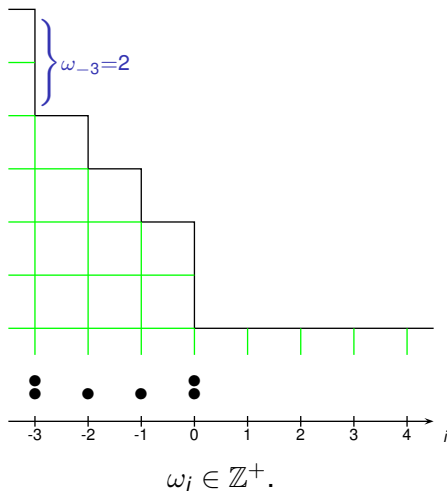
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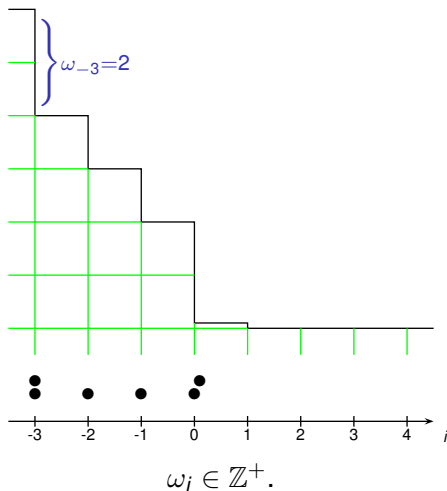
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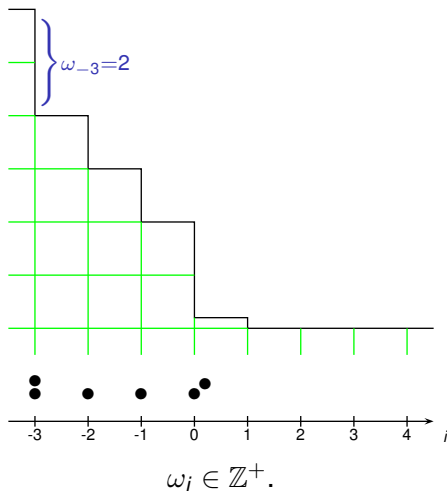
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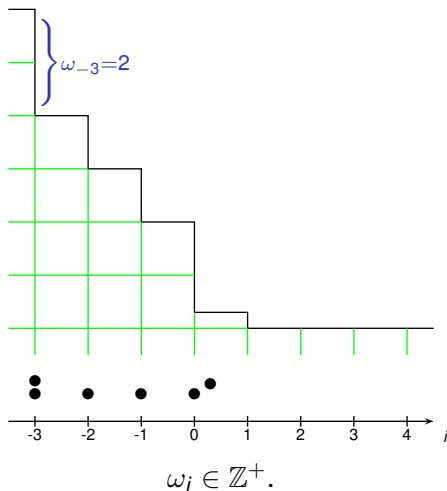
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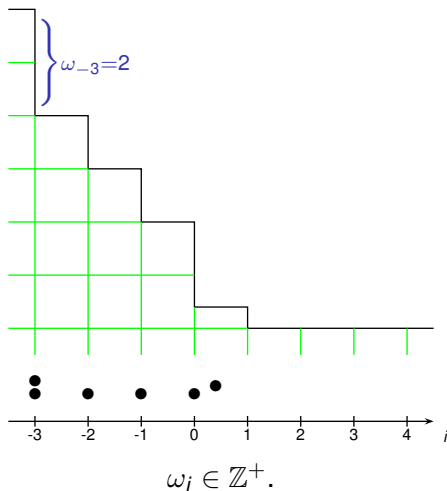
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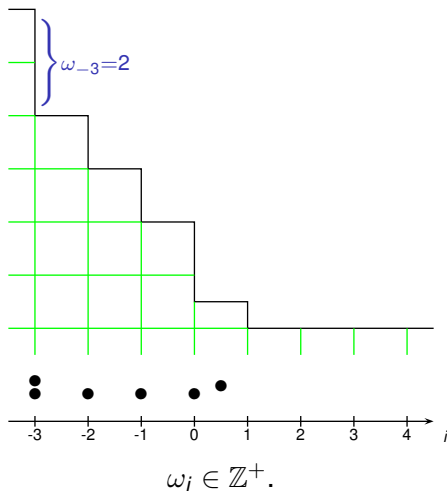
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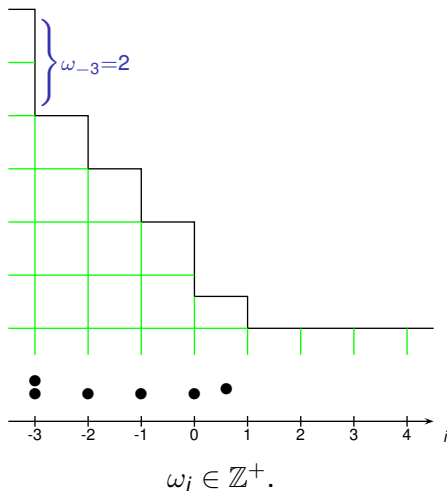
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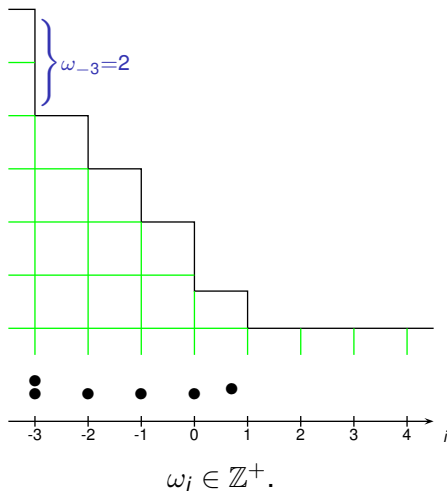
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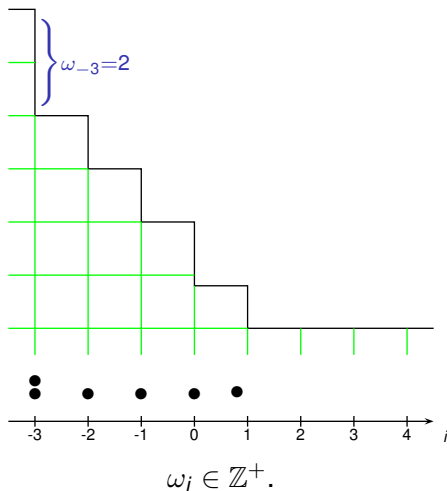
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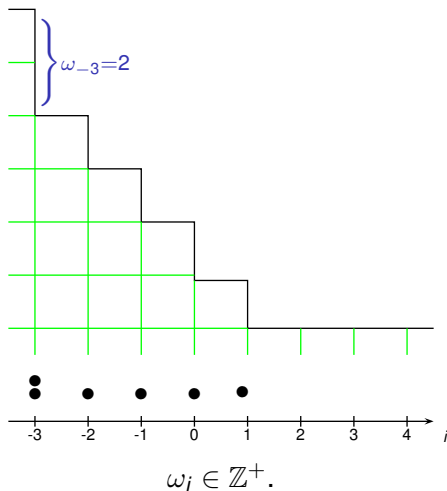
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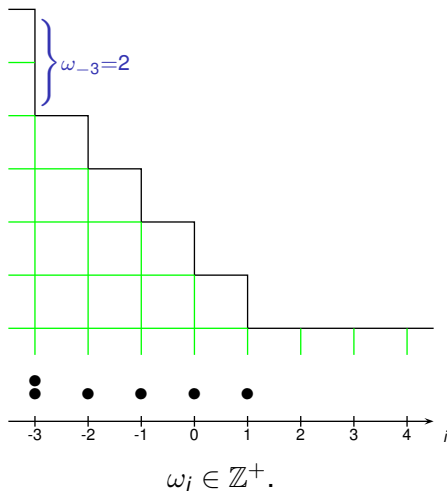
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Particles jump to the right with rate $p \cdot r(\omega_j)$
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The asymmetric zero range process



Particles jump to the right with rate $p \cdot r(\omega_j)$
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The asymmetric zero range process

We need r **non-decreasing** and assume, as before,
 $q = 1 - p < p$.

Examples:

- ▶ 'Classical' ZRP: $r(\omega_i) = \mathbf{1}\{\omega_i > 0\}$.
- ▶ Independent walkers: $r(\omega_i) = \omega_i$.

Product blocking measures

Can we have a reversible stationary distribution in product form:

$$\underline{\mu}(\underline{\omega}) = \bigotimes_i \mu_i(\omega_i);$$

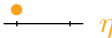
$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \rightsquigarrow i+1}) = \underline{\mu}(\underline{\omega}^{i \rightsquigarrow i+1}) \cdot \text{rate}(\underline{\omega}^{i \rightsquigarrow i+1} \rightarrow \underline{\omega}) \quad ?$$

Here

$$\underline{\omega}^{i \rightsquigarrow i+1} = \underline{\omega} - \underline{\delta}_i + \underline{\delta}_{i+1}.$$

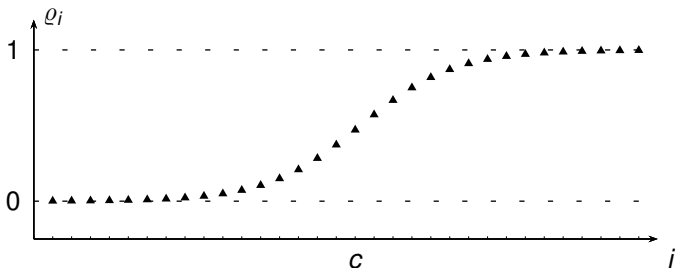
Asymmetric simple exclusion

$$\underline{\mu}(\underline{\eta}) \cdot \text{rate}(\underline{\eta} \rightarrow \underline{\eta}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\eta}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\eta}^{i \curvearrowright i+1} \rightarrow \underline{\eta})$$

ASEP: $\mu_i \sim \text{Bernoulli}(\varrho_i)$;  η

$$\varrho_i(1 - \varrho_{i+1}) \cdot p = (1 - \varrho_i)\varrho_{i+1} \cdot q$$

Solution:
$$\varrho_i = \frac{(\frac{p}{q})^{i-c}}{1 + (\frac{p}{q})^{i-c}} = \frac{1}{(\frac{q}{p})^{i-c} + 1}$$



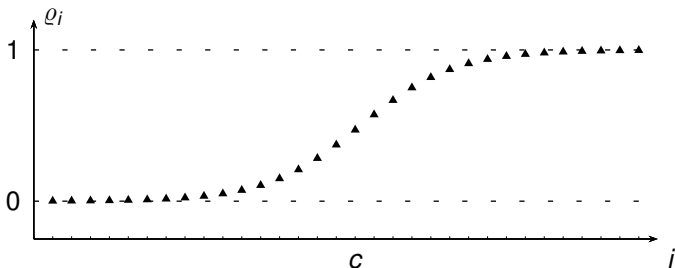
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Asymmetric zero range process

$$\underline{\mu}(\underline{\omega}) \cdot \text{rate}(\underline{\omega} \rightarrow \underline{\omega}^{i \curvearrowright i+1}) = \underline{\mu}(\underline{\omega}^{i \curvearrowright i+1}) \cdot \text{rate}(\underline{\omega}^{i \curvearrowright i+1} \rightarrow \underline{\omega}) \quad ?$$

AZRP:

$$\mu_i(\omega_i) \mu_{i+1}(\omega_{i+1}) \cdot p \mathbf{1}\{\omega_i > 0\} = \mu_i(\omega_i - 1) \mu_{i+1}(\omega_{i+1} + 1) \cdot q$$

Solution: $\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i - \text{const}}\right).$

State space: ASEP

Notice:

$$\mathbf{P}\{\eta_i = 0\} = 1 - \varrho_i = \frac{1}{1 + \left(\frac{\rho}{q}\right)^{i-c}} \quad \text{as } i \rightarrow \infty$$

$$\mathbf{P}\{\eta_i = 1\} = \varrho_i = \frac{1}{\left(\frac{q}{\rho}\right)^{i-c} + 1} \quad \text{as } i \rightarrow -\infty$$

are both summable. Hence by Borel-Cantelli there is $\underline{\mu}$ -a.s. a rightmost hole and a leftmost particle,

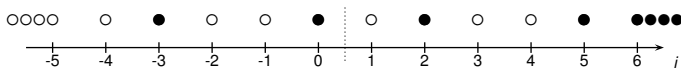
$$N := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

is $\underline{\mu}$ -a.s. finite.

State space: ASEP

$$N(\underline{\eta}) := \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$1 = 3 - 2$$



Also: it is **conserved**. Define

$$\Omega^n := \{0, 1\}^{\mathbb{Z}} \cap \{N(\underline{\eta}) = n\},$$

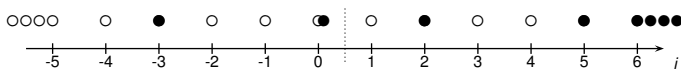
the **irreducible components** of the state space.

$$\underline{\mu} \left(\bigcup_{n=-\infty}^{\infty} \Omega^n \right) = 1.$$

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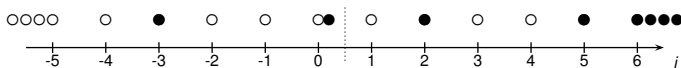
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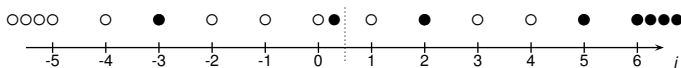
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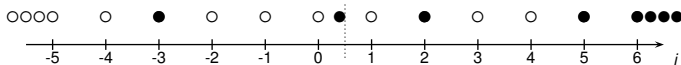
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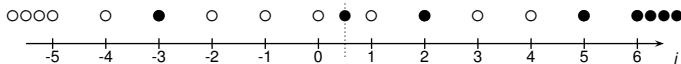
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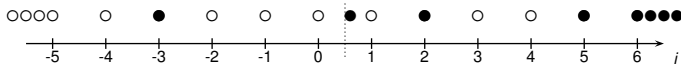
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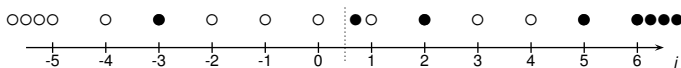
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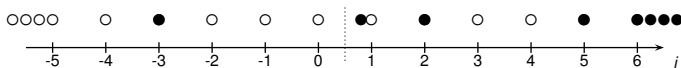
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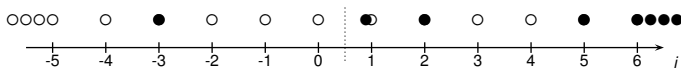
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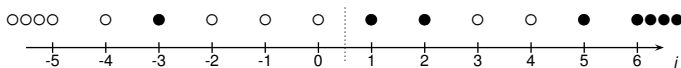
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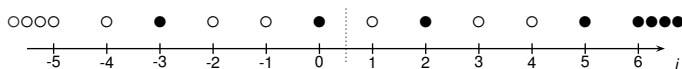
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Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

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$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

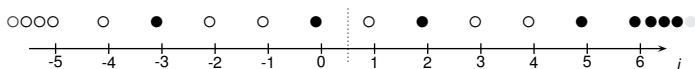
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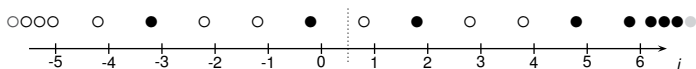
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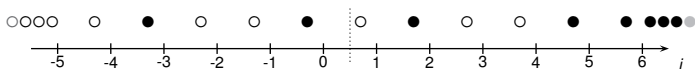
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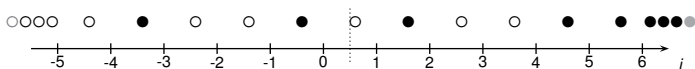
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

$$N(\underline{\eta}) = \sum_{i=1}^{\infty} (1 - \eta_i) - \sum_{i=-\infty}^0 \eta_i$$

$$N = 3 - 2 = 1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

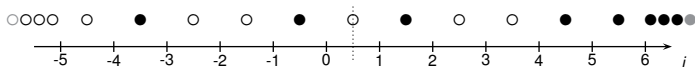
State space: ASEP

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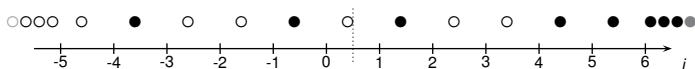
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$$N = 2 - 2 = 0$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

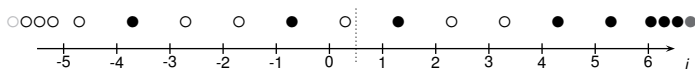
State space: ASEP

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$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

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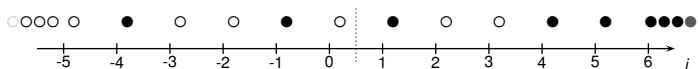
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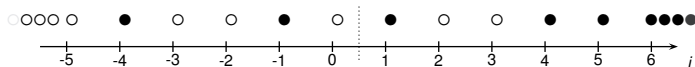
State space: ASEP

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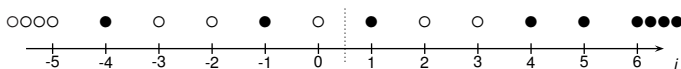
State space: ASEP

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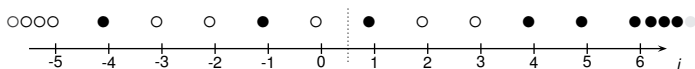
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

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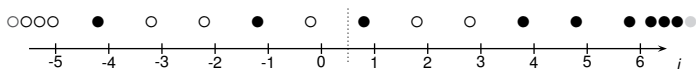
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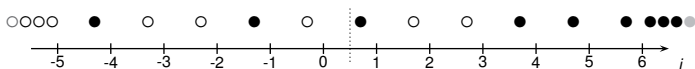
State space: ASEP

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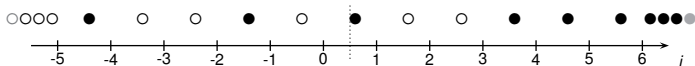
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

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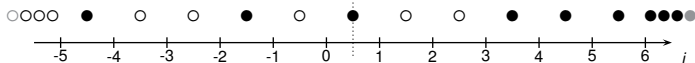
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

$$N(\tau\underline{\eta}) = N(\underline{\eta}) - 1$$

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$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

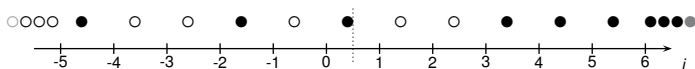
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$$N = 2 - 3 = -1$$



$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

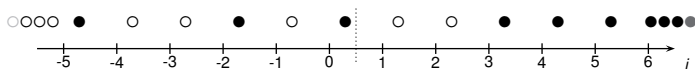
State space: ASEP

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$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

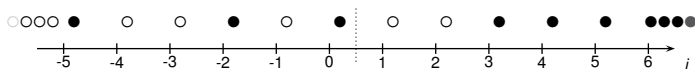
State space: ASEP

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$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

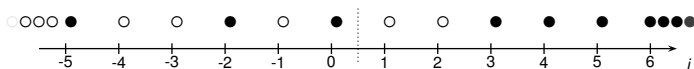
State space: ASEP

Left shift: $(\tau\underline{\eta})_i = \eta_{i+1}$.

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$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

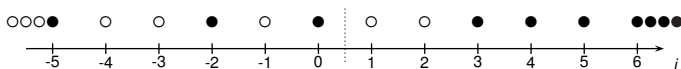
State space: ASEP

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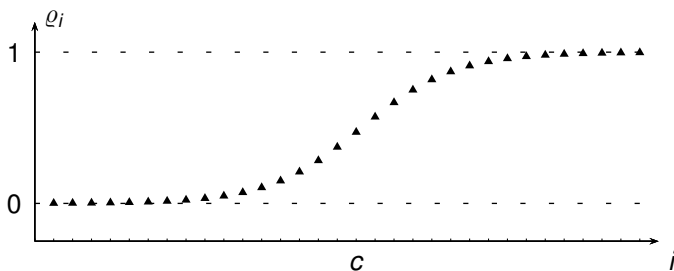


$$\tau : \Omega^n \rightarrow \Omega^{n-1}$$

State space: ASEP

Recall

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}},$$

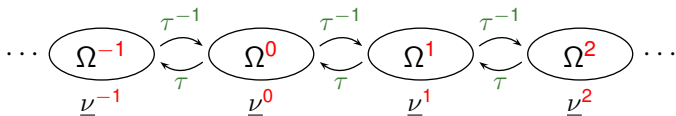


but

$$\underline{\nu}^n(\cdot) := \underline{\mu}(\cdot \mid N(\cdot) = n)$$

doesn't depend on c anymore. **Stationary measure on Ω^n .**

State space: ASEP



$$\underline{\mu}(\cdot) = \sum_{n=-\infty}^{\infty} \underline{\mu}(\cdot | N(\cdot) = n) \underline{\mu}(N(\cdot) = n) = \sum_{n=-\infty}^{\infty} \underline{\nu}^n(\cdot) \underline{\mu}(N(\cdot) = n).$$

Ergodic decomposition of $\underline{\mu}$.

Let's find the coefficients $\underline{\mu}(N(\cdot) = n)$!

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$\underline{\mu}(\underline{\eta}) = \prod_{i \leq 0} \frac{\left(\frac{p}{q}\right)^{(i-c)\eta_i}}{1 + \left(\frac{p}{q}\right)^{i-c}} \cdot \prod_{i > 0} \frac{\left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

Recall:

$$\varrho_i = \frac{\left(\frac{p}{q}\right)^{i-c}}{1 + \left(\frac{p}{q}\right)^{i-c}} = \frac{1}{\left(\frac{q}{p}\right)^{i-c} + 1}$$

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$$= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\underline{\mu}(\underline{\tau\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)}
 \end{aligned}$$

State space: ASEP

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 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
 &= \frac{\prod_{i \leq 1} \left(\frac{p}{q}\right)^{(i-c-1)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 1} \left(\frac{q}{p}\right)^{(i-c-1)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
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 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
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 \end{aligned}$$

State space: ASEP

$$\begin{aligned}
 \underline{\mu}(\underline{\tau\eta}) &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_{i+1}}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_{i+1})}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \\
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 &= \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{N(\underline{\eta})-c}.
 \end{aligned}$$

State space: ASEP

So,

$$\underline{\mu}(N = n - 1) = \sum_{\underline{\eta}: N(\underline{\eta}) = n - 1} \underline{\mu}(\underline{\eta})$$

State space: ASEP

So,

$$\begin{aligned}\underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\ &= \sum_{\underline{\eta}: N(\underline{\tau}\underline{\eta})=n-1} \underline{\mu}(\underline{\tau}\underline{\eta})\end{aligned}$$

State space: ASEP

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State space: ASEP

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State space: ASEP

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State space: ASEP

So,

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 \underline{\mu}(N = n - 1) &= \sum_{\underline{\eta}: N(\underline{\eta})=n-1} \underline{\mu}(\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\tau\underline{\eta})=n-1} \underline{\mu}(\tau\underline{\eta}) \\
 &= \sum_{\underline{\eta}: N(\underline{\eta})=n} \underline{\mu}(\underline{\eta}) \cdot \left(\frac{p}{q}\right)^{n-c} \\
 &= \underline{\mu}(N = n) \cdot \left(\frac{p}{q}\right)^{n-c}.
 \end{aligned}$$

Solution:

$$\underline{\mu}(N = n) = \frac{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}$$

discrete Gaussian.

State space: ASEP

and, if $N(\underline{\eta}) = n$,

$$\begin{aligned} \underline{\nu}^n(\underline{\eta}) &= \underline{\mu}(\underline{\eta} \mid N(\underline{\eta}) = n) = \frac{\underline{\mu}(\underline{\eta})}{\underline{\mu}(N(\underline{\eta}) = n)} \\ &= \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c)\eta_i}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c)(1-\eta_i)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{n^2+n}{2} - cn}}. \end{aligned}$$

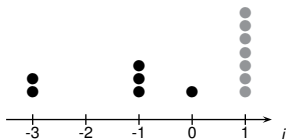
This is the unique stationary distribution on Ω^n .

State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

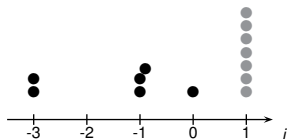


State space: AZRP

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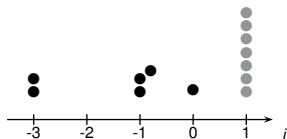


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

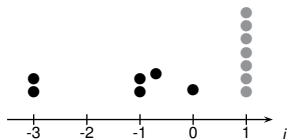


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

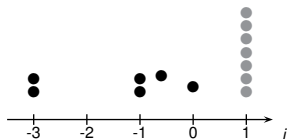


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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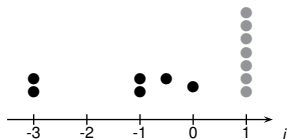


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.

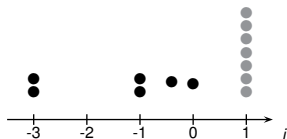


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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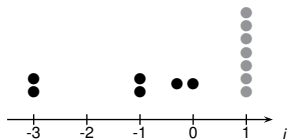


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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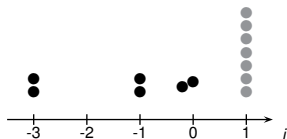


State space: AZRP

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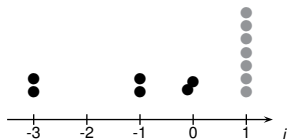


State space: AZRP

Recall: Stationary distribution with marginals

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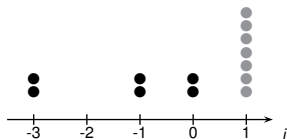


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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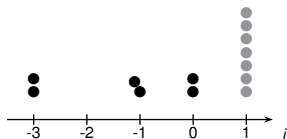


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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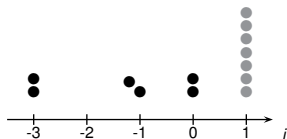


State space: AZRP

Recall: Stationary distribution with marginals

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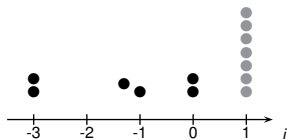


State space: AZRP

Recall: Stationary distribution with marginals

$$\mu_i \sim \text{Geometric}\left(1 - \left(\frac{p}{q}\right)^{i-\text{const}}\right).$$

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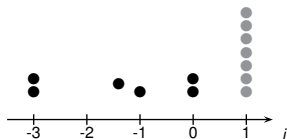


State space: AZRP

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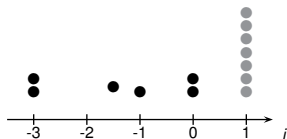


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Recall: Stationary distribution with marginals

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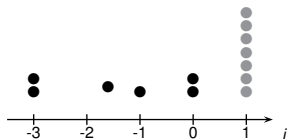


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Recall: Stationary distribution with marginals

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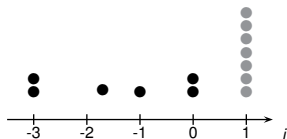


State space: AZRP

Recall: Stationary distribution with marginals

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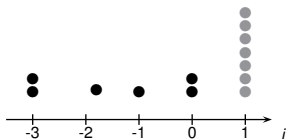


State space: AZRP

Recall: Stationary distribution with marginals

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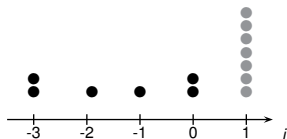


State space: AZRP

Recall: Stationary distribution with marginals

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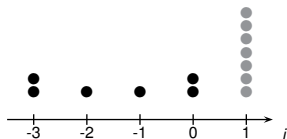


State space: AZRP

Recall: Stationary distribution with marginals

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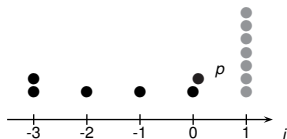


State space: AZRP

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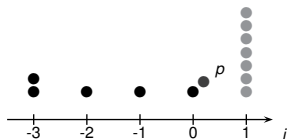


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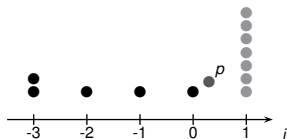


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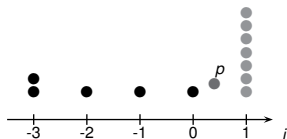


State space: AZRP

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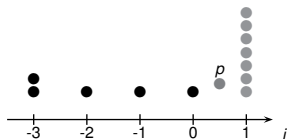


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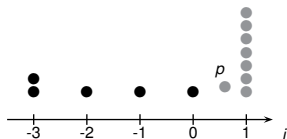


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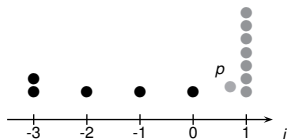


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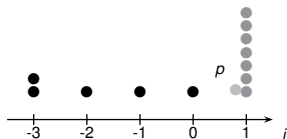


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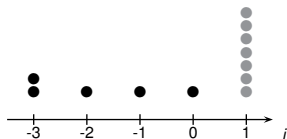


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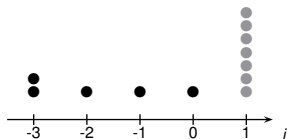


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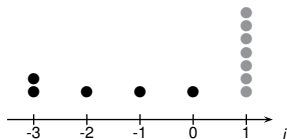


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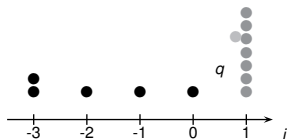


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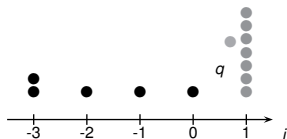


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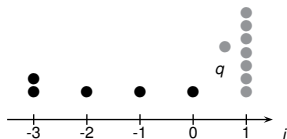


State space: AZRP

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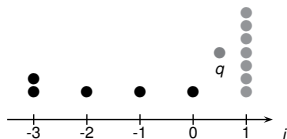


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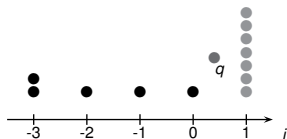


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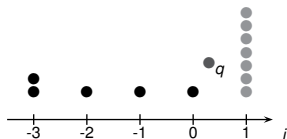


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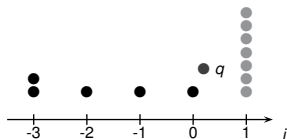


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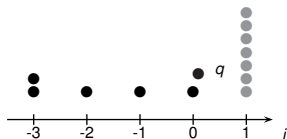


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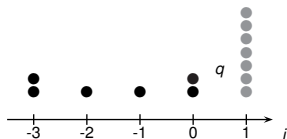


State space: AZRP

Recall: Stationary distribution with marginals

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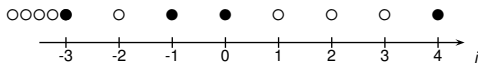
\rightsquigarrow we have a problem: cannot do this for all i ! We'll pick $\text{const} = 1$ and have a *right boundary* instead.



\rightsquigarrow The product measure stays stationary on the half-line.

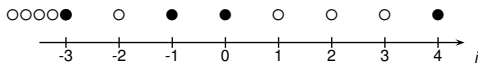
Lay down / stand up

ASEP



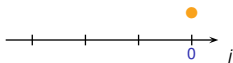
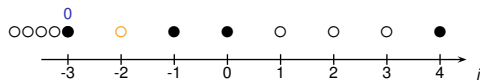
Lay down / stand up

ASEP



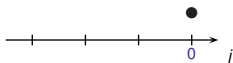
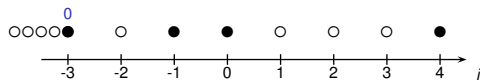
Lay down / stand up

ASEP



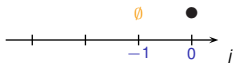
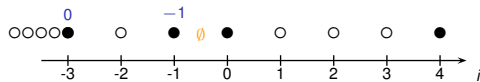
Lay down / stand up

ASEP



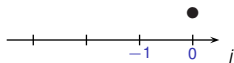
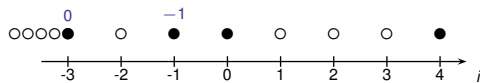
Lay down / stand up

ASEP



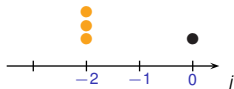
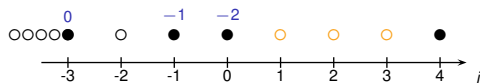
Lay down / stand up

ASEP



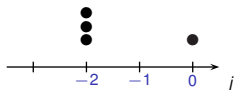
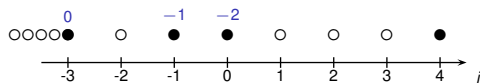
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ASEP



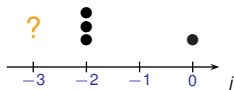
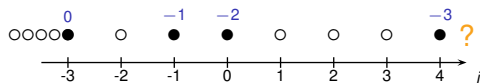
Lay down / stand up

ASEP



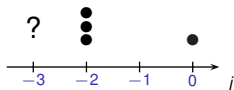
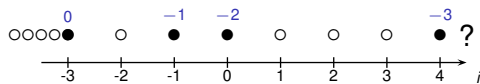
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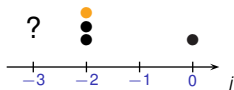
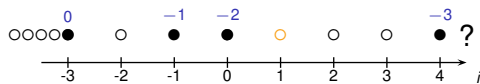
Lay down / stand up

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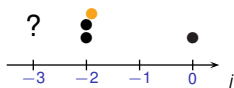
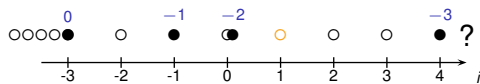
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ASEP



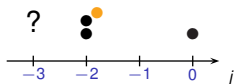
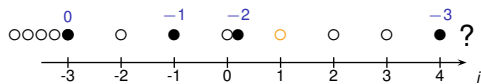
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ASEP



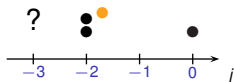
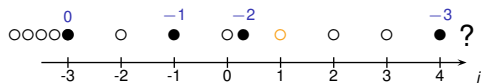
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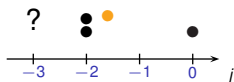
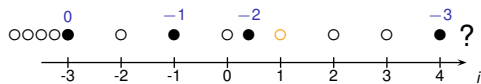
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ASEP



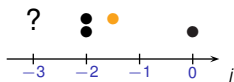
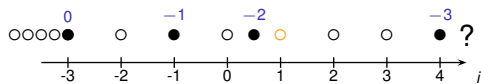
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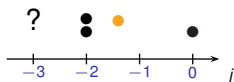
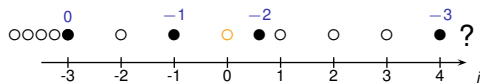
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ASEP



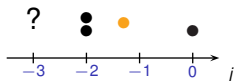
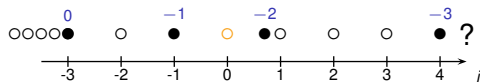
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ASEP



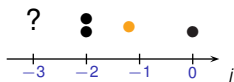
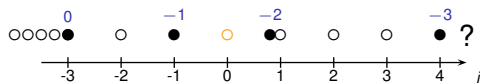
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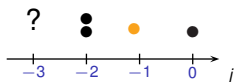
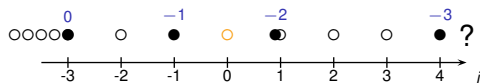
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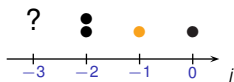
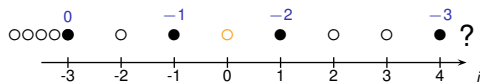
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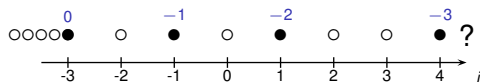
Lay down / stand up

ASEP

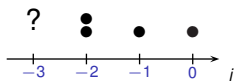


Lay down / stand up

ASEP

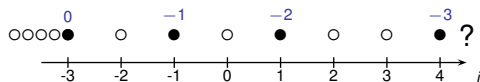


AZRP

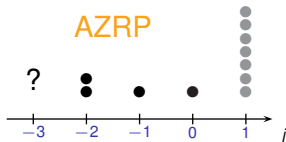


Lay down / stand up

ASEP

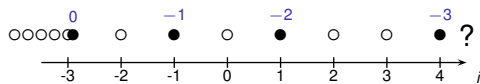


AZRP

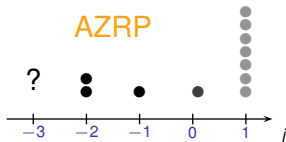


Lay down / stand up

ASEP

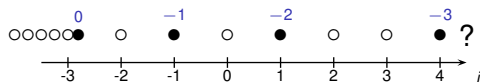


AZRP

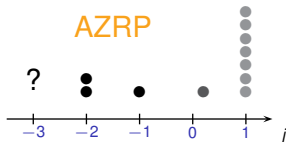


Lay down / stand up

ASEP

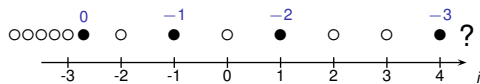


AZRP

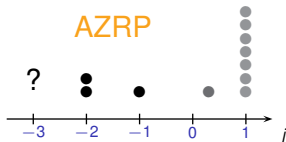


Lay down / stand up

ASEP

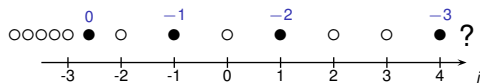


AZRP

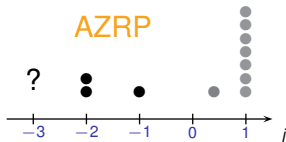


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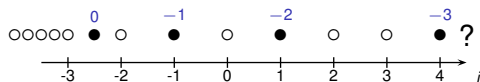


AZRP

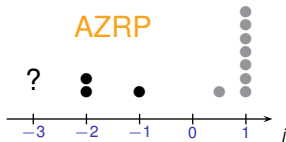


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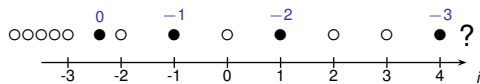


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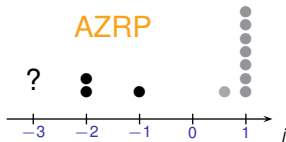


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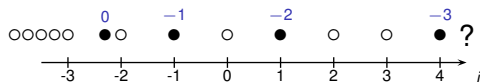


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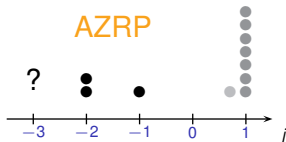


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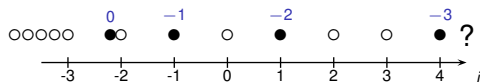


AZRP

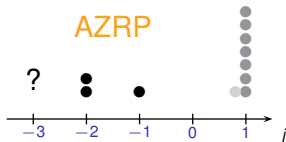


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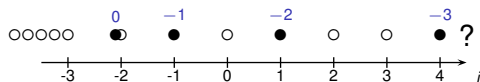


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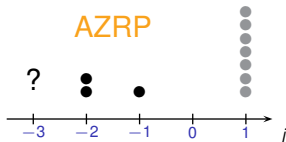


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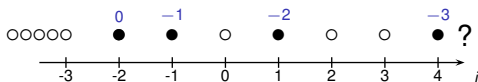


AZRP

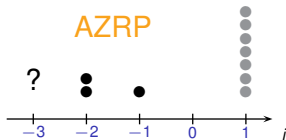


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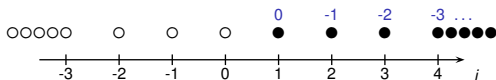
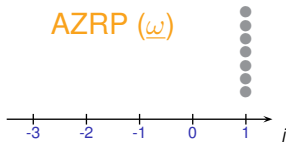


$$\text{ASEP} \stackrel{T^n}{=} \text{AZRP}$$

$$\underline{\nu}^n = \prod_{i \leq 0} \text{Geometric} \left(1 - \left(\frac{p}{q} \right)^{i-1} \right)$$

since stationary distributions of countable irreducible Markov chains are unique.

Jacobi triple product

ASEP ($\underline{\eta}$)AZRP ($\underline{\omega}$)

$$\eta_i = \mathbf{1}\{i \geq 1\}, \quad N(\underline{\eta}) = 0, \quad \omega_i \equiv 0.$$

$$\underline{\nu}^0(\underline{\eta}) = \frac{\prod_{i \leq 0} \left(\frac{p}{q}\right)^{(i-c) \cdot 0}}{\prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right)} \cdot \frac{\prod_{i > 0} \left(\frac{q}{p}\right)^{(i-c) \cdot (1-1)}}{\prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right)} \cdot \frac{\sum \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}}{\left(\frac{q}{p}\right)^{\frac{0^2+0}{2} - c \cdot 0}}$$

$$\underline{\mu}(\underline{\omega}) = \prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right)$$

Jacobi triple product

$$\prod_{i \leq 0} \left(1 - \left(\frac{p}{q}\right)^{i-1}\right) \cdot \prod_{i \leq 0} \left(1 + \left(\frac{p}{q}\right)^{i-c}\right) \cdot \prod_{i > 0} \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) = \sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2+m}{2} - cm}$$

LHS:

$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 - \left(\frac{q}{p}\right)^i\right) \cdot \left(1 + \left(\frac{q}{p}\right)^{i-1+c}\right) \cdot \left(\left(\frac{q}{p}\right)^{i-c} + 1\right) \\ = \prod_{i=1}^{\infty} \left(1 - x^{2i}\right) \left(1 + \frac{x^{2i-1}}{y^2}\right) \left(1 + x^{2i-1}y^2\right) \end{aligned}$$

with $x = \left(\frac{q}{p}\right)^{\frac{1}{2}}$, $y = \left(\frac{q}{p}\right)^{\frac{1}{4}-\frac{c}{2}}$.

RHS:

$$\sum_{m=-\infty}^{\infty} \left(\frac{q}{p}\right)^{\frac{m^2}{2}} \left(\frac{q}{p}\right)^{m(\frac{1}{2}-c)} = \sum_{m=-\infty}^{\infty} x^{m^2} y^{2m}.$$



Further models

Product blocking measures are very general.

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The point was that ASEP is in both lists.

Thank you.