Anomalous scaling of current fluctuations in interacting particle systems

Joint with
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The models

- Asymmetric simple exclusion process
- Zero range
- Bricklayers

Hydrodynamics

- Characteristics

Tool: the second class particle

- Single
- Many second class particles

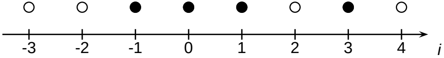
Results

- Normal fluctuations
- Abnormal fluctuations

Proof

- Upper bound
- Lower bound
- Microscopic concavity/convexity

Asymmetric simple exclusion



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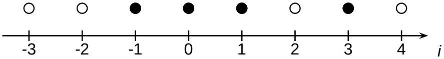

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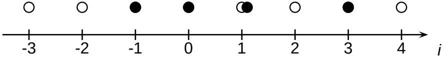

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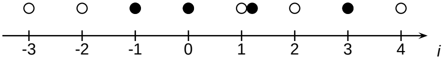

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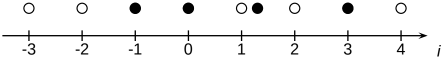

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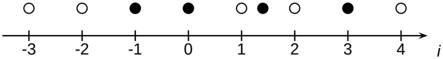

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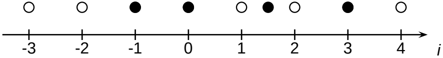

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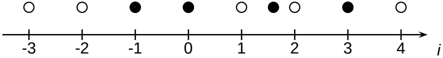

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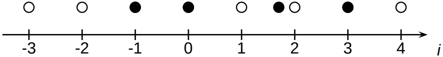

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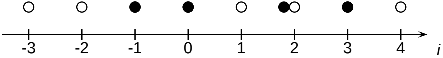

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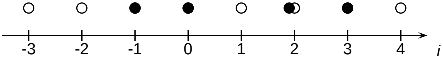

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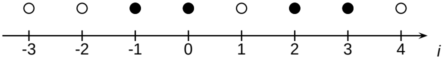

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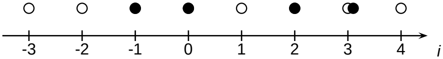

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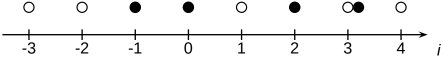

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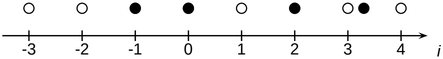

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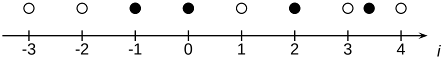

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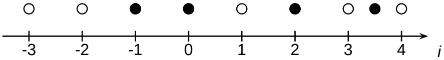

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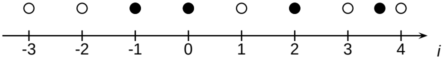

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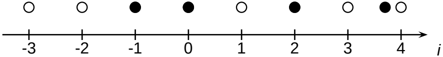

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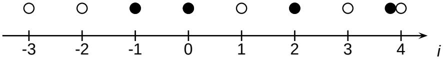

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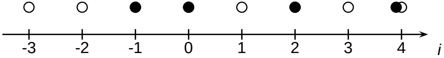

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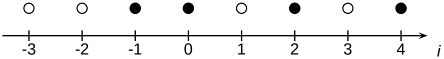

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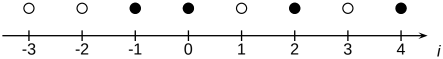

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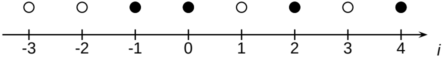

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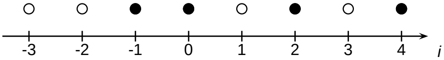

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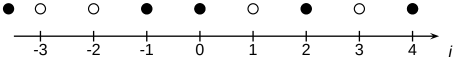

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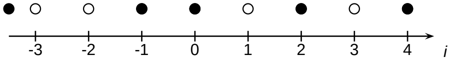

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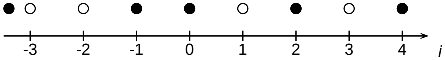

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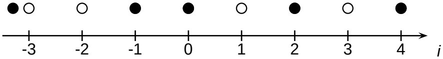

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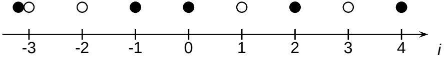

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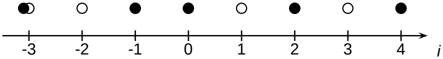

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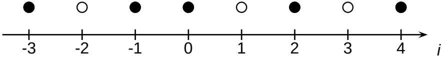

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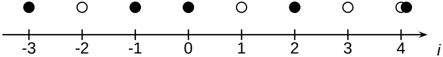

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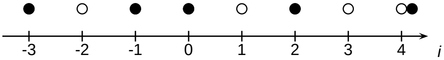

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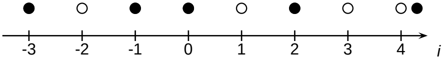

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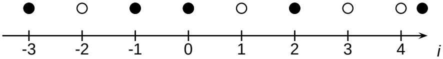

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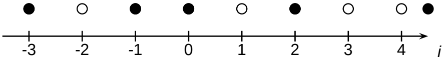

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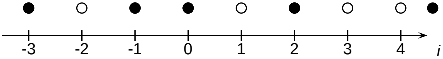

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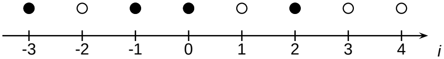

Bernoulli(ϱ) distribution; $\omega_i = 0$ or 1.

Particles try to jump

to the right with rate p
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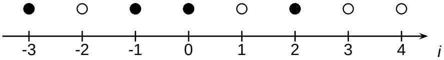

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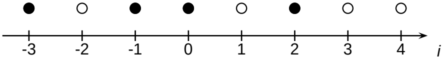

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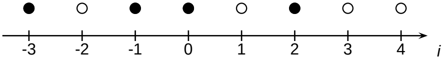

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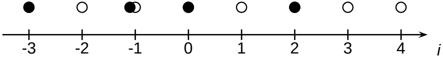

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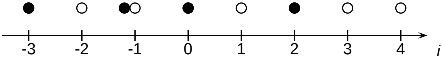

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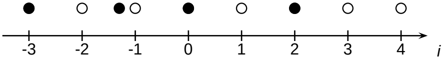

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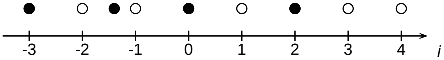

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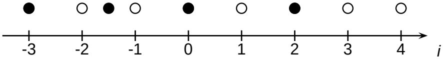

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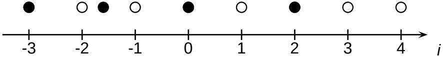

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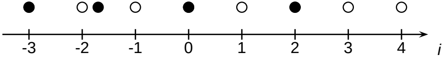

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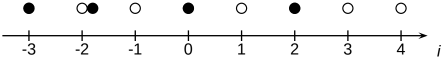

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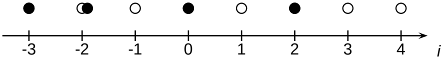

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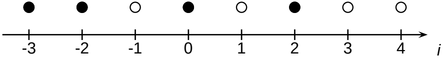

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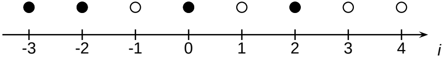

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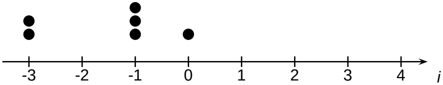
The Bernoulli(ϱ) distribution is time-stationary for any $(0 \leq \varrho \leq 1)$.

The asymmetric zero range process



Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

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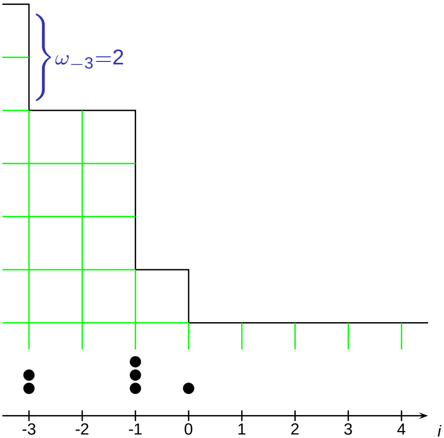

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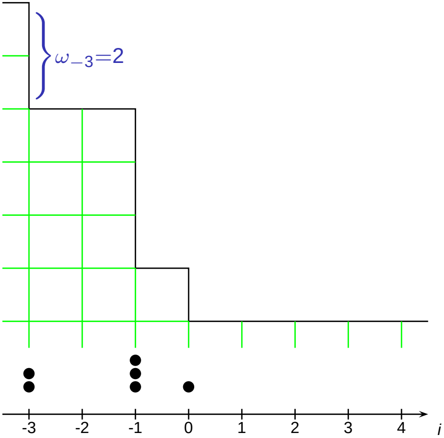

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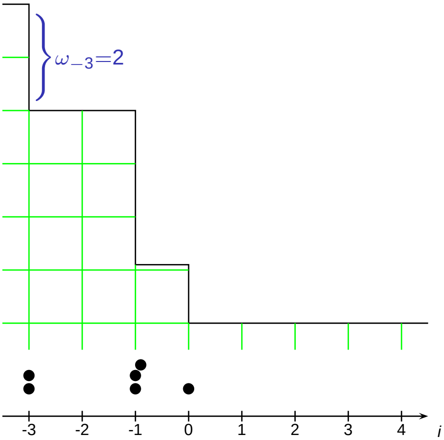

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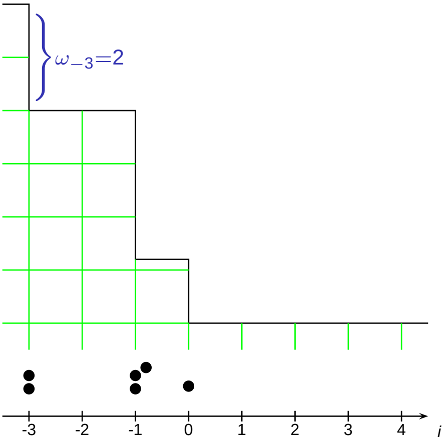

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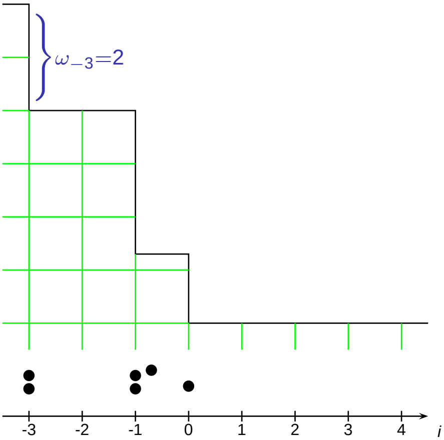

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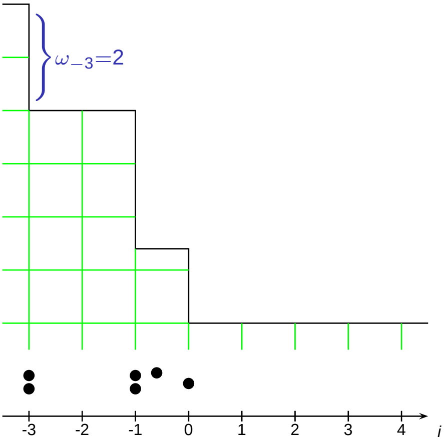

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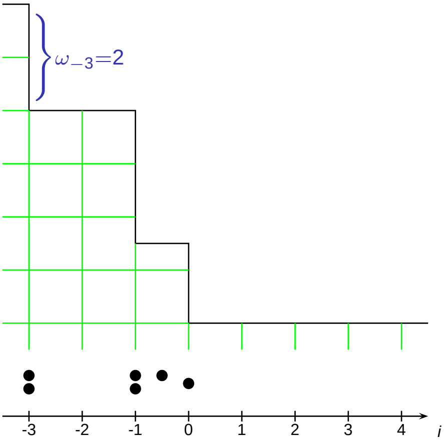

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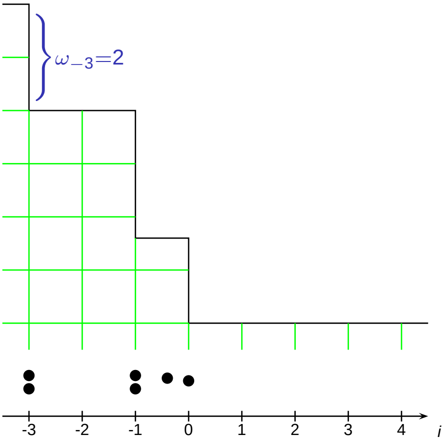

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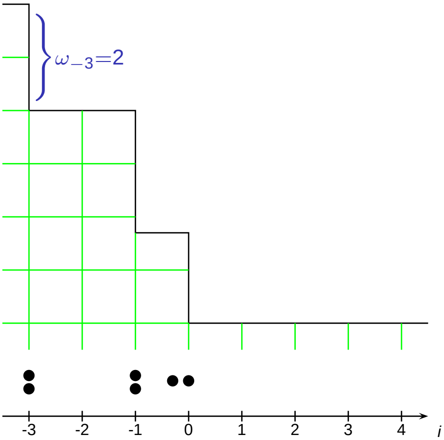

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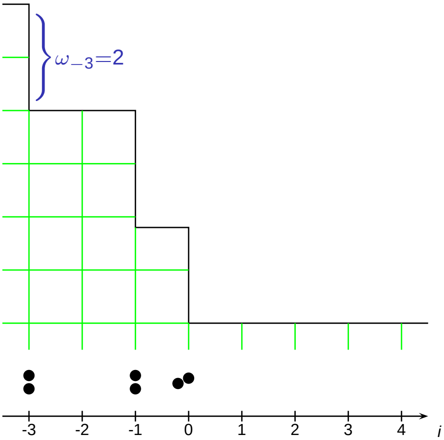

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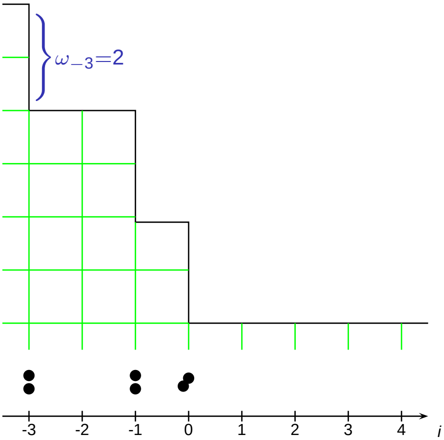

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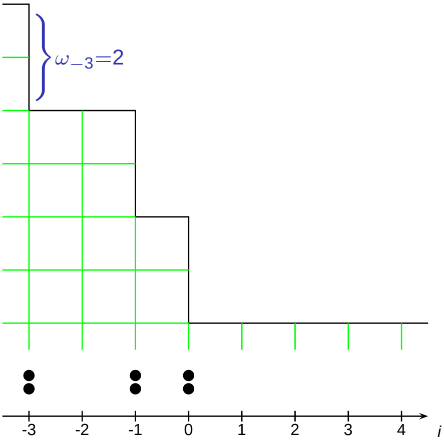

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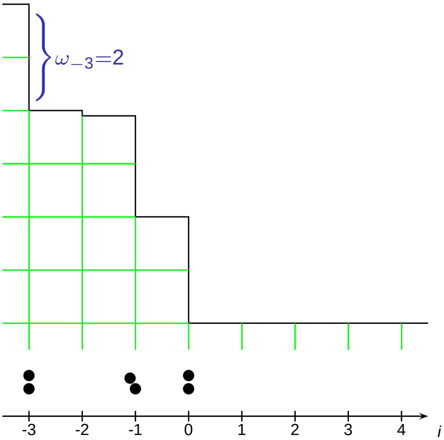

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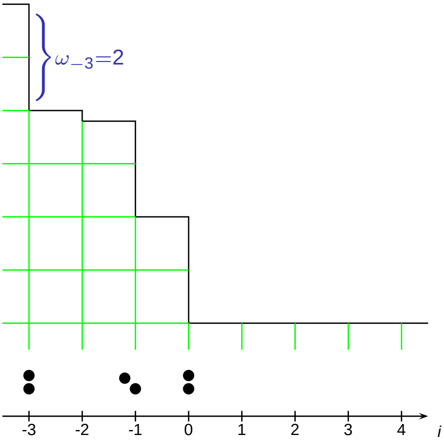

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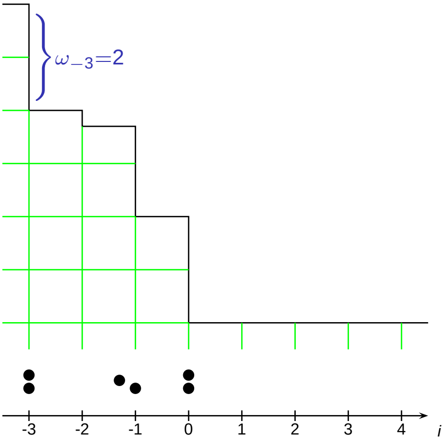

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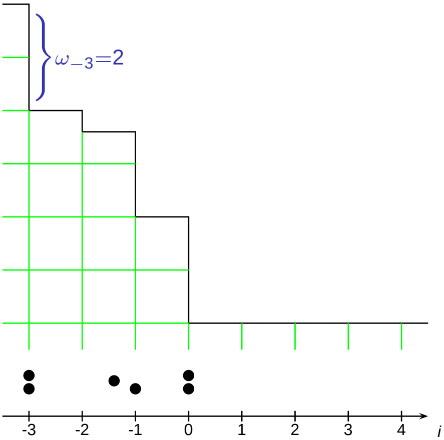

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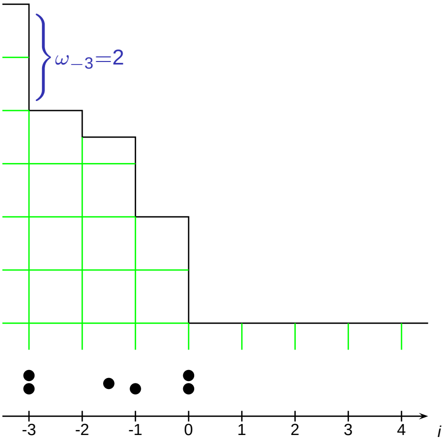

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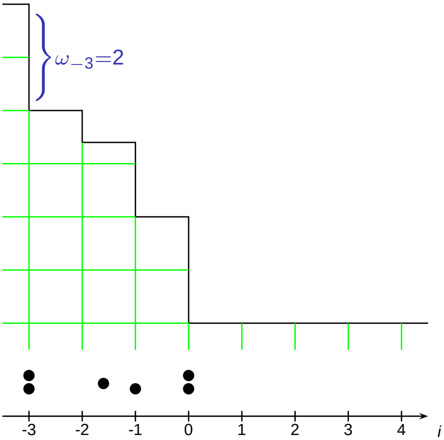

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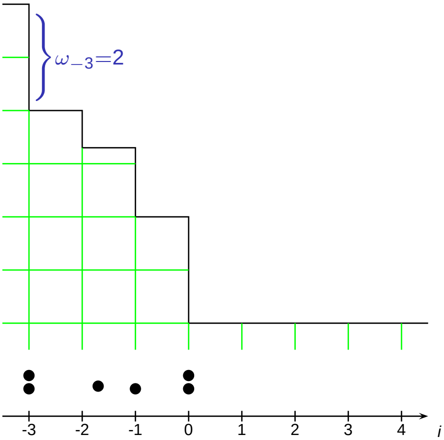

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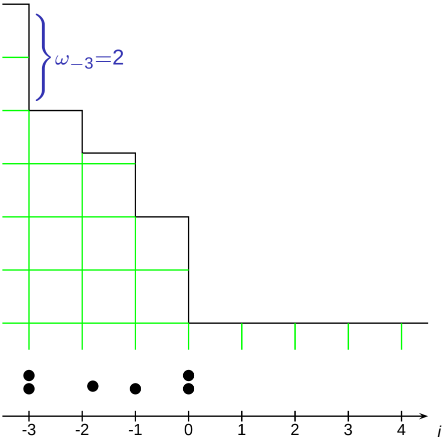

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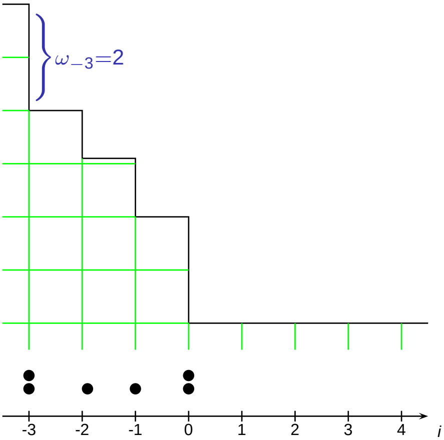

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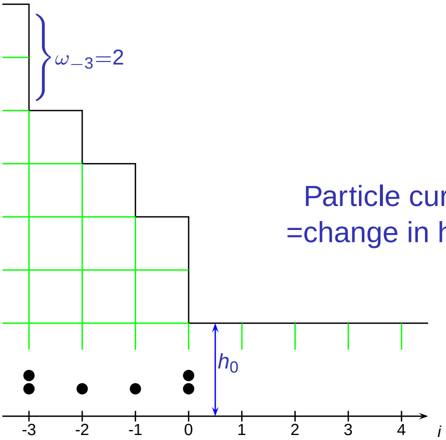

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=change in height

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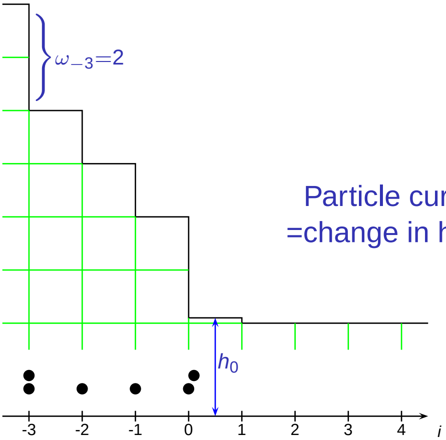

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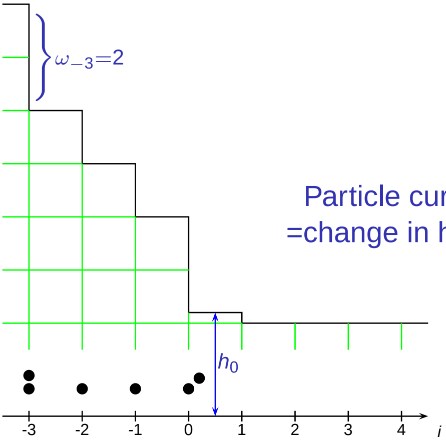

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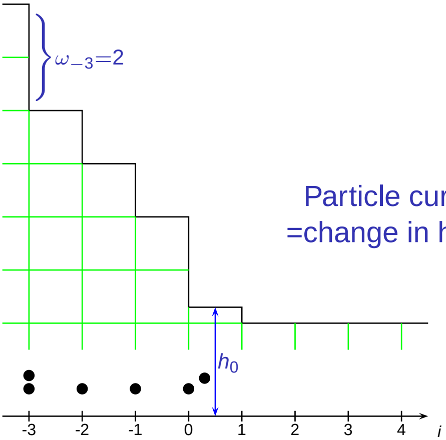

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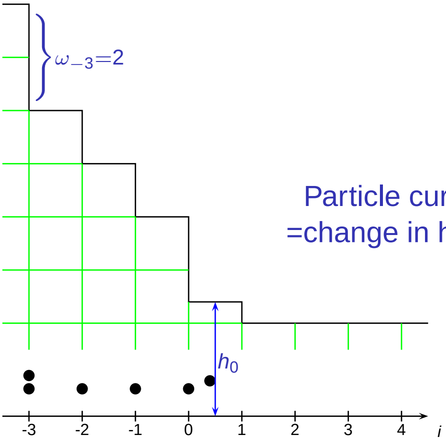

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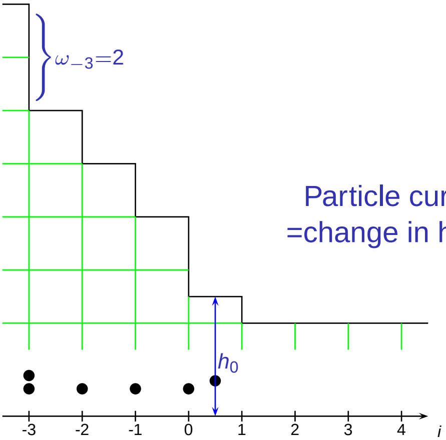
The asymmetric zero range process

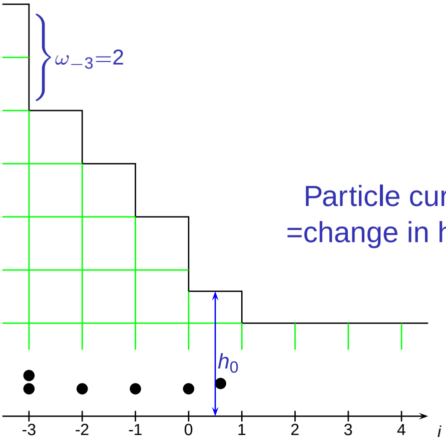


Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

Particles jump
to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
to the left with rate $q \cdot r(\omega_i)$ ($q = 1 - p < p$).

The asymmetric zero range process



Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

Particles jump
to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
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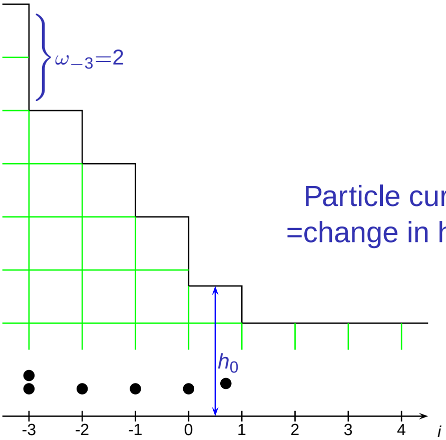

Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

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to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
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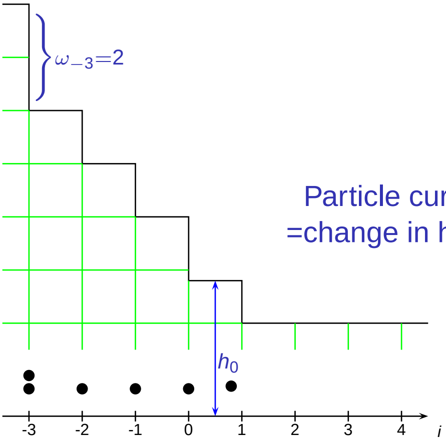
The asymmetric zero range process

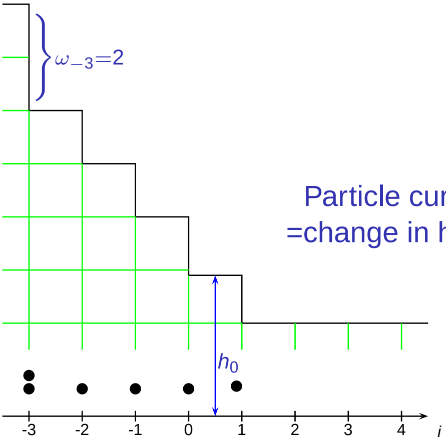


Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

Particles jump
to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
to the left with rate $q \cdot r(\omega_i)$ ($q = 1 - p < p$).

The asymmetric zero range process

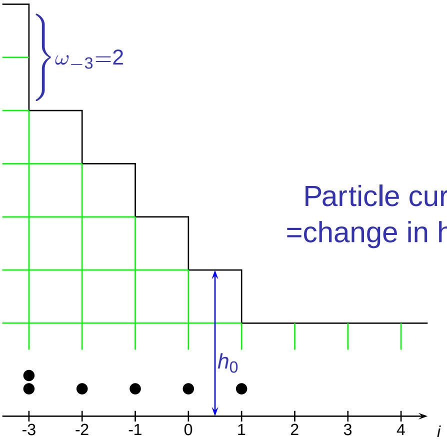


Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

Particles jump
to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
to the left with rate $q \cdot r(\omega_i)$ ($q = 1 - p < p$).

The asymmetric zero range process

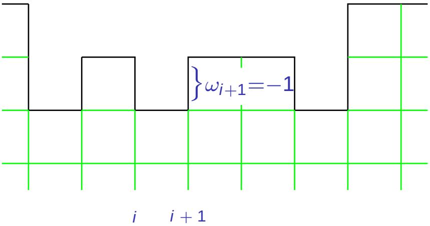


Particle current
=change in height

Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

Particles jump
 to the right with rate $p \cdot r(\omega_i)$ (*r* non-decreasing)
 to the left with rate $q \cdot r(\omega_i)$ ($q = 1 - p < p$).

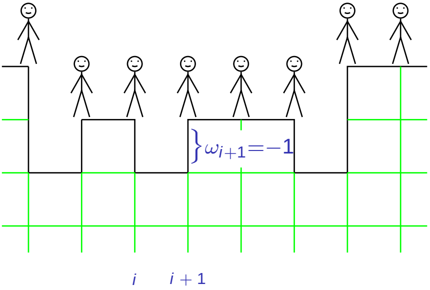
The asymmetric bricklayers process



$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

Poisson-type distribution; $\omega_i \in \mathbb{Z}$.

The asymmetric bricklayers process



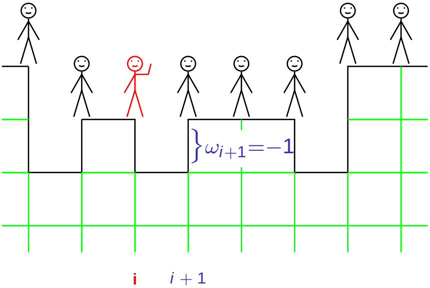
$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \,\|\, \begin{pmatrix}1\\ -1\end{pmatrix}$$

Poisson-type distribution; $\omega_i \in \mathbb{Z}$.

a brick is added with rate $p \cdot [r(\omega_i) + r(-\omega_{i+1})]$
a brick is removed with rate $q \cdot [r(-\omega_i) + r(\omega_{i+1})]$.

$(r(\omega) \cdot r(1-\omega) = 1;\quad r$ non-decreasing; $\quad q = 1-p < p)$.

The asymmetric bricklayers process



$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \| \begin{pmatrix}1\\ -1\end{pmatrix}$$

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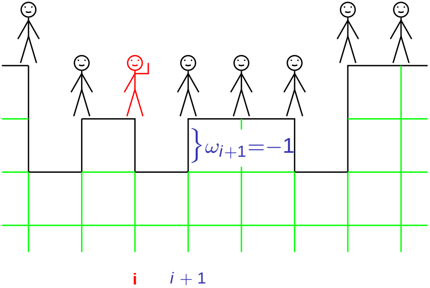

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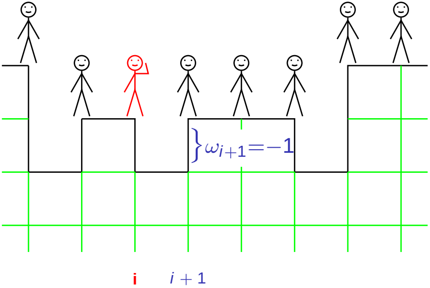

$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \\ || \\ \begin{pmatrix}1\\ -1\end{pmatrix}$$

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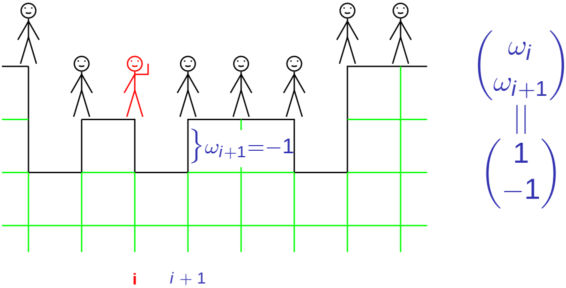

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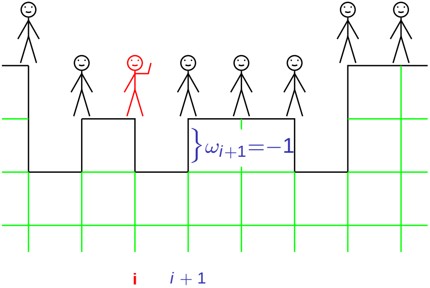

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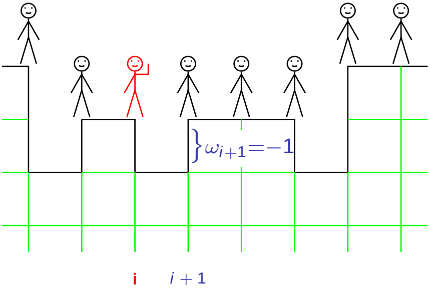

$$\begin{array}{c}\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix}\\ \| \\ \begin{pmatrix}1\\ -1\end{pmatrix}\end{array}$$

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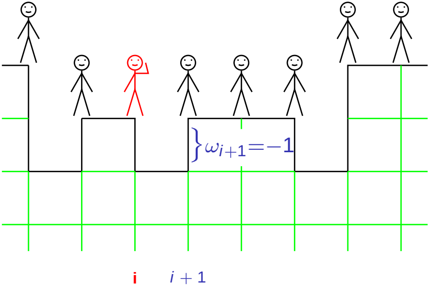

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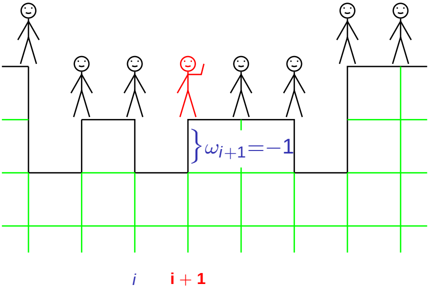

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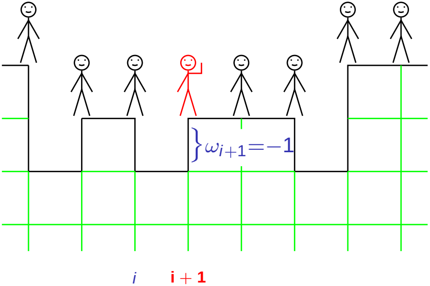

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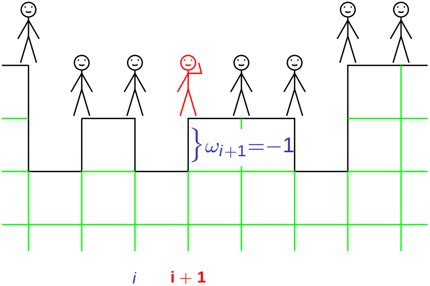

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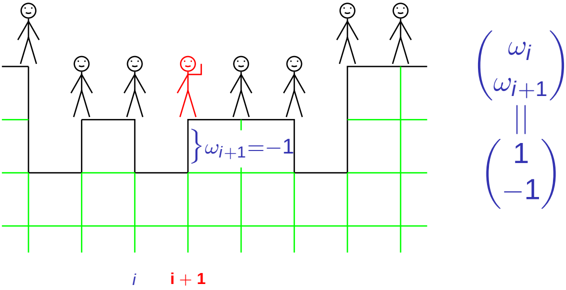

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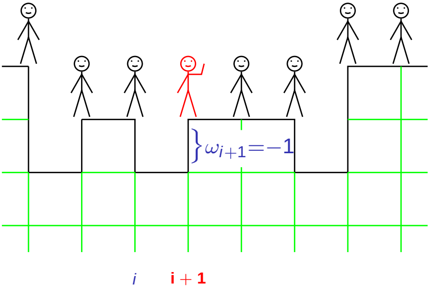

$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \begin{array}{c}\|\end{array} \begin{pmatrix}1\\ -1\end{pmatrix}$$

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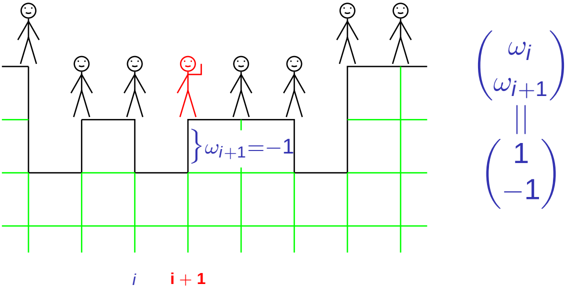

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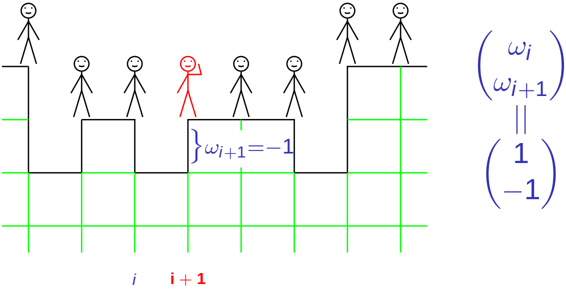

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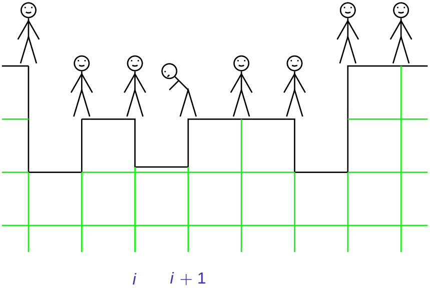

$$\begin{array}{ccc}\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} & \longrightarrow & \begin{pmatrix}\omega_i-1\\ \omega_{i+1}+1\end{pmatrix}\\ \| & & \| \\ \begin{pmatrix}1\\ -1\end{pmatrix} & \longrightarrow & \begin{pmatrix}0\\ 0\end{pmatrix}\end{array}$$

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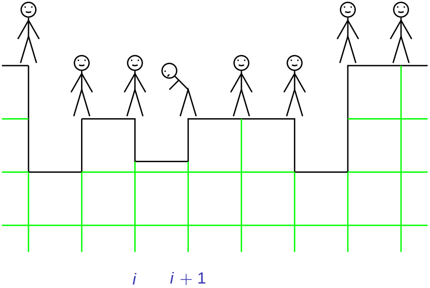

$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \to \begin{pmatrix}\omega_i-1\\ \omega_{i+1}+1\end{pmatrix}$$

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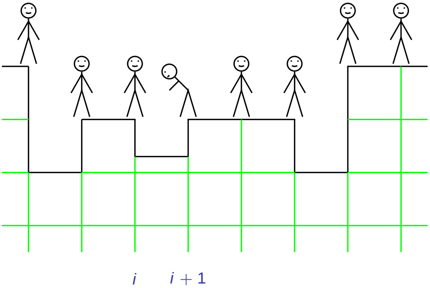
The asymmetric bricklayers process



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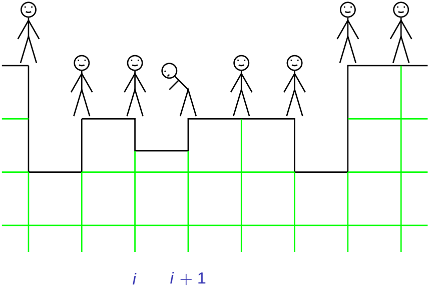

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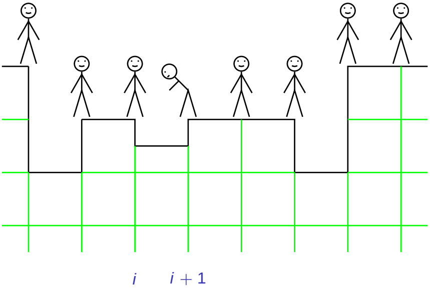

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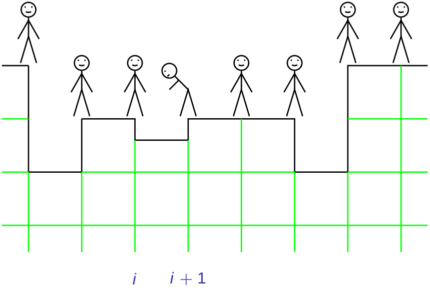

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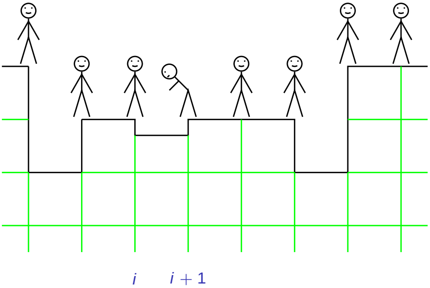

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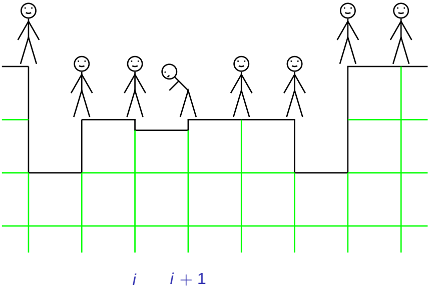

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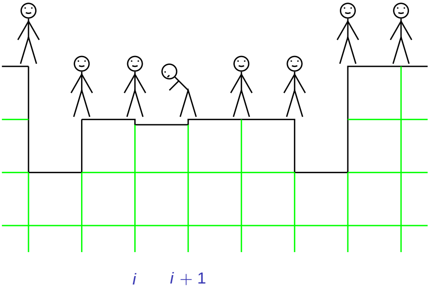

$$\begin{array}{ccc}
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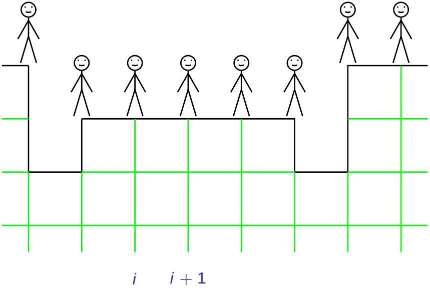
The asymmetric bricklayers process



$$\begin{array}{ccc} \begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} & \to & \begin{pmatrix}\omega_i-1\\ \omega_{i+1}+1\end{pmatrix}\\ \| & & \| \\ \begin{pmatrix}1\\ -1\end{pmatrix} & \to & \begin{pmatrix}0\\ 0\end{pmatrix}\end{array}$$

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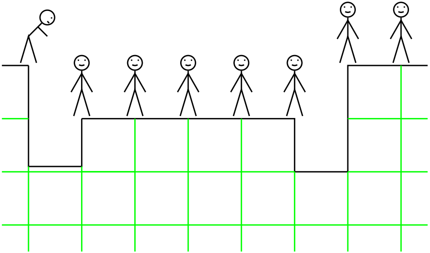

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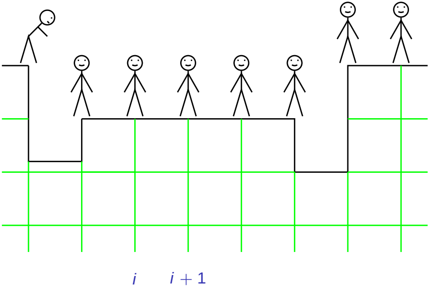

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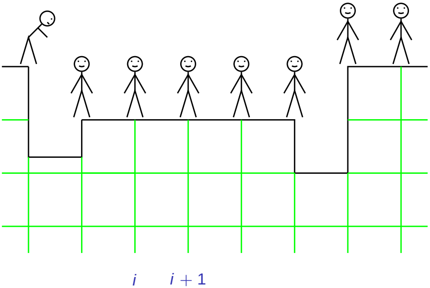

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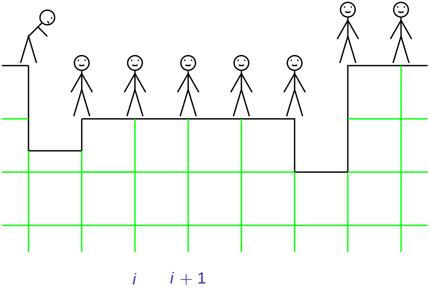

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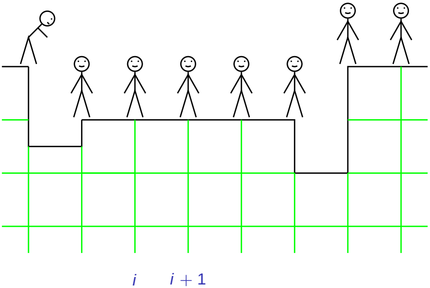

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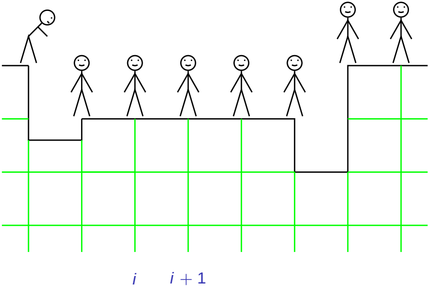

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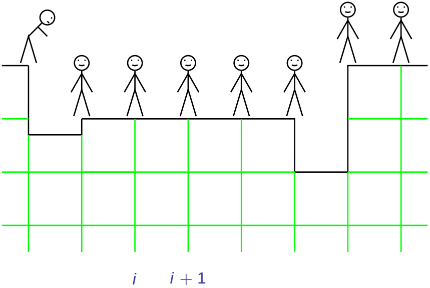

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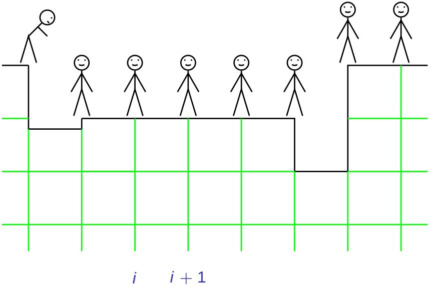

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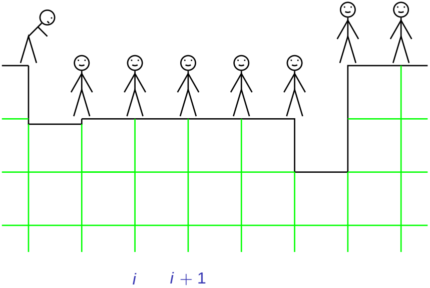

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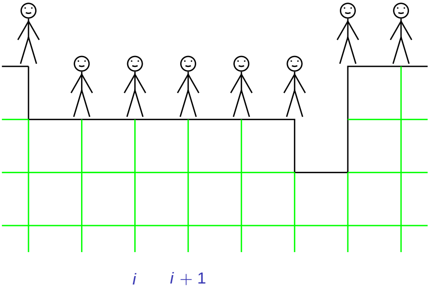

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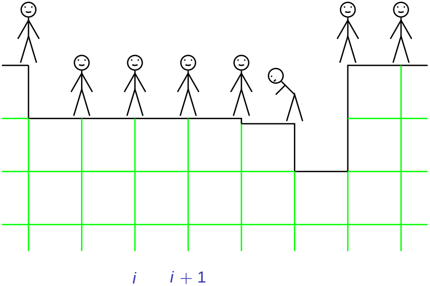

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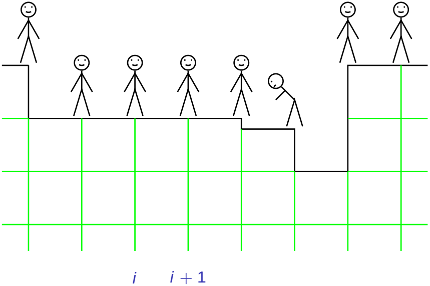

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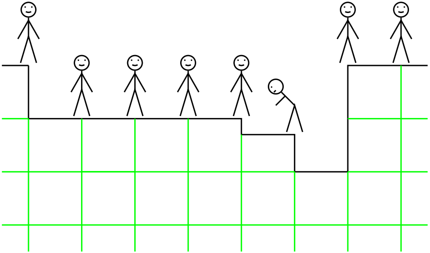

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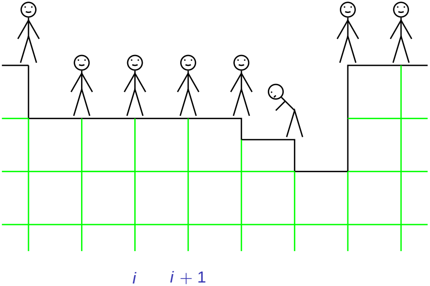

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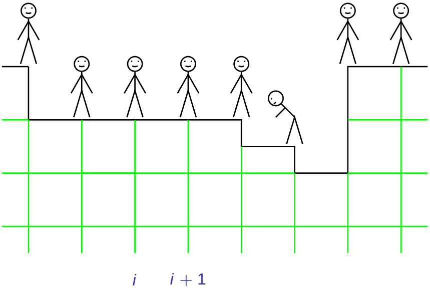

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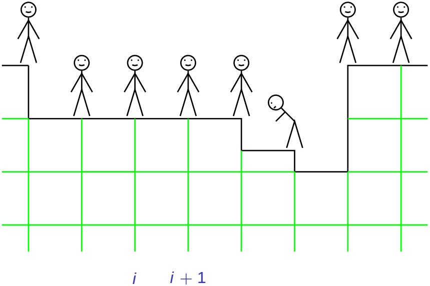

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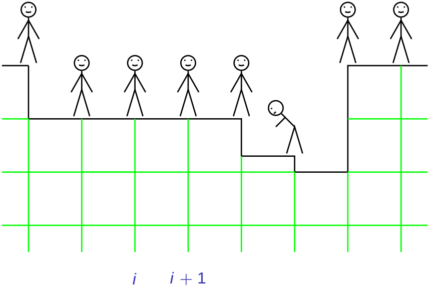

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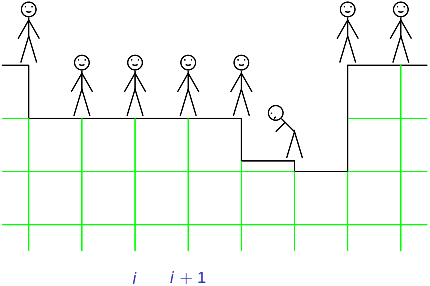

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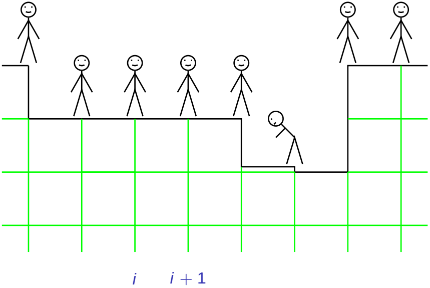

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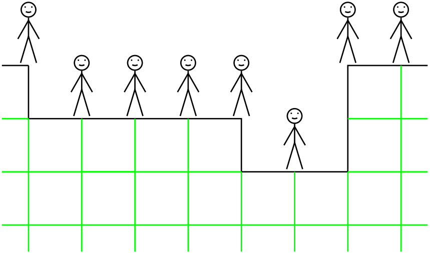

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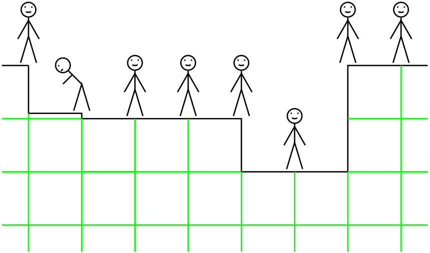

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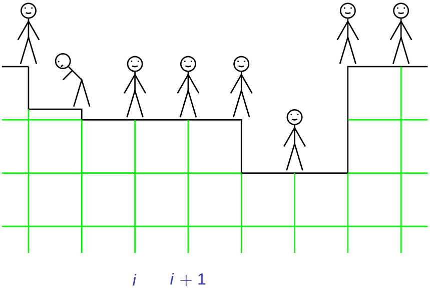

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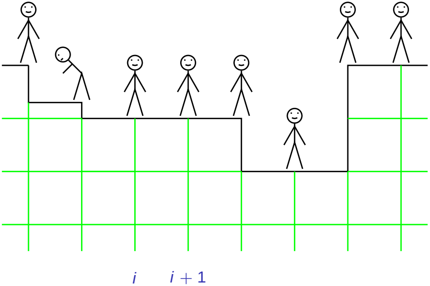

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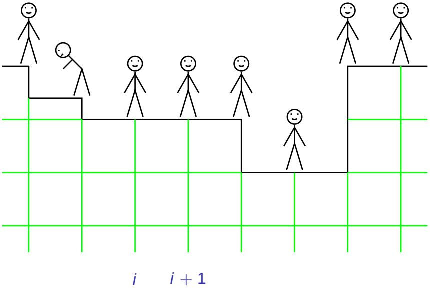

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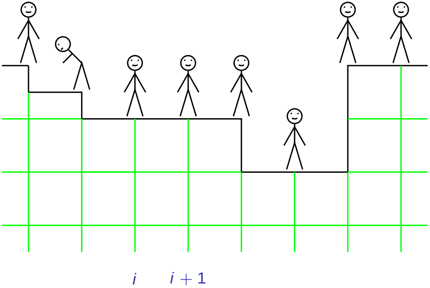

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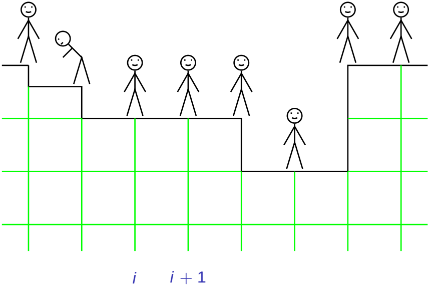

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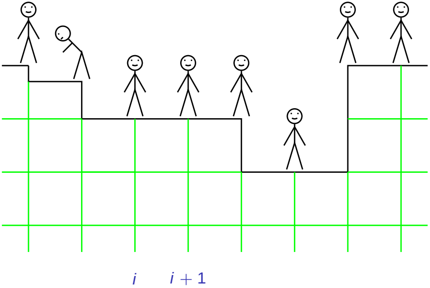

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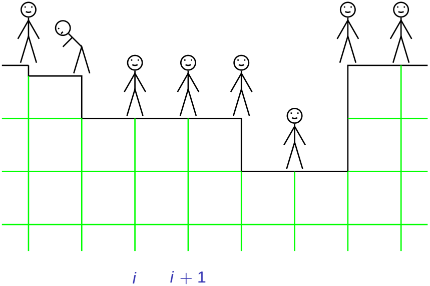

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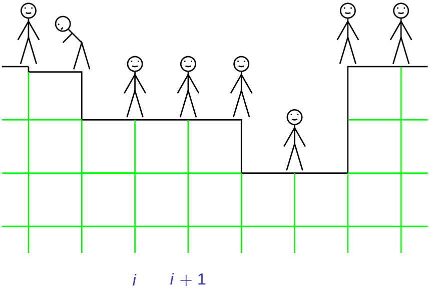

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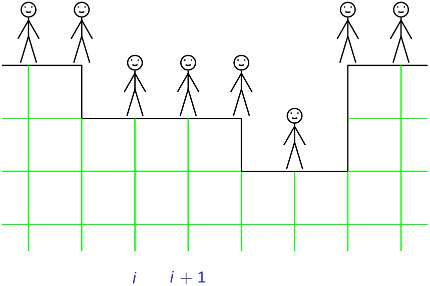

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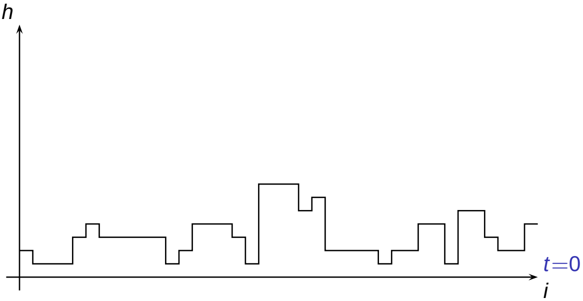

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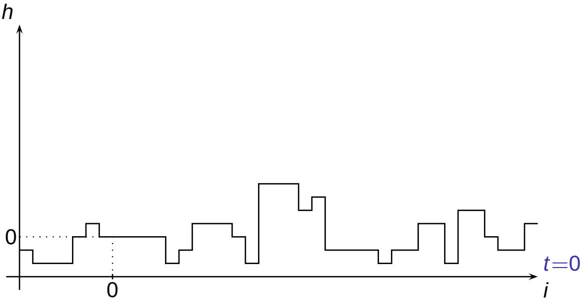

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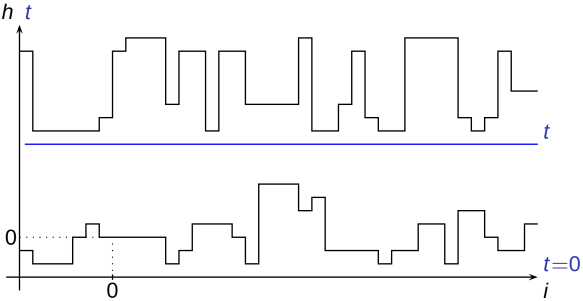

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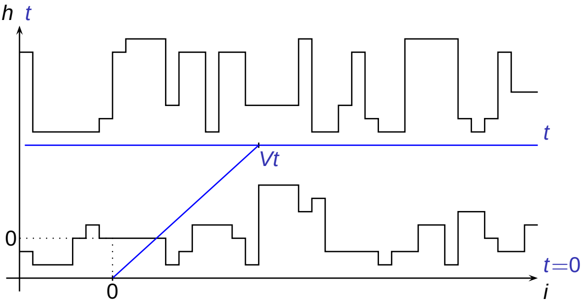

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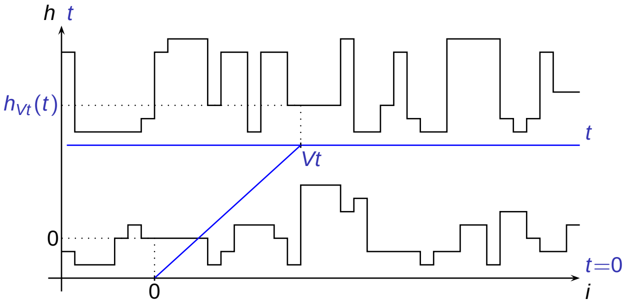

$h_{Vt}(t)$ = height as seen by a moving observer of velocity V.

= net number of particles passing the window $s \mapsto Vs$.

(Remember: particle current=change in height.)

The question

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This would be the speed of a shock of densities ϱ and λ.

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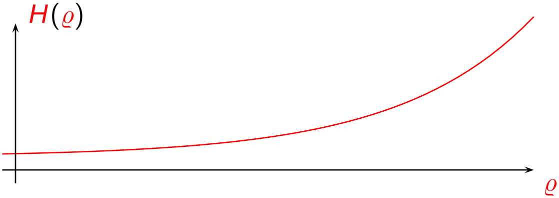

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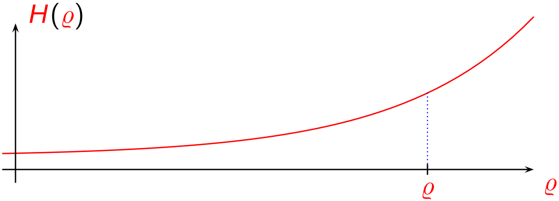

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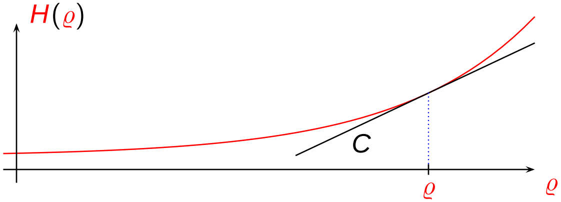

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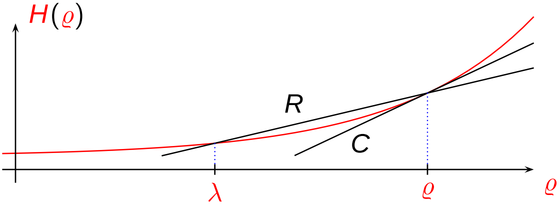

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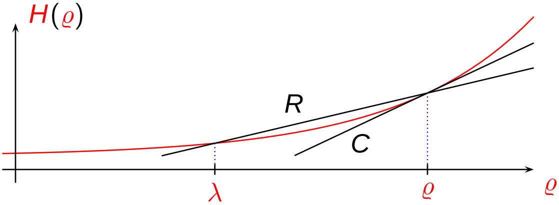

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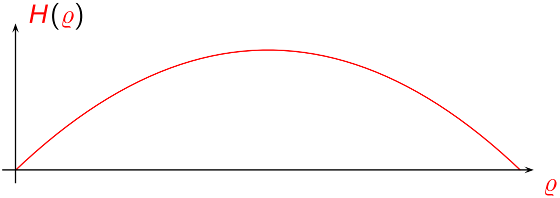

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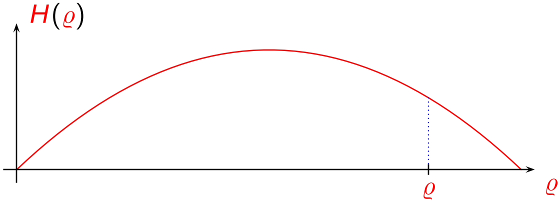

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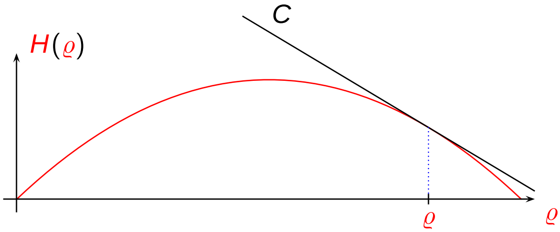

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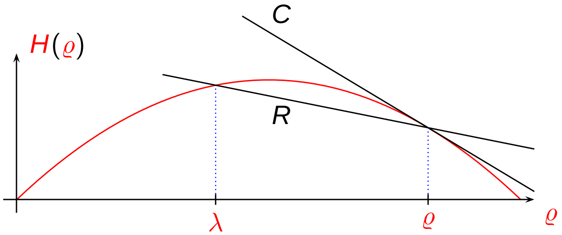

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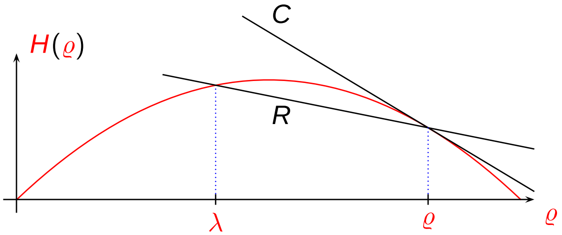

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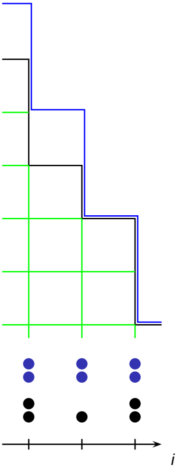

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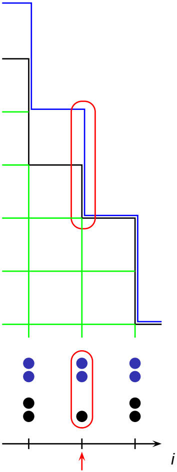

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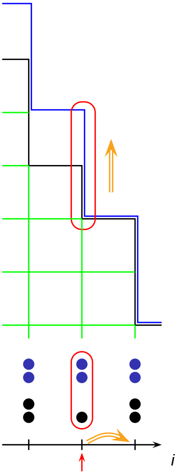

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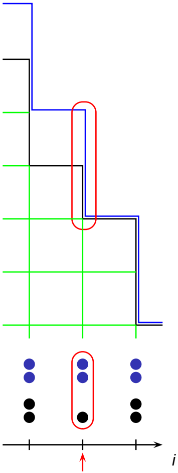

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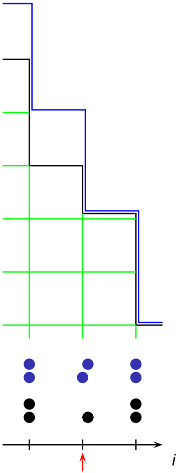

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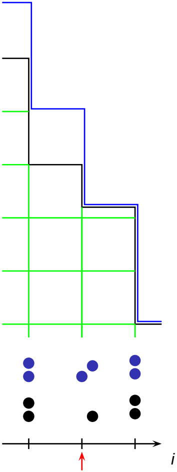

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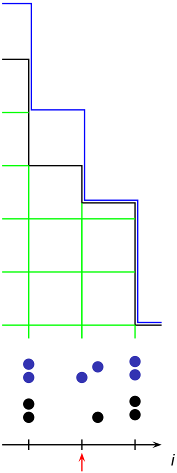

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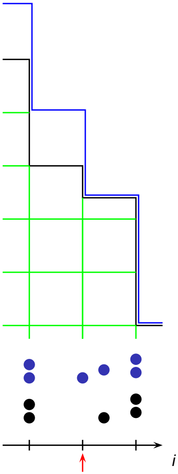

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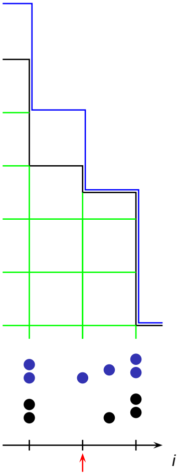

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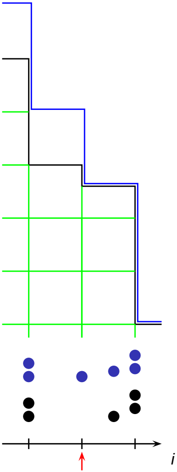

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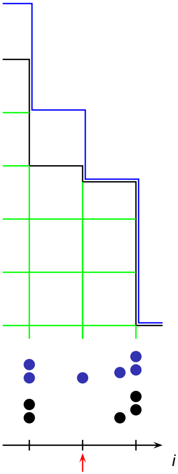

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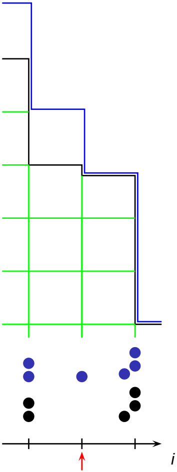

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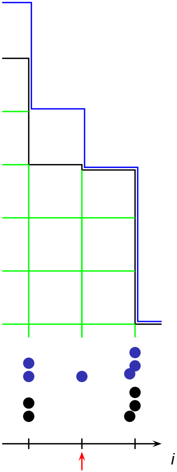

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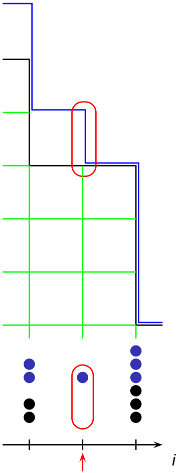

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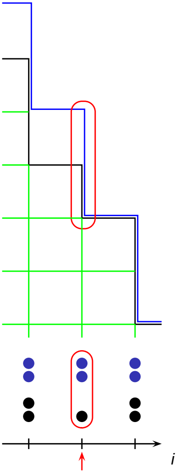

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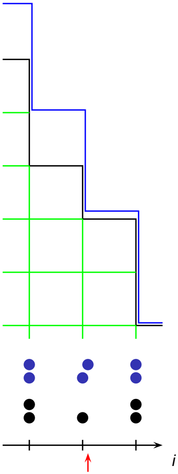

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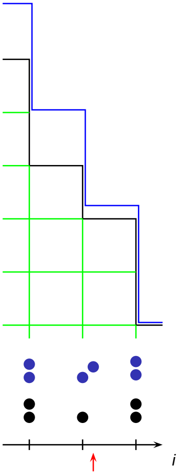

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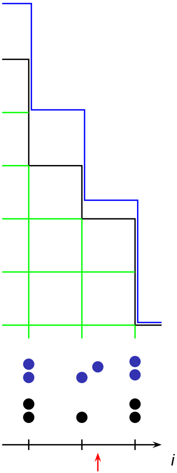

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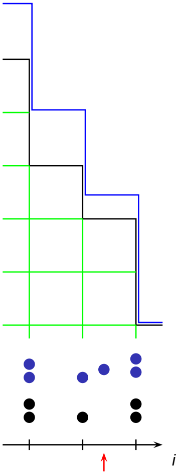

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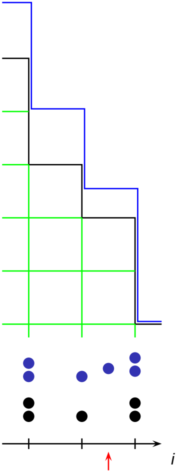

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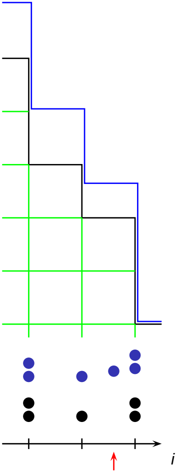

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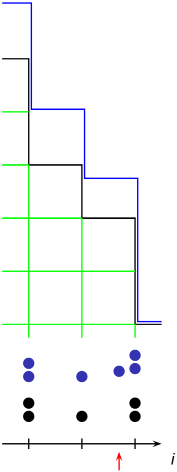

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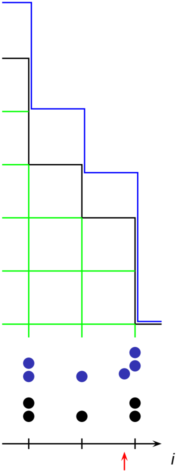

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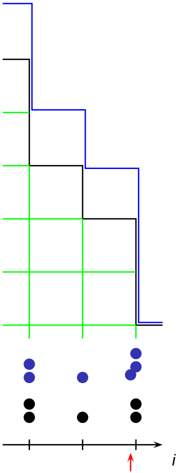

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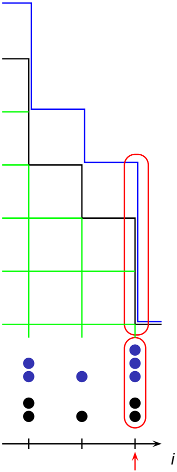

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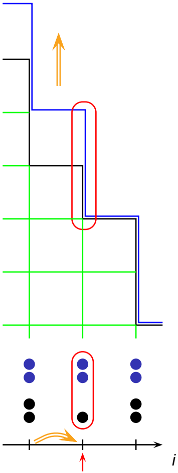

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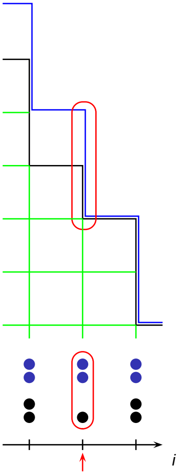

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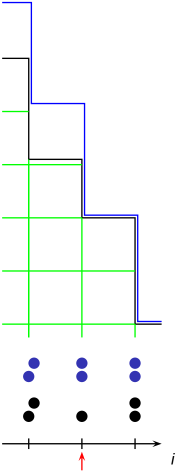

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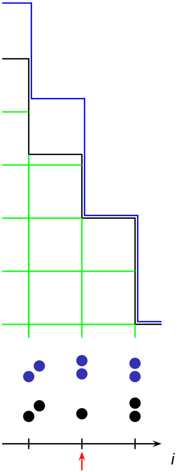

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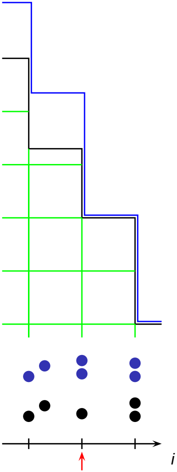

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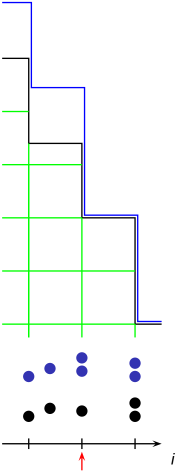

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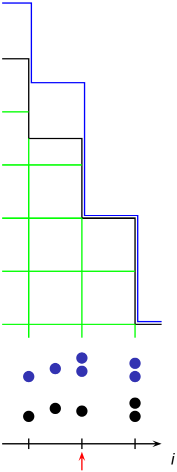

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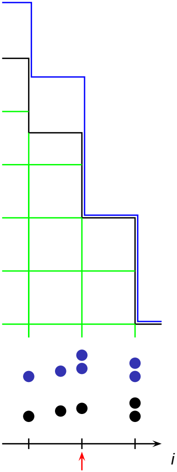

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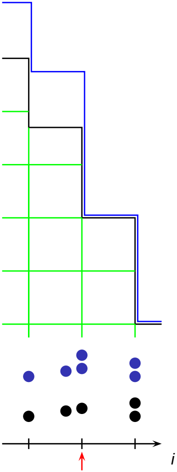

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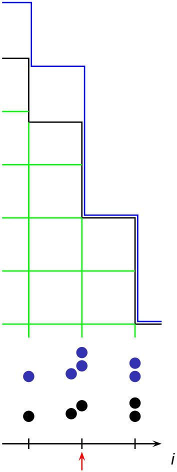


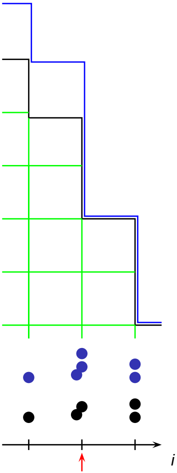
Tool: the second class particle

States ω and ω only differ at one site.

Growth on the left:

rate$\geq$rate

with rate:



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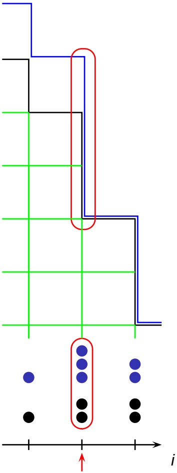

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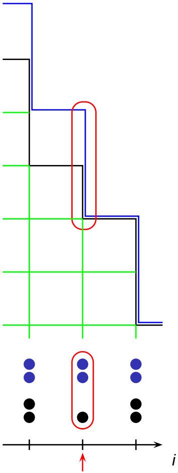

Growth on the left:

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with rate-rate:

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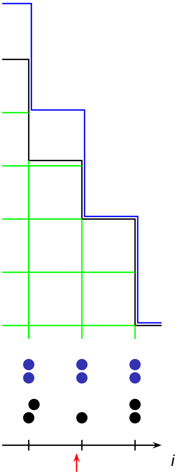

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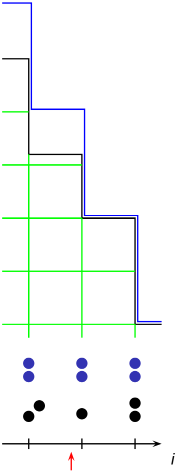

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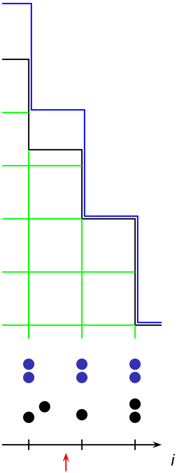

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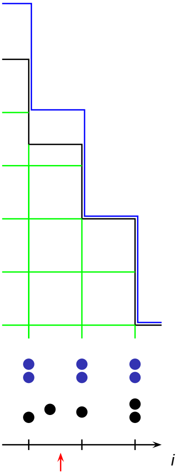

Growth on the left:

rate≥rate

with rate-rate:

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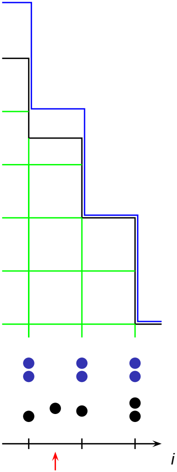

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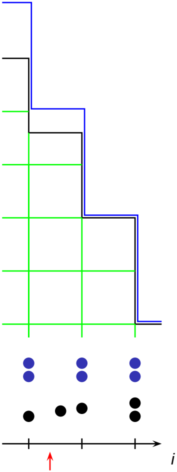

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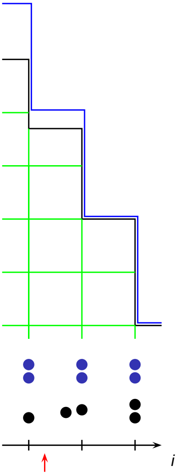

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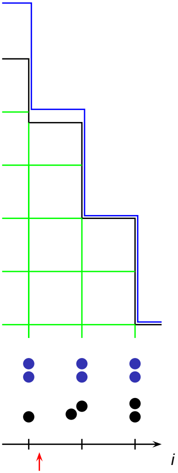

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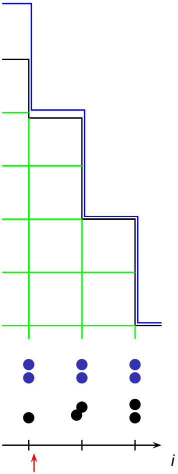

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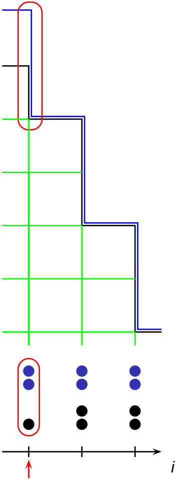

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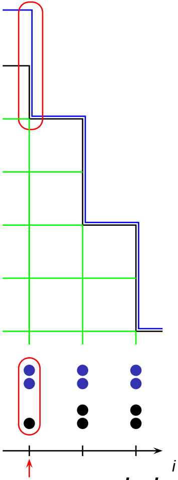

Growth on the left:

rate≥rate

with rate-rate:

A single discrepancy↑, the *second class particle*, is conserved.

Its position at time t is $Q(t)$.

Tool: the second class particle

Theorem (B. - Seppäläinen; also ideas from B. Tóth, H. Spohn and M. Prähofer)

Started from (almost) equilibrium,

$$\mathbf{E}(Q(t)) = C \cdot t$$

in the whole family of processes.

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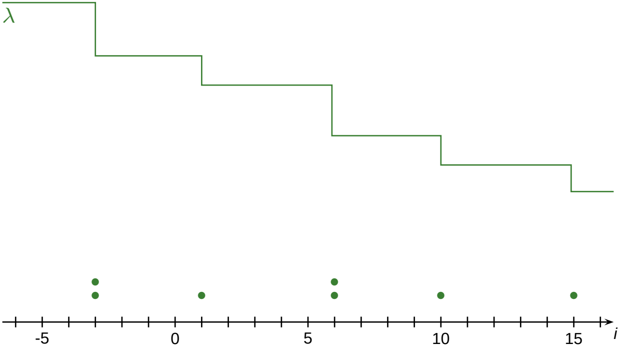
C is the characteristic speed.

The second class particle follows the characteristics, people have known this for a long time.

Many second class particles



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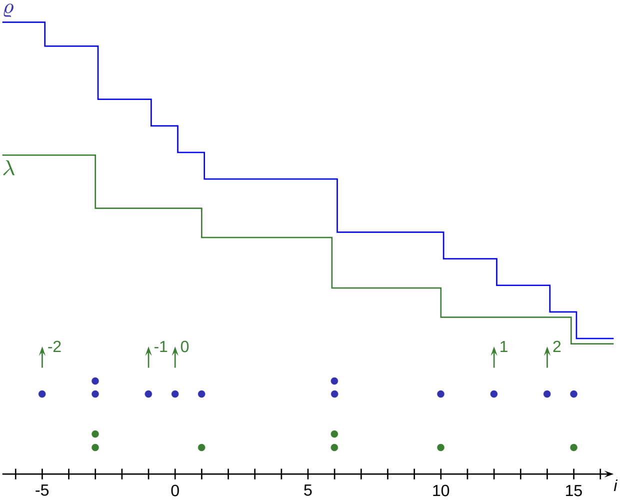

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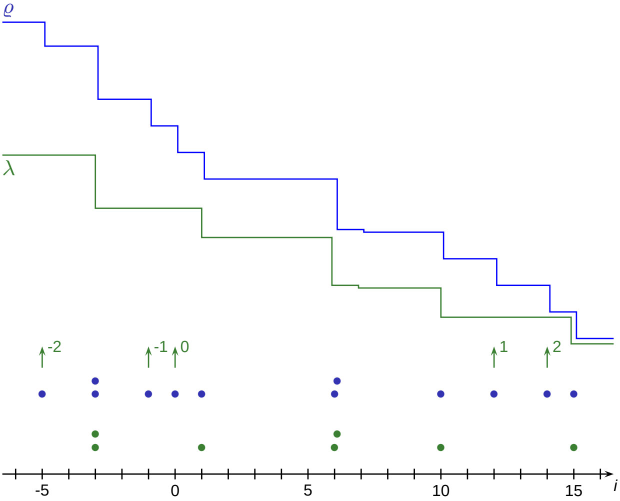

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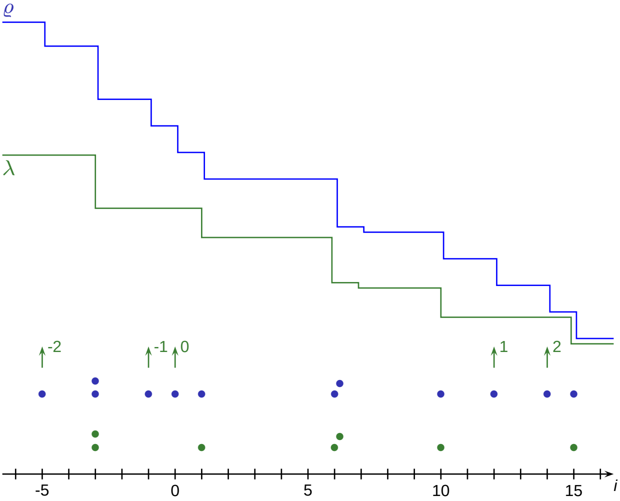

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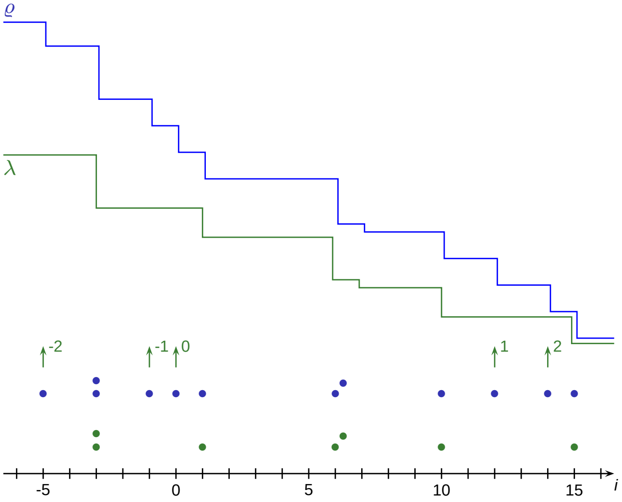

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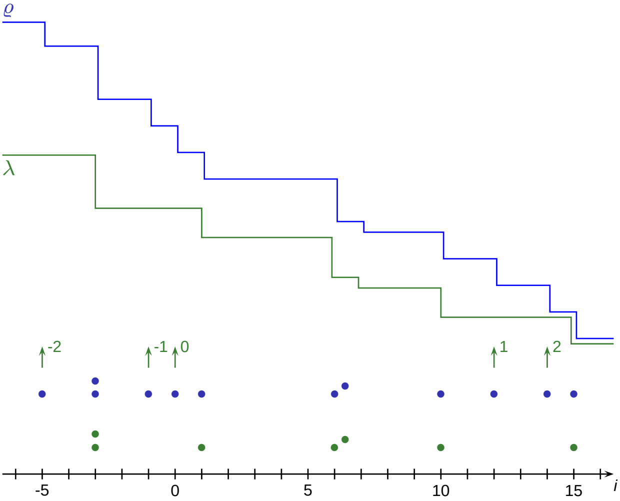

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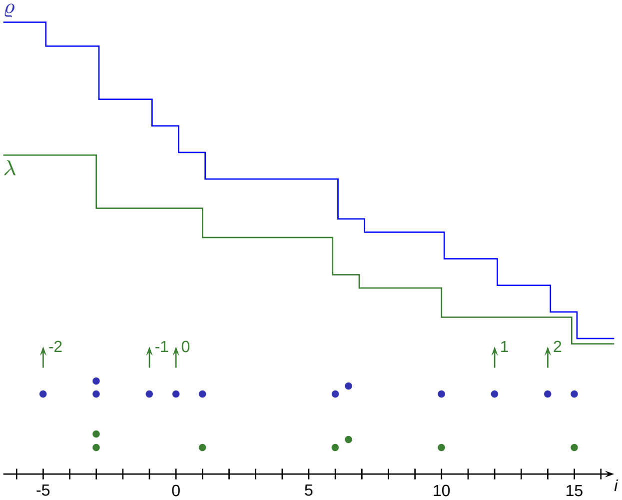

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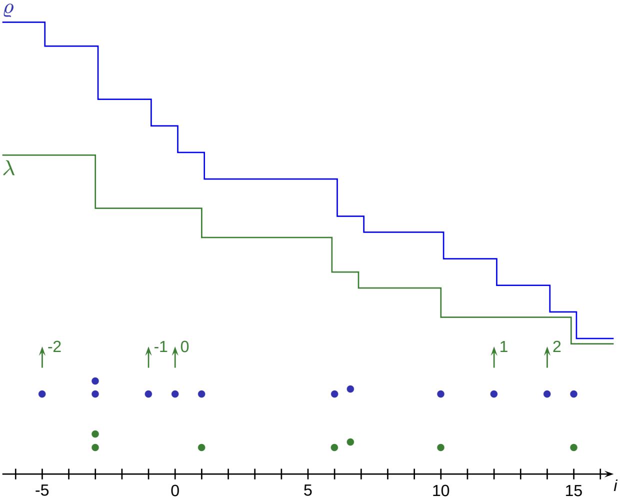

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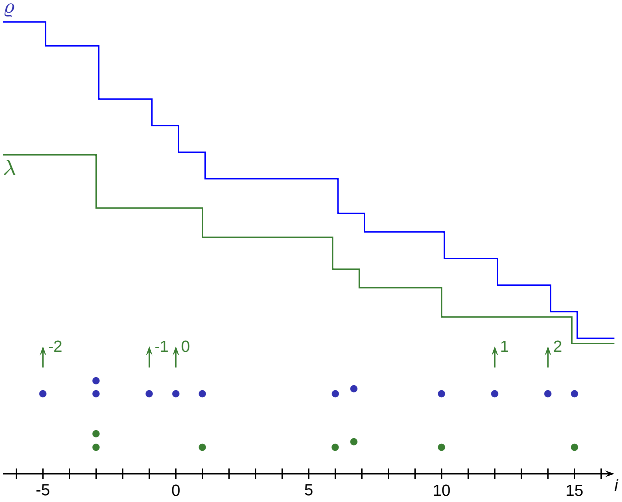

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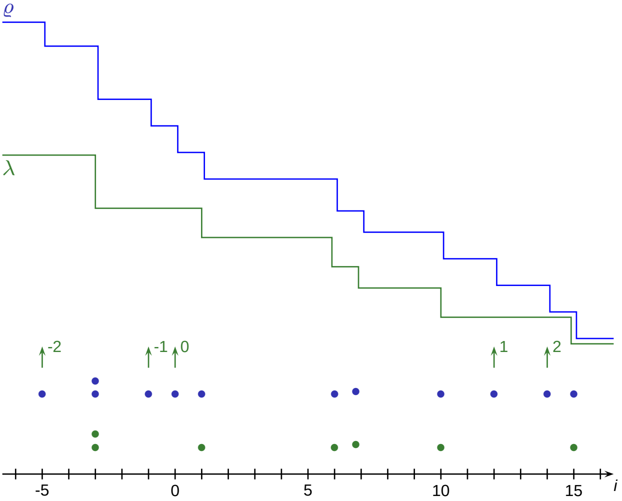

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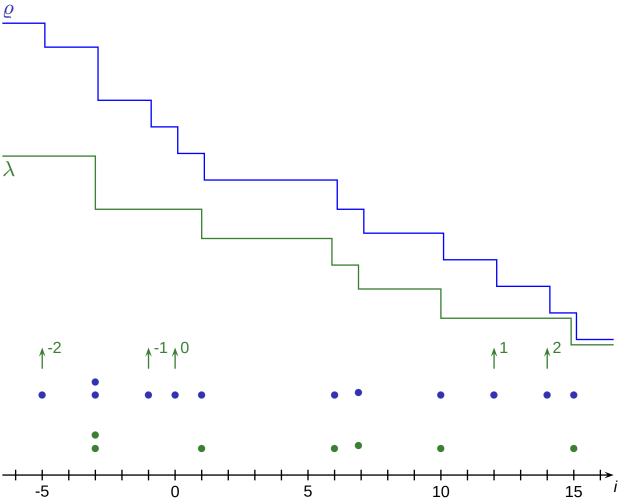

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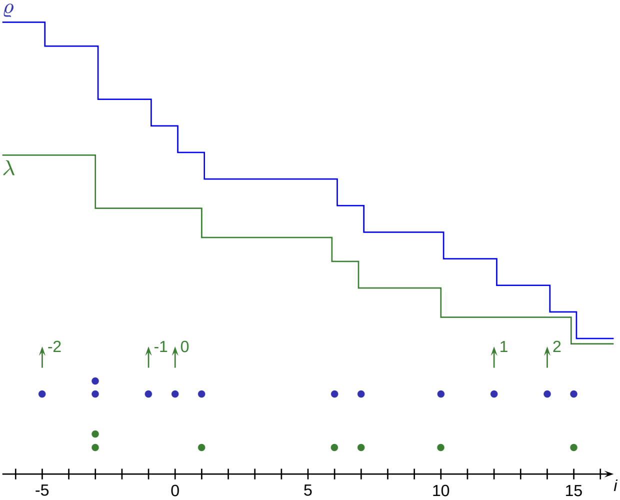

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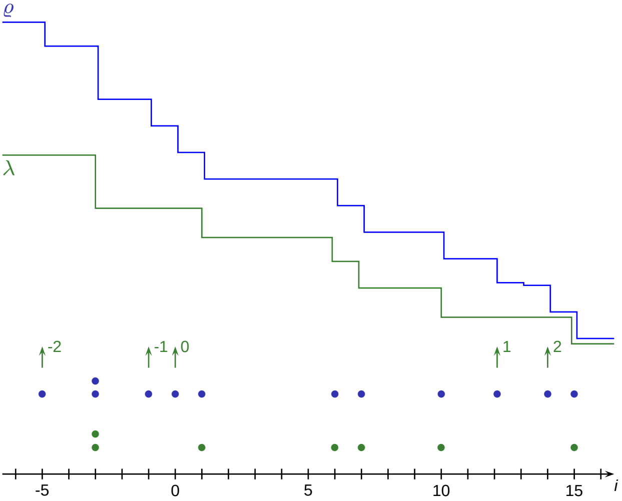

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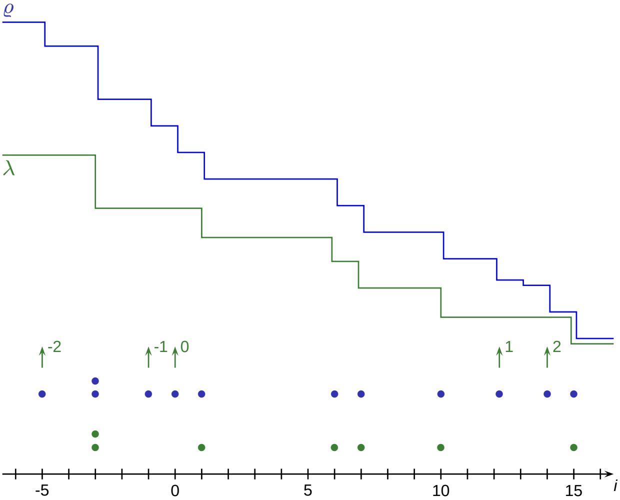

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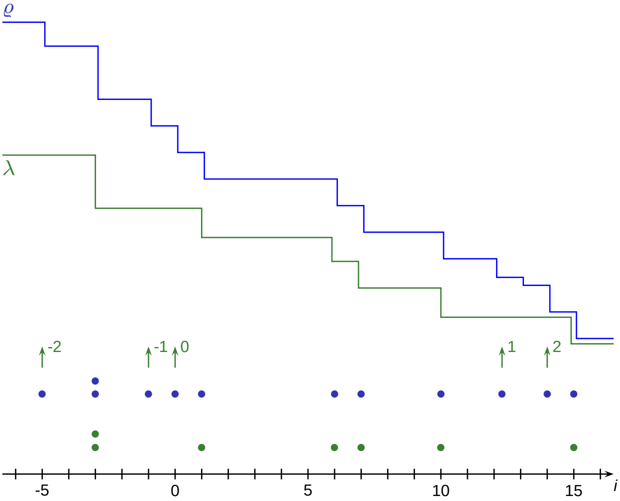

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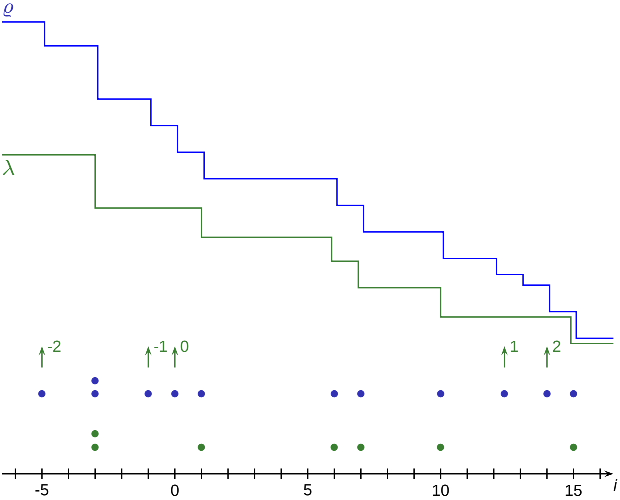

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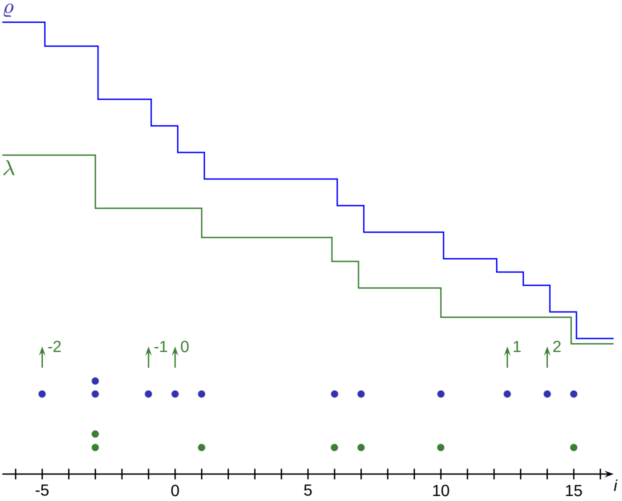

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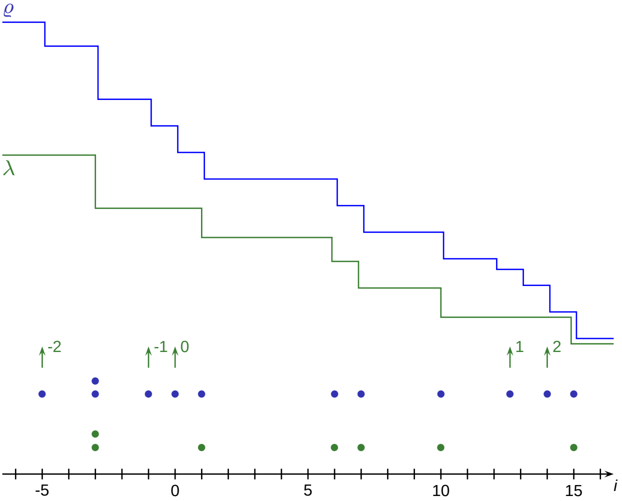

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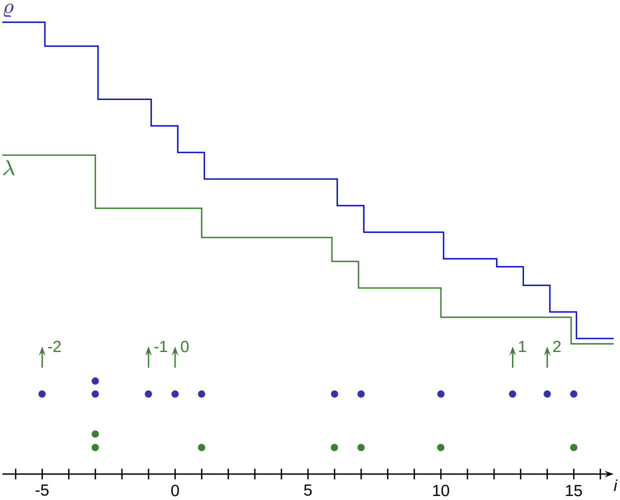

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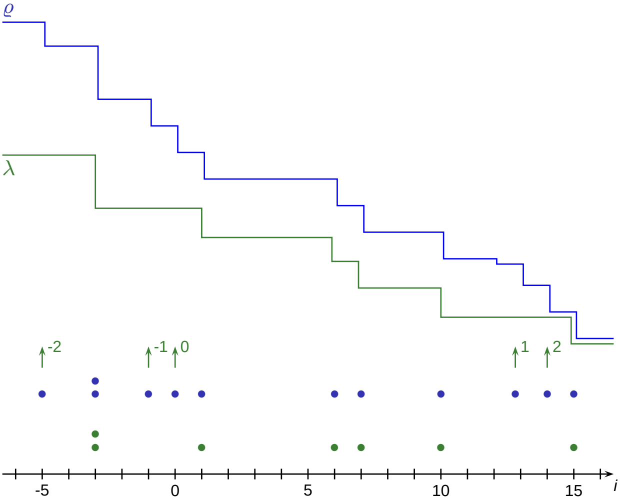

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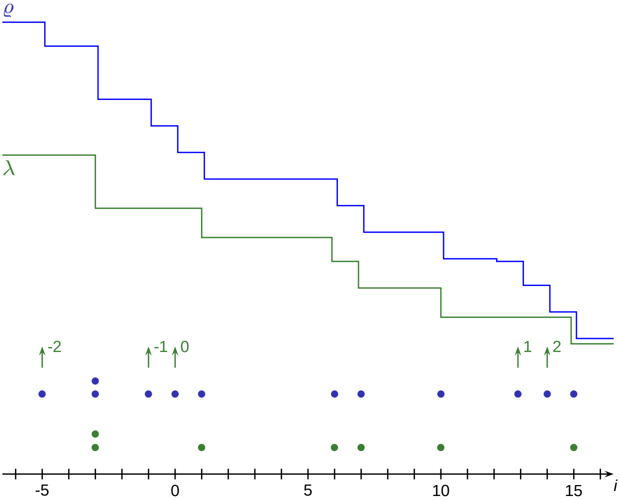

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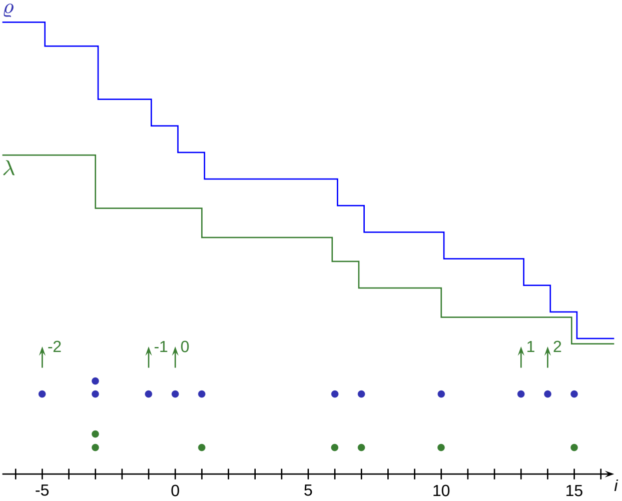

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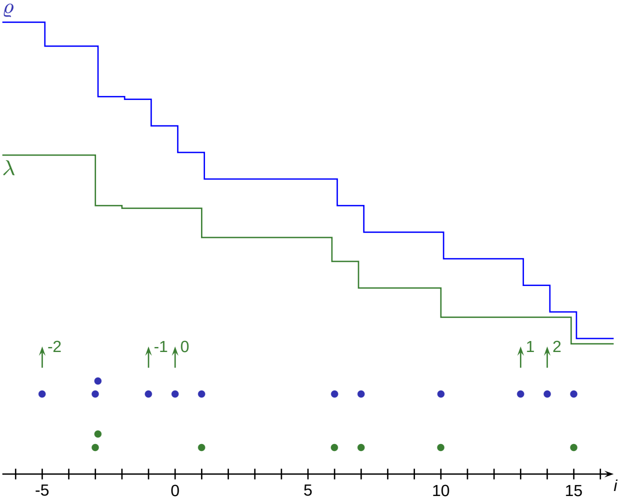

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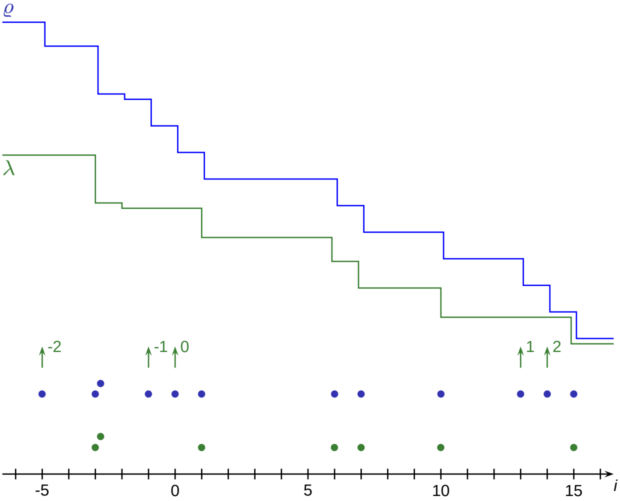

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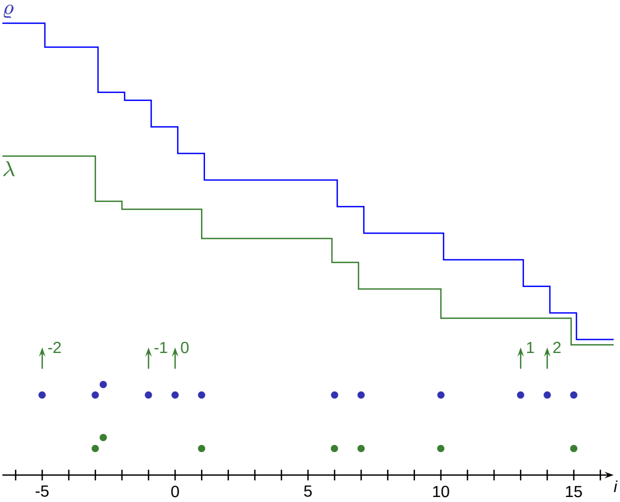

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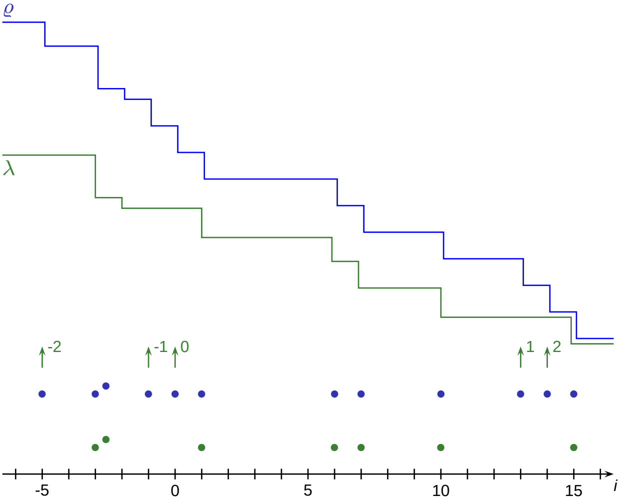

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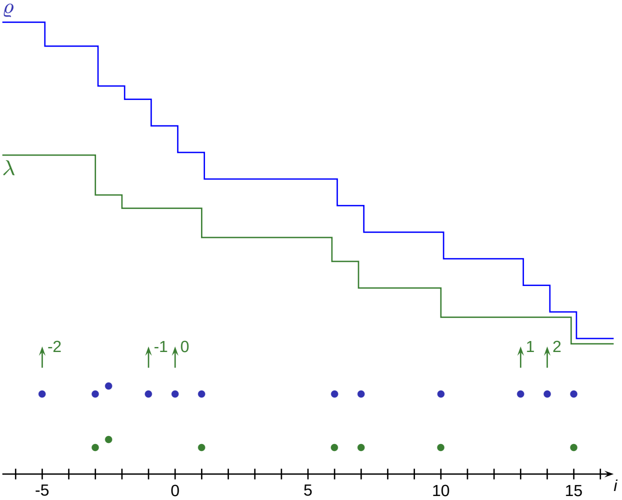

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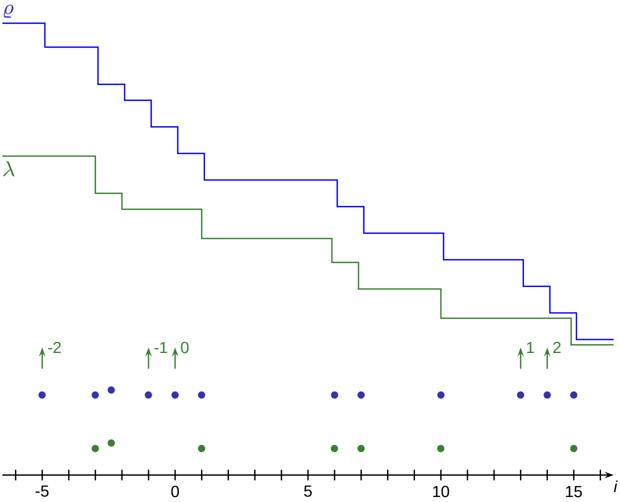

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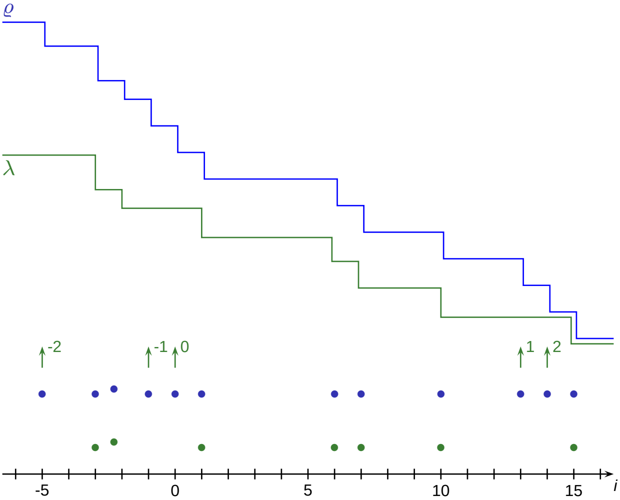

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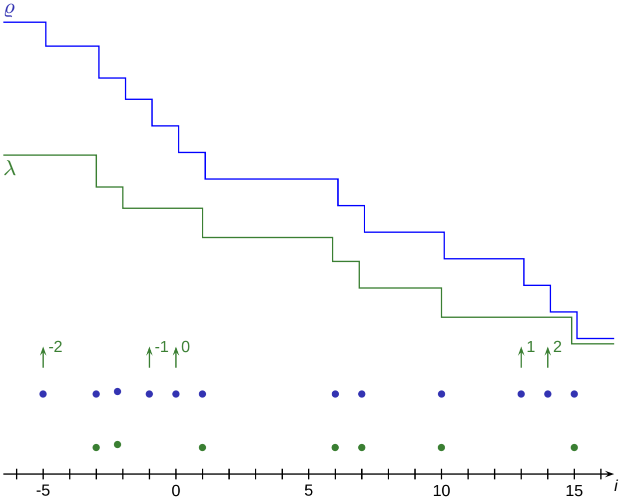

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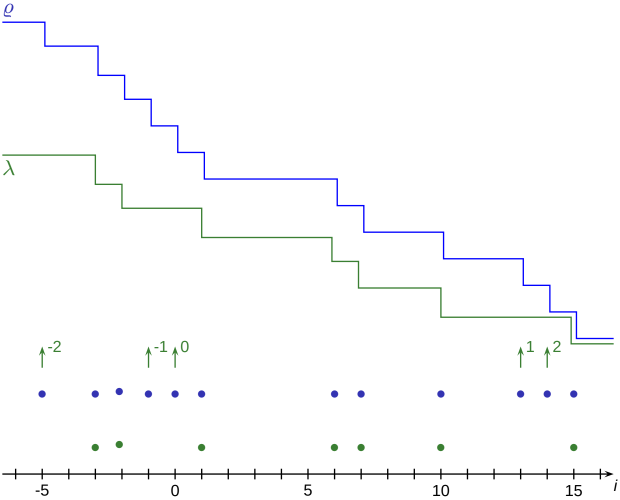

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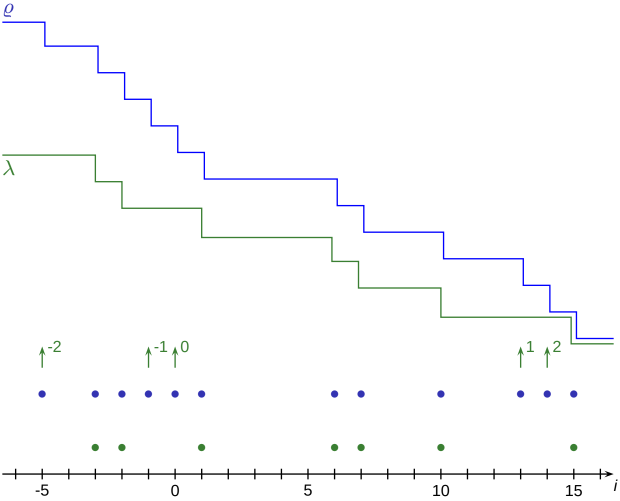

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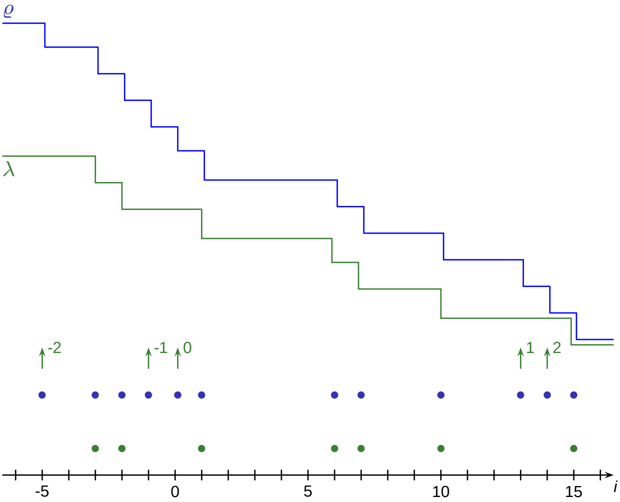

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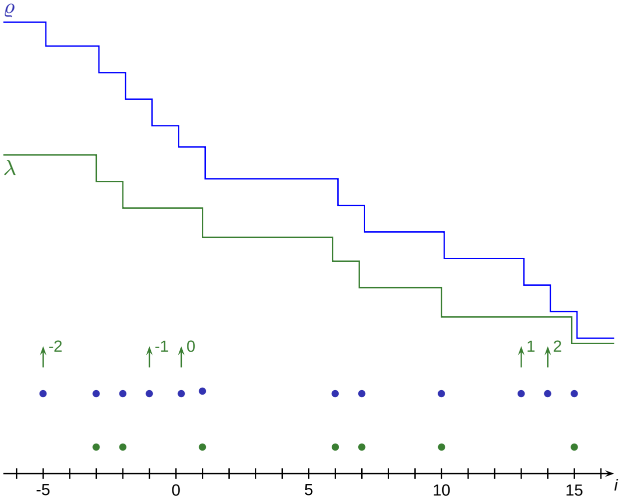

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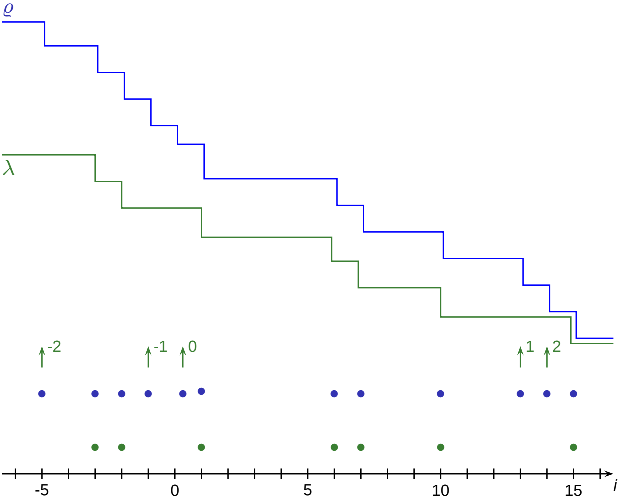

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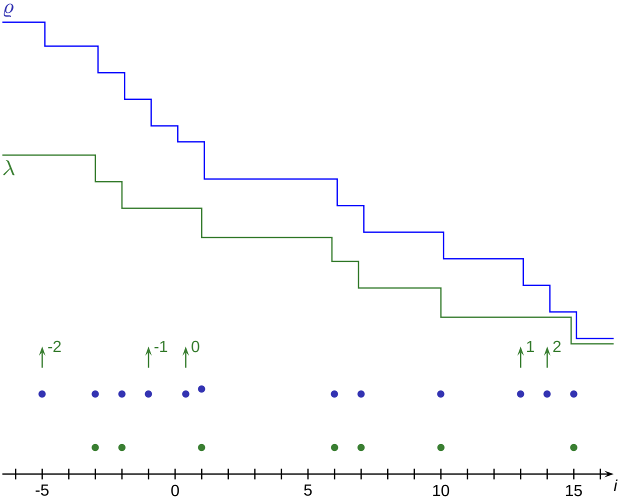

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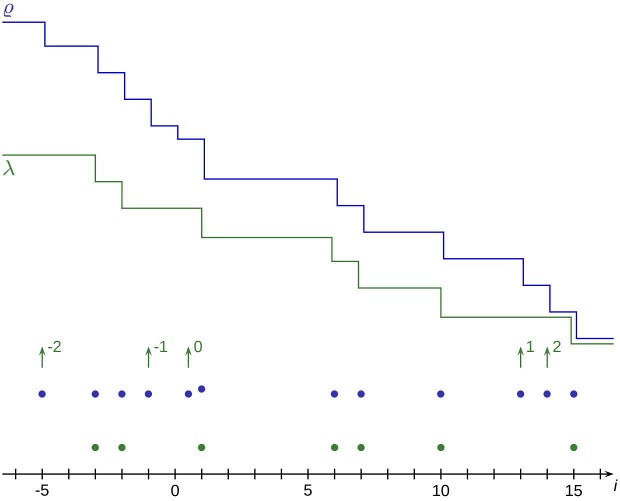

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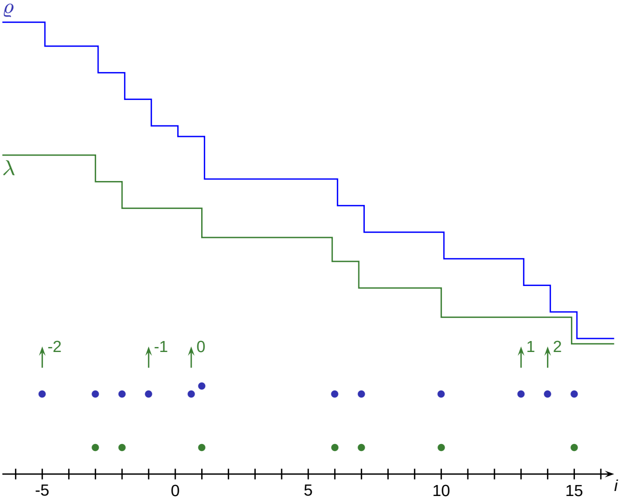

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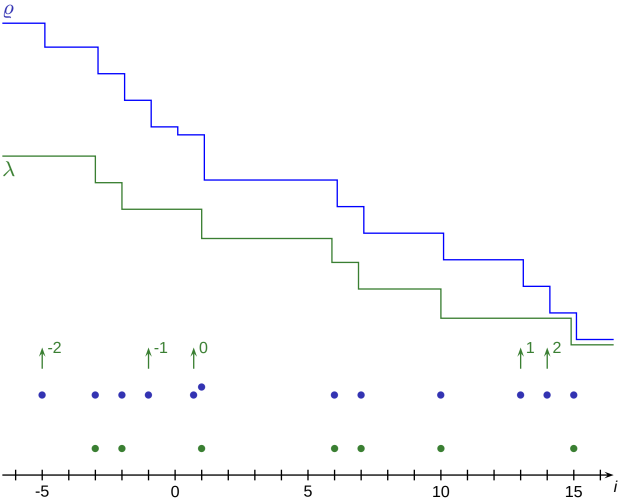

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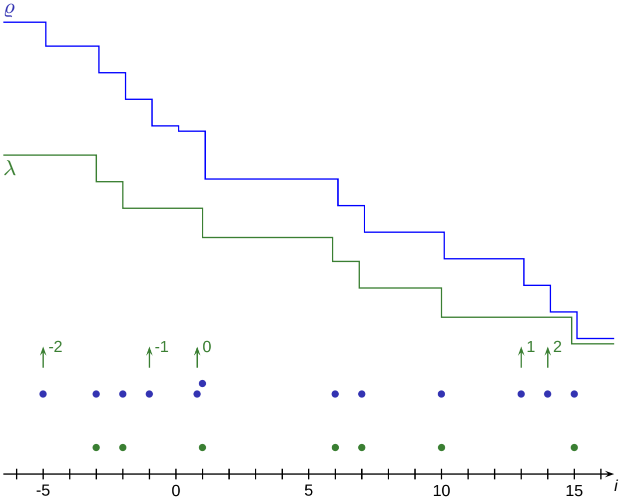

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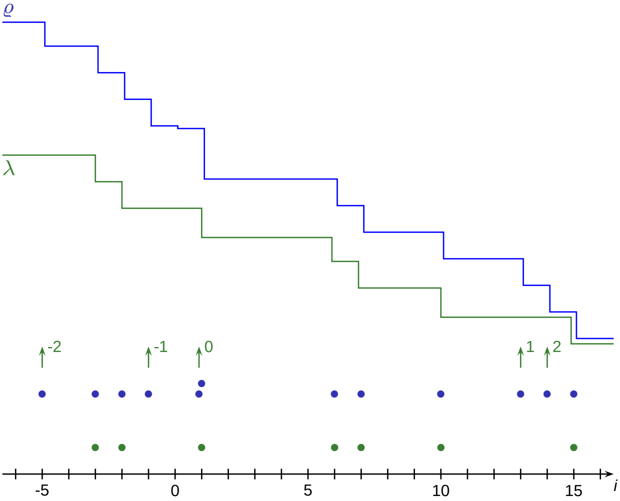

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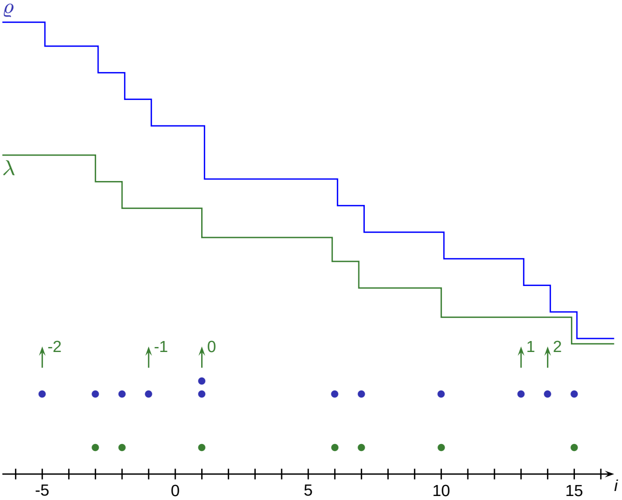

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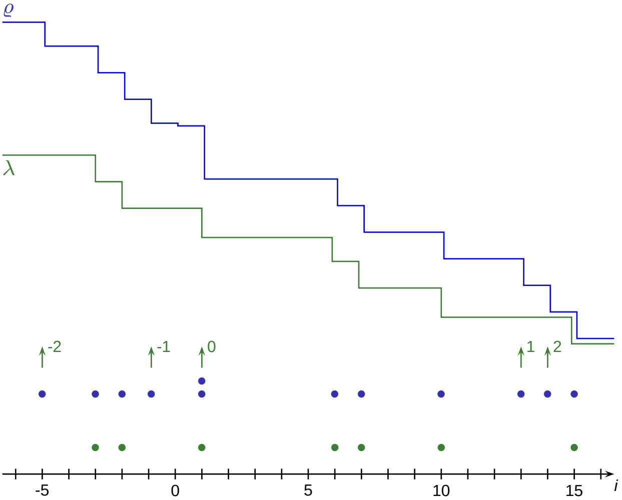

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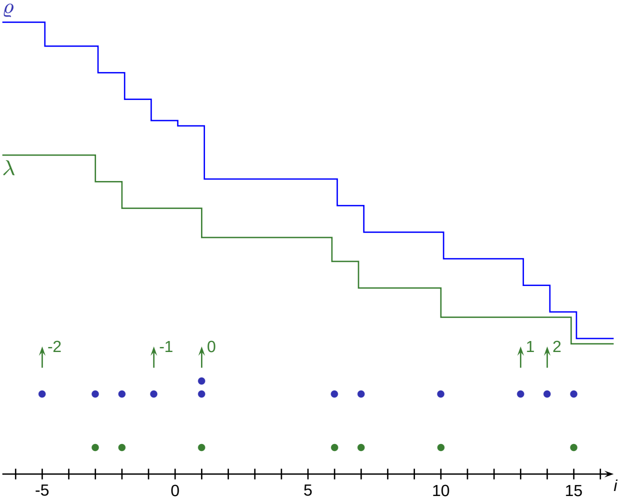

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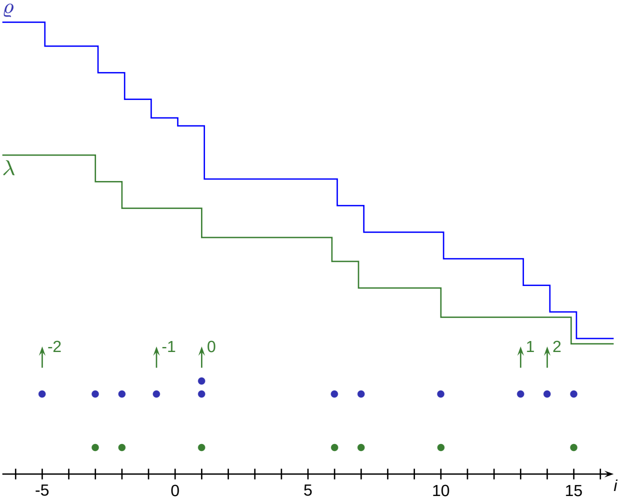

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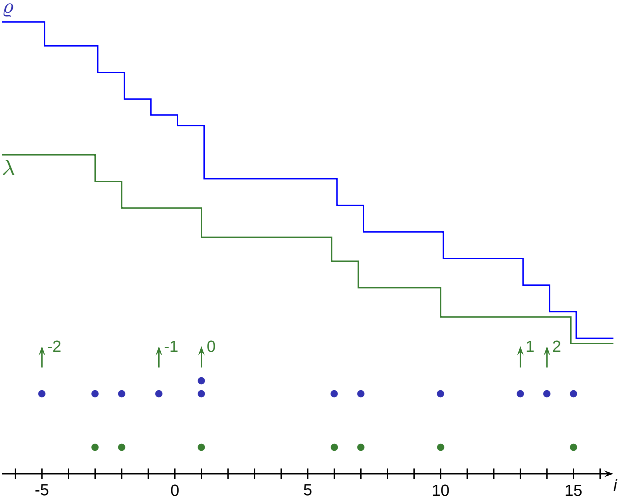

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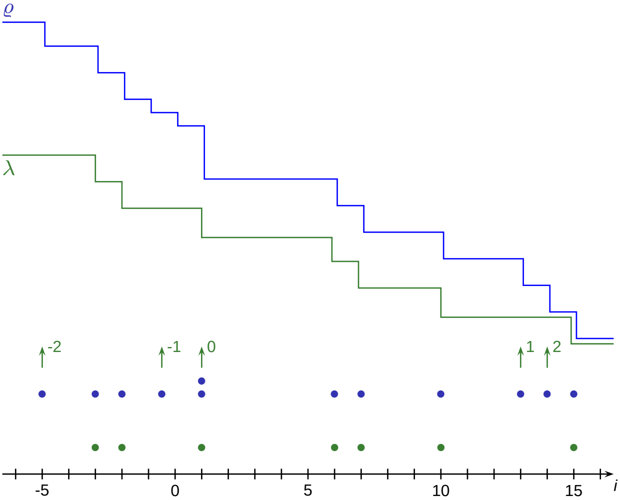

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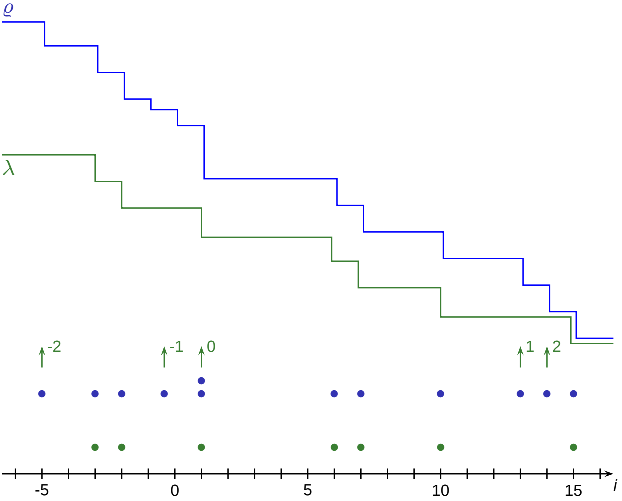

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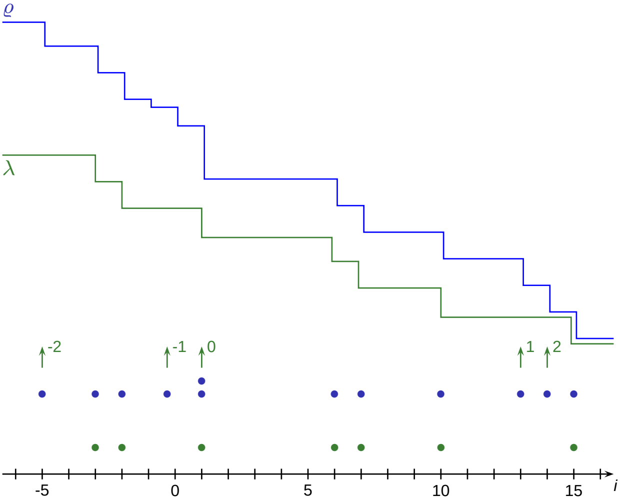

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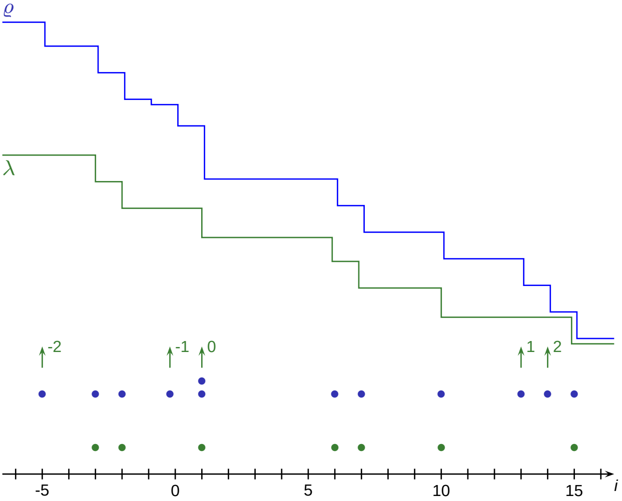
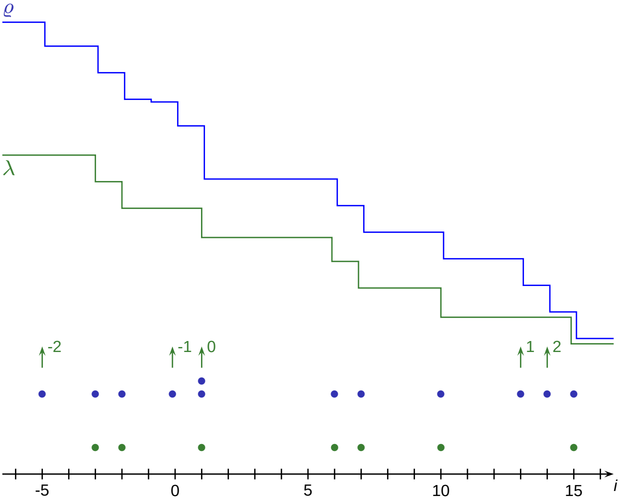


Many second class particles



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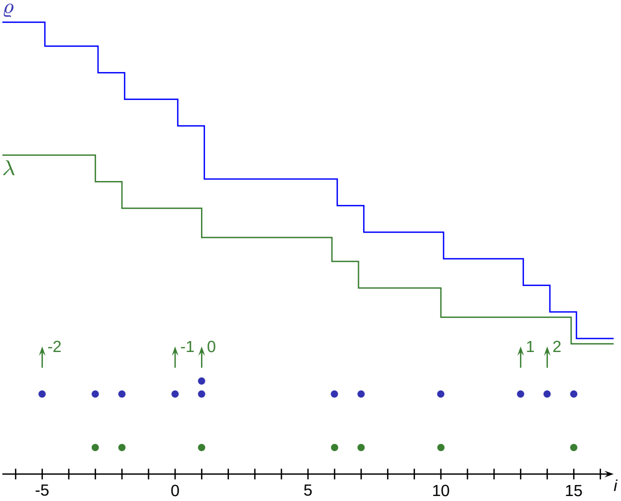

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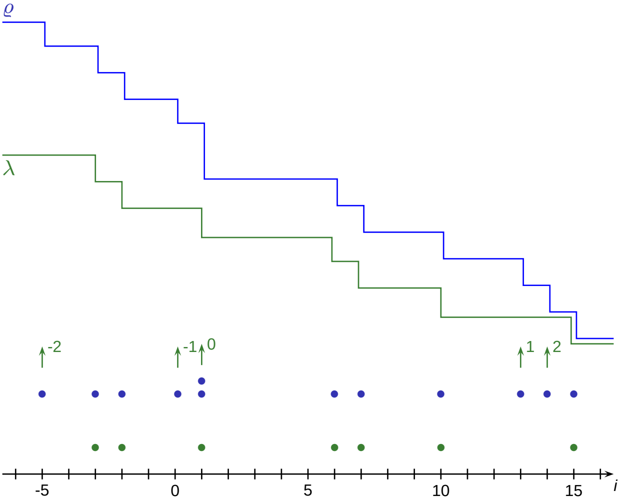

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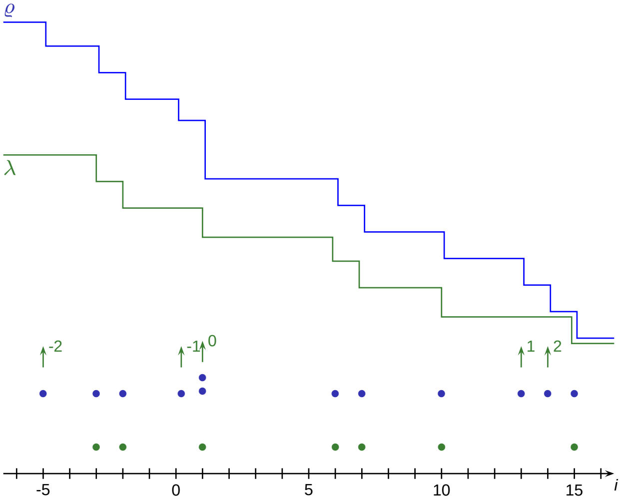

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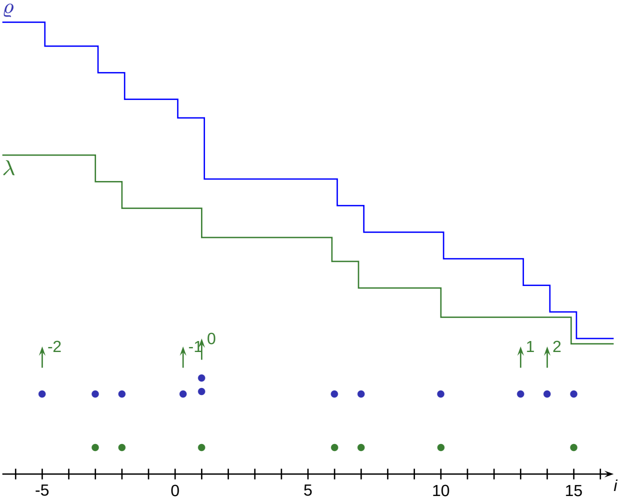

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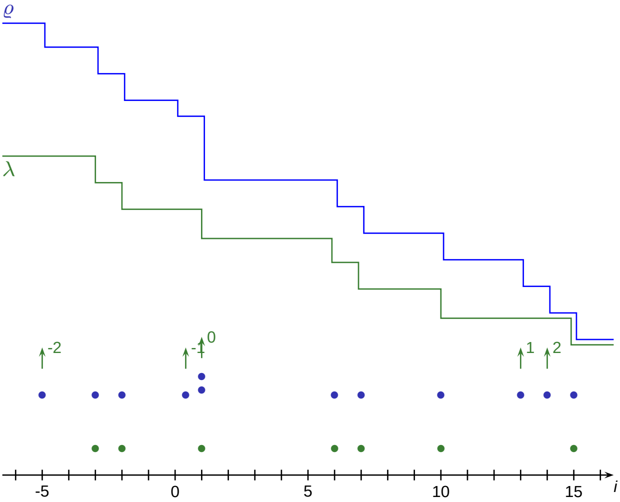

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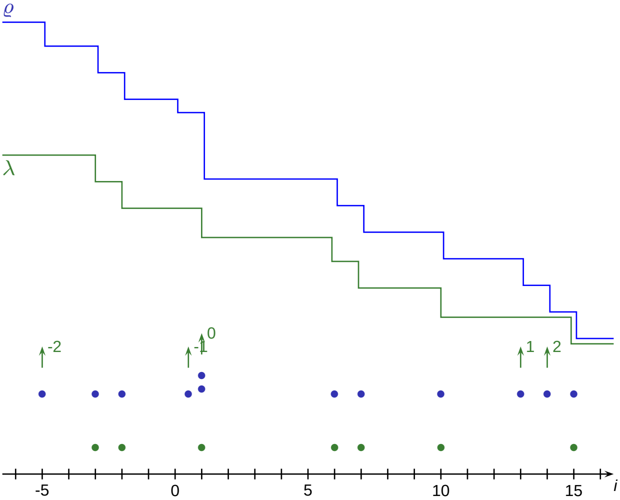

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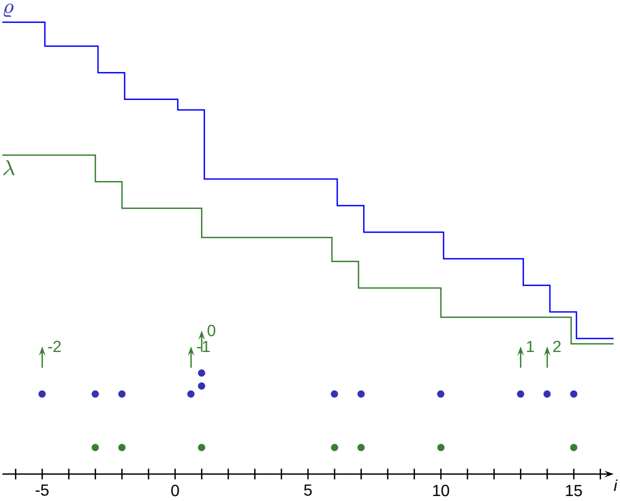

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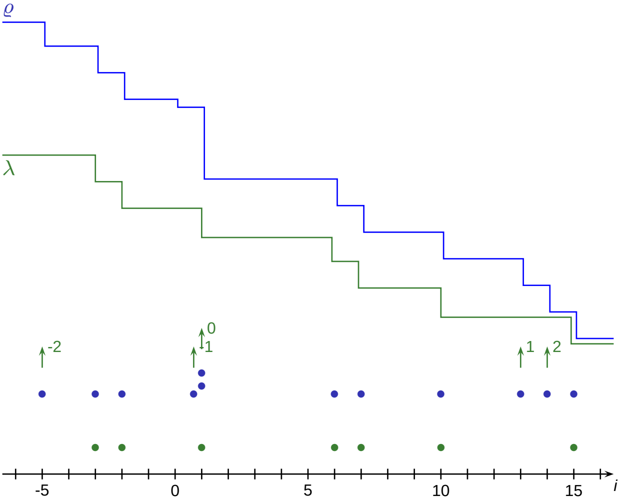

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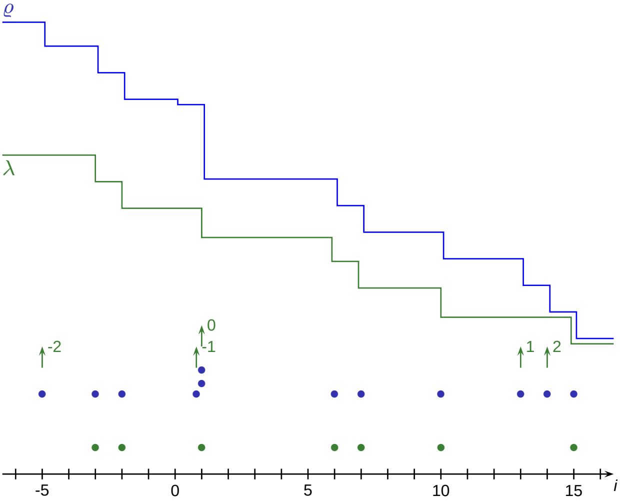

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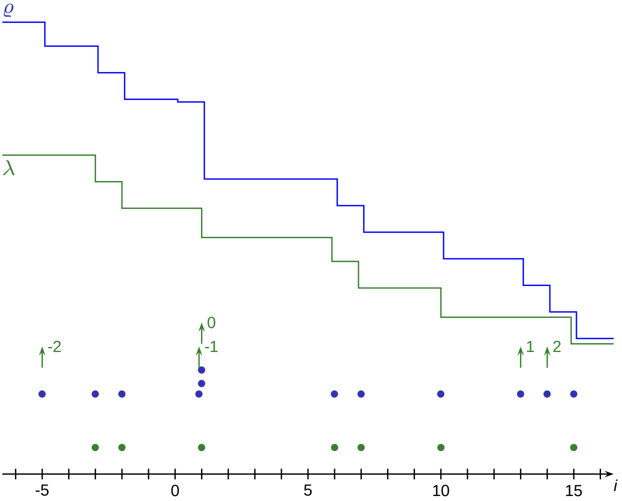

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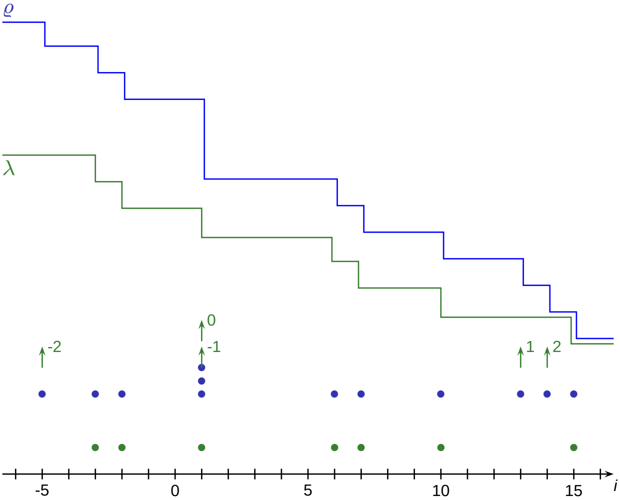

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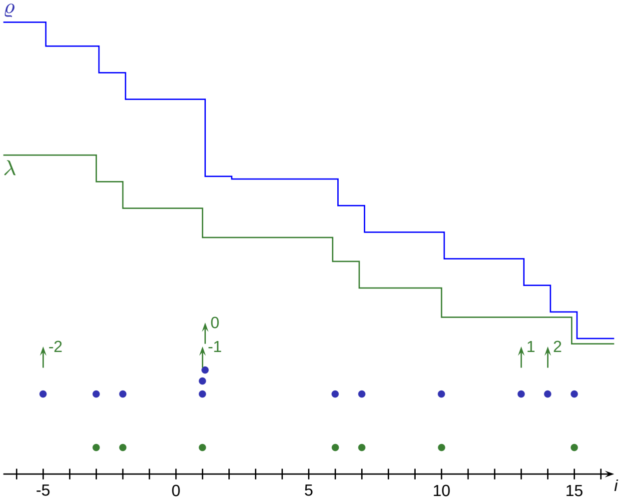

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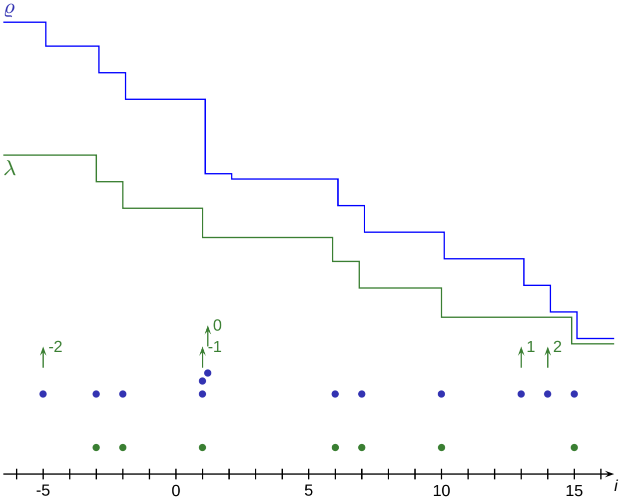

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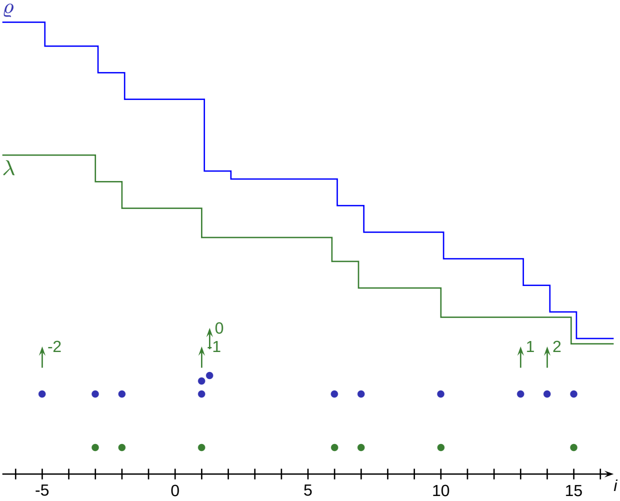

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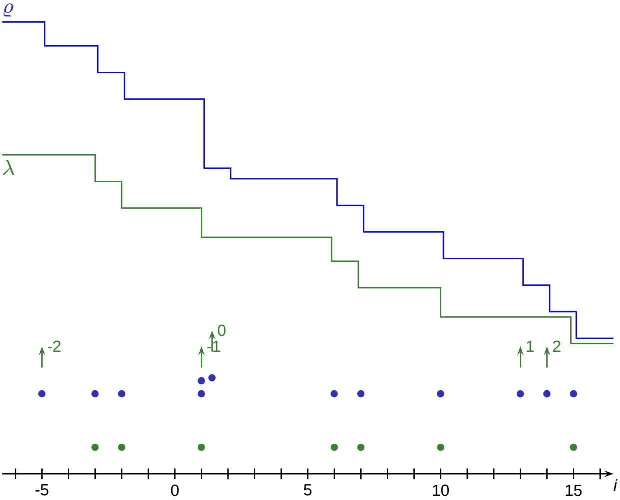

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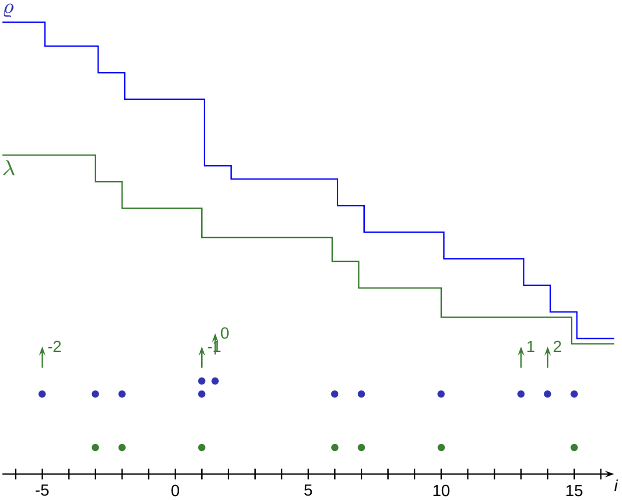

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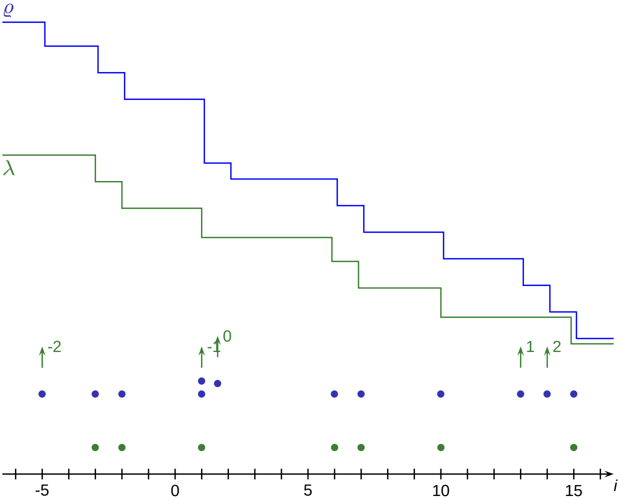

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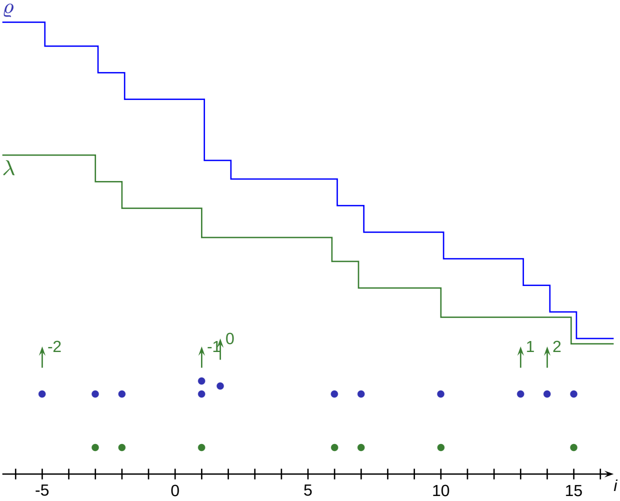

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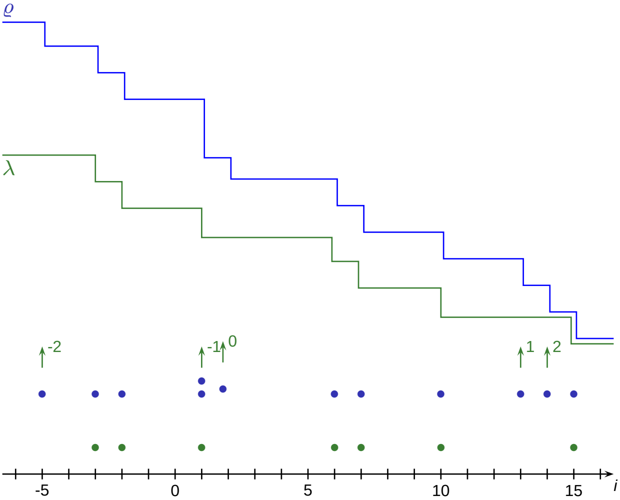

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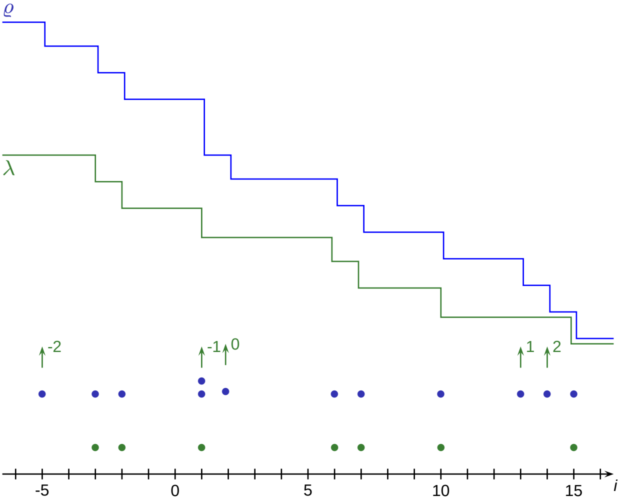

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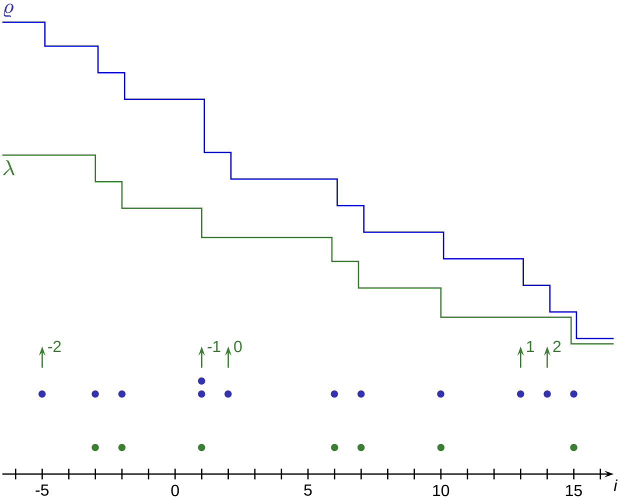

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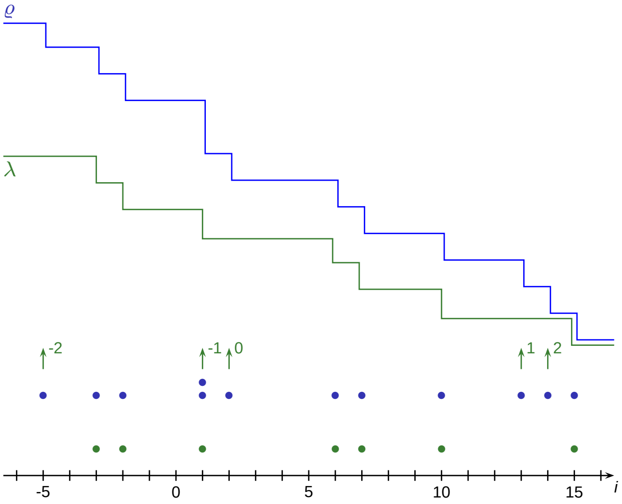
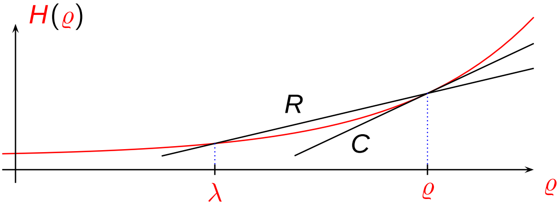


Picture:

The position $X(t)$ of $\uparrow^0$ follows the Rankine-Hugoniot speed R.

Characteristics (very briefly)

Convex flux (some cases of AZRP, ABLP):



Recall $$C = H'(\varrho) > R = \frac{H(\varrho) - H(\lambda)}{\varrho - \lambda}$$

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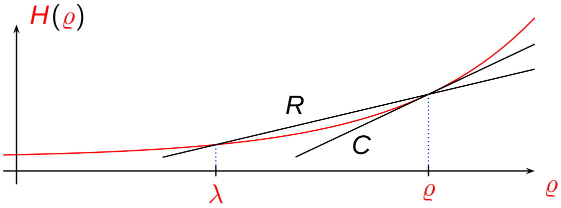

Recall $C = H'(\varrho) > R = \dfrac{H(\varrho) - H(\lambda)}{\varrho - \lambda}$

Do we have $Q(t) \overset{?}{\geq} X(t)$

Characteristics (very briefly)

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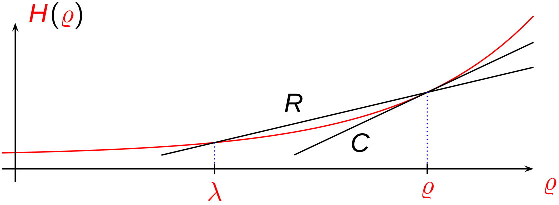
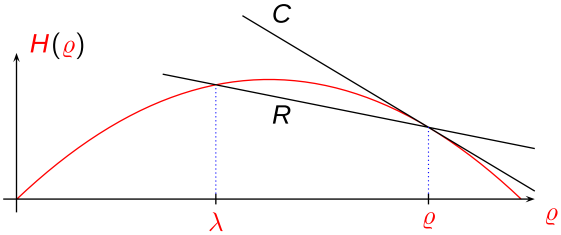


Recall $C = H'(\varrho) > R = \dfrac{H(\varrho) - H(\lambda)}{\varrho - \lambda}$

Do we have $Q(t) \overset{?}{\geq} X(t) -$ tight error

Characteristics (very briefly)

Concave flux (ASEP, AZRP):



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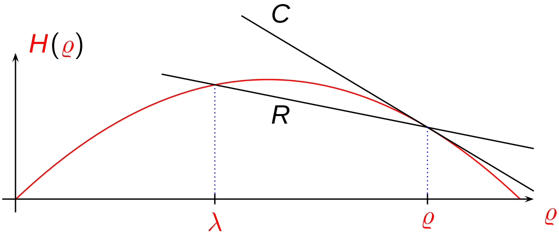

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Do we have $Q(t) \overset{?}{\leq} X(t)$

Characteristics (very briefly)

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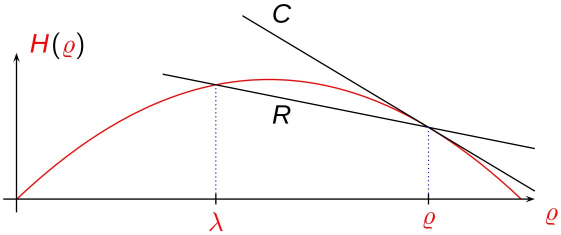
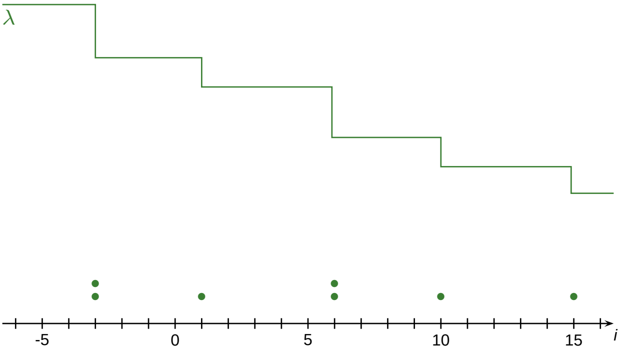


$$C = H'(\varrho) < R = \frac{H(\varrho) - H(\lambda)}{\varrho - \lambda}$$

Do we have $Q(t) \overset{?}{\leq} X(t) +$ tight error

Many second class particles



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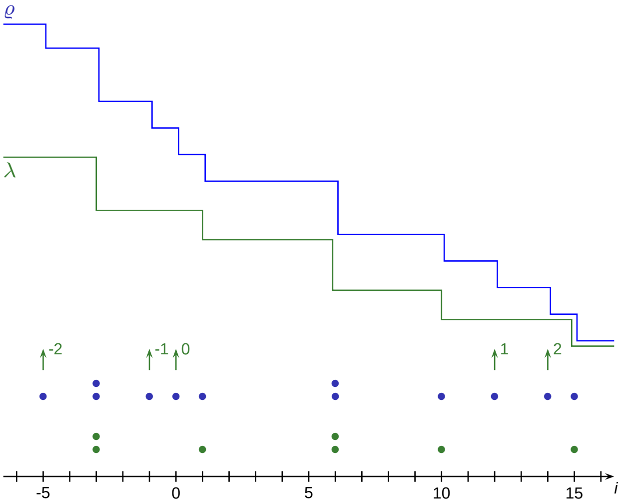
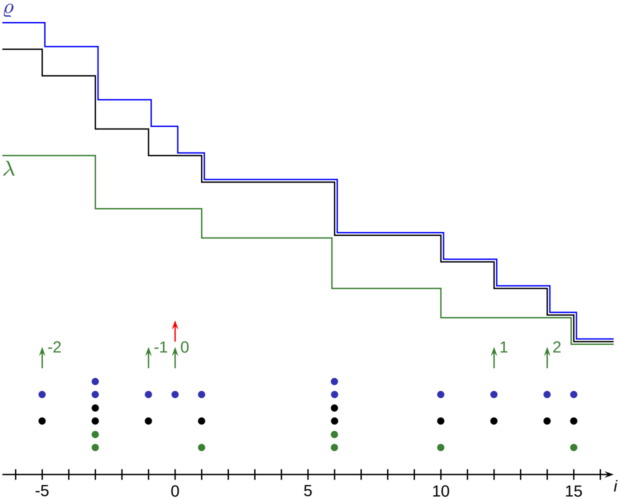
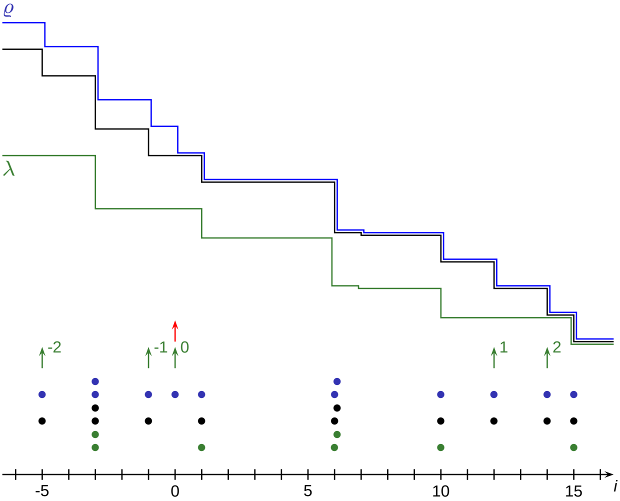


Many second class particles plus one



Couple three processes, and $X(t)$ to $Q(t)$.

Many second class particles plus one



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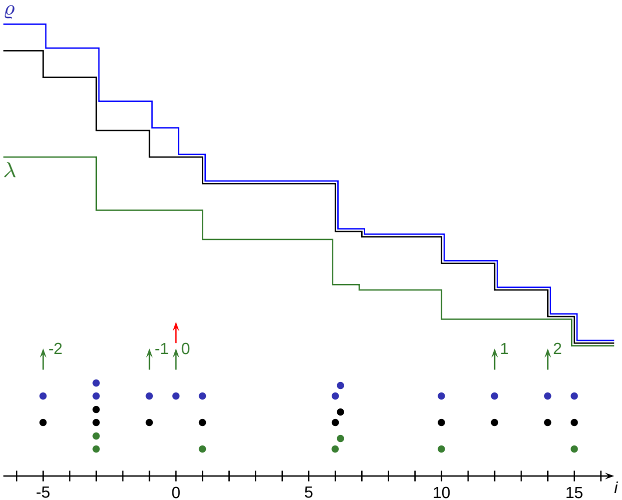

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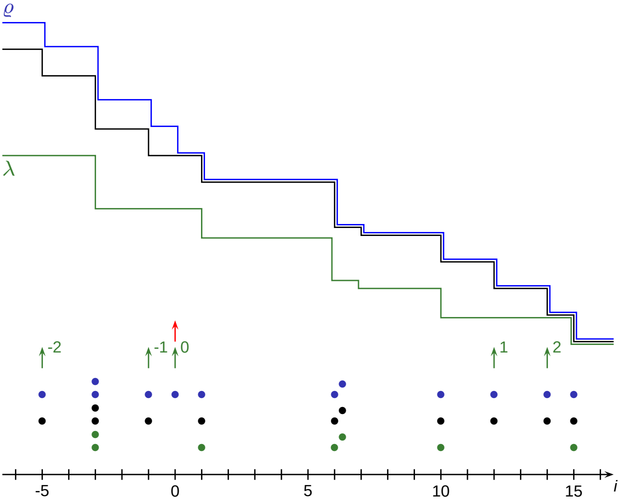

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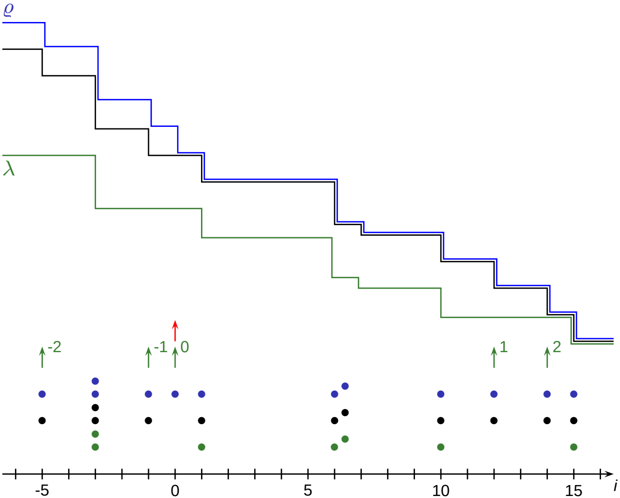

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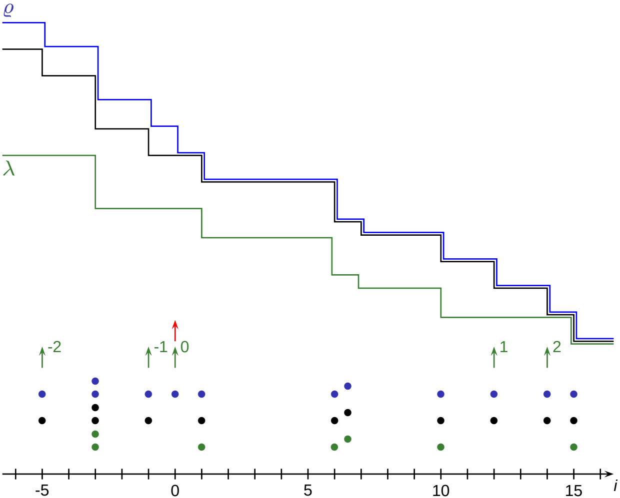

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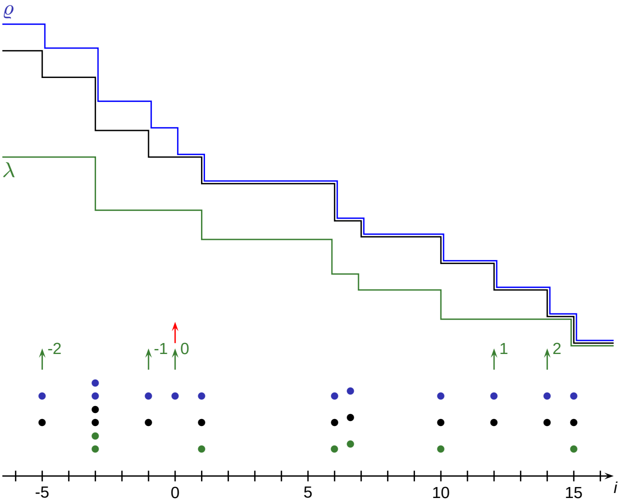

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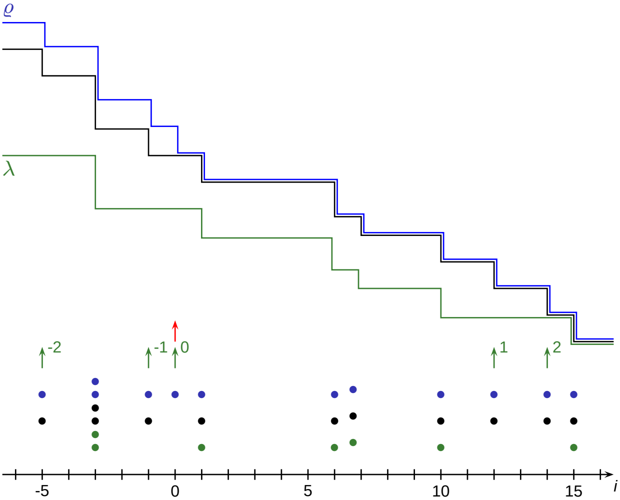

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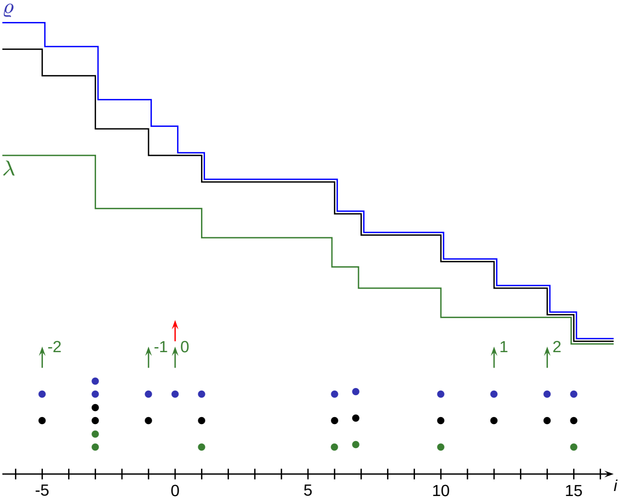

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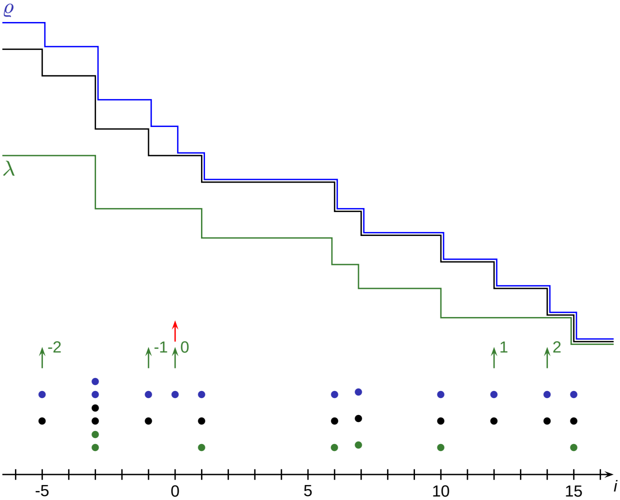

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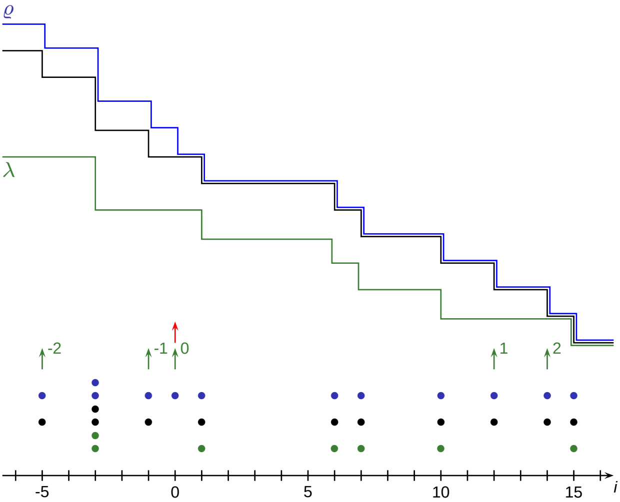

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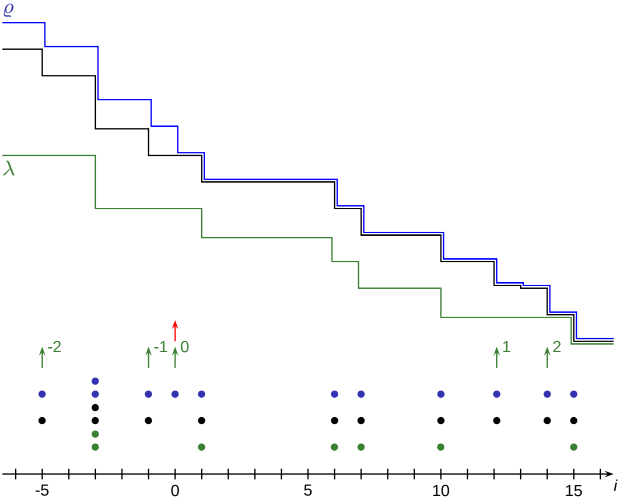

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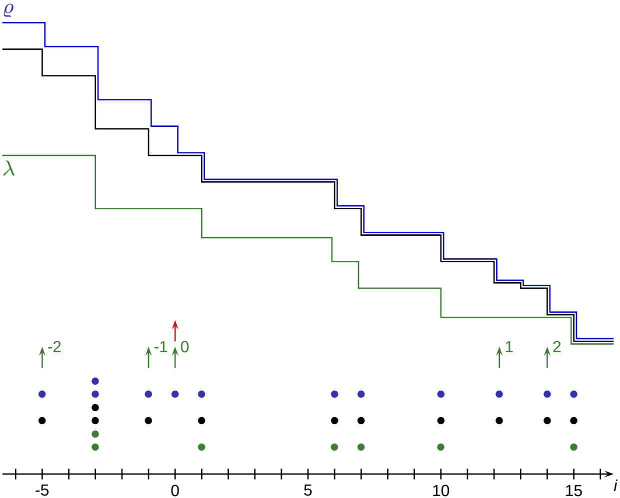

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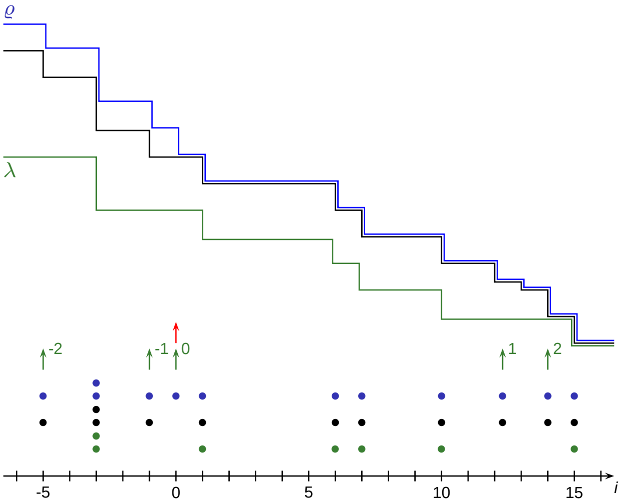

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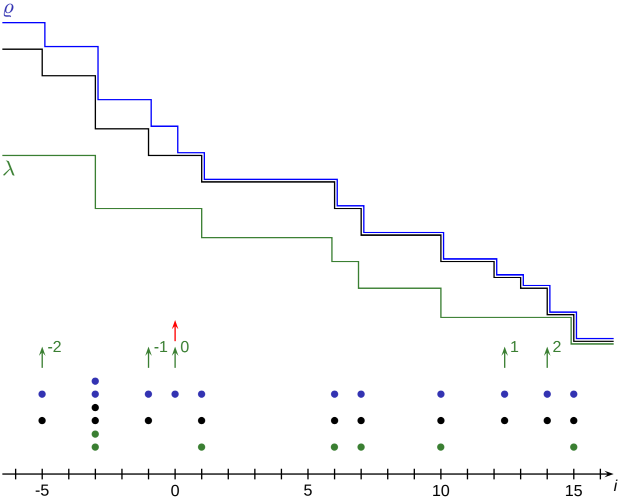

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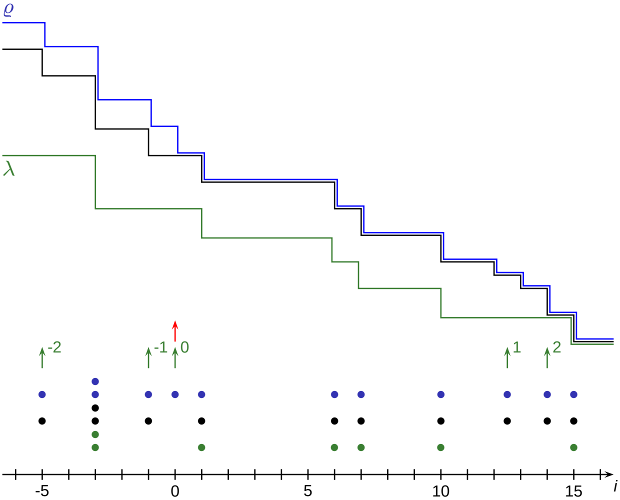

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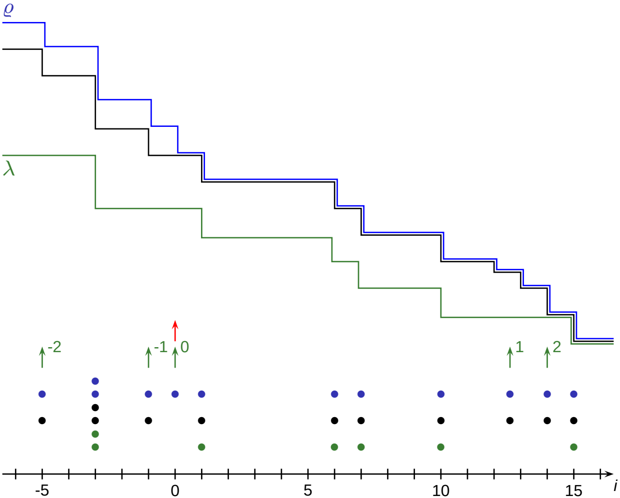

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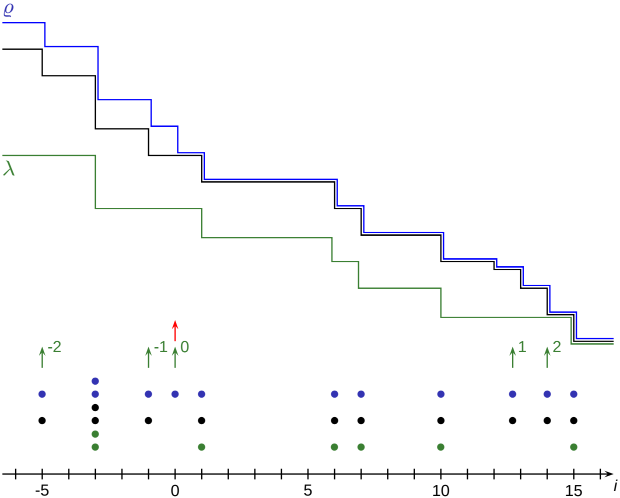

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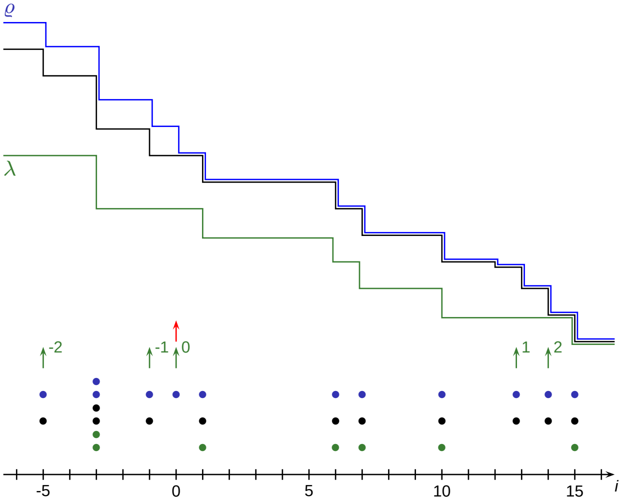

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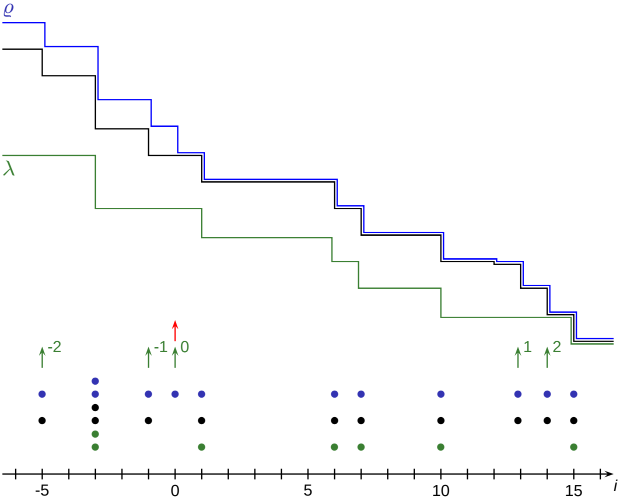

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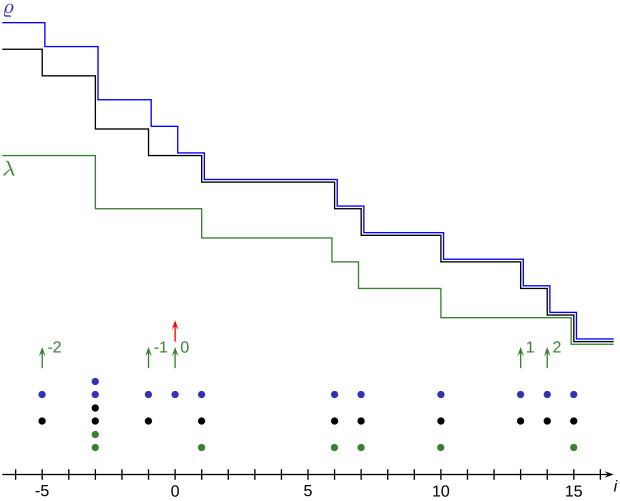

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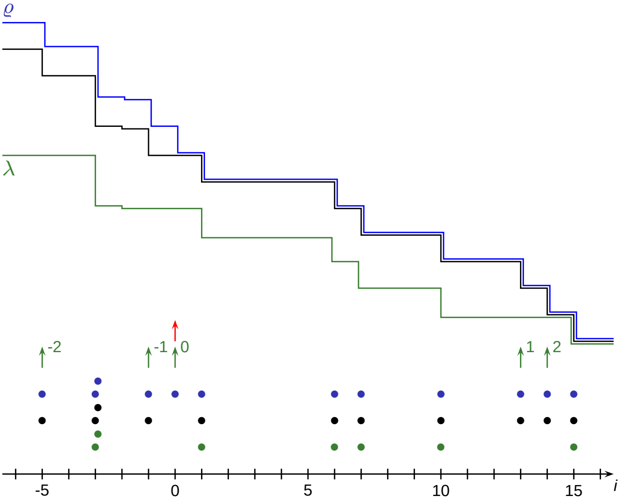

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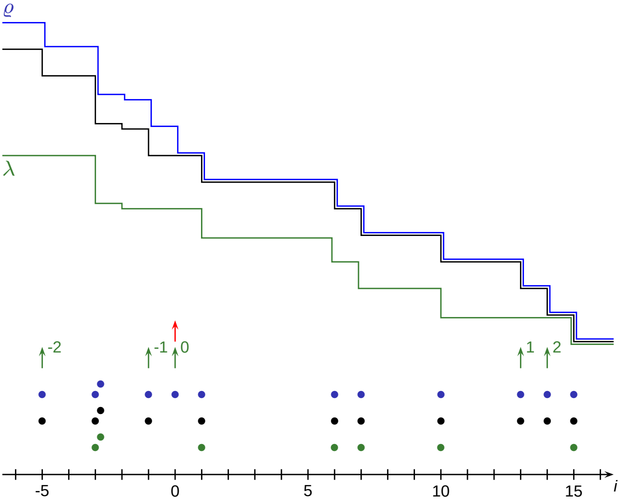

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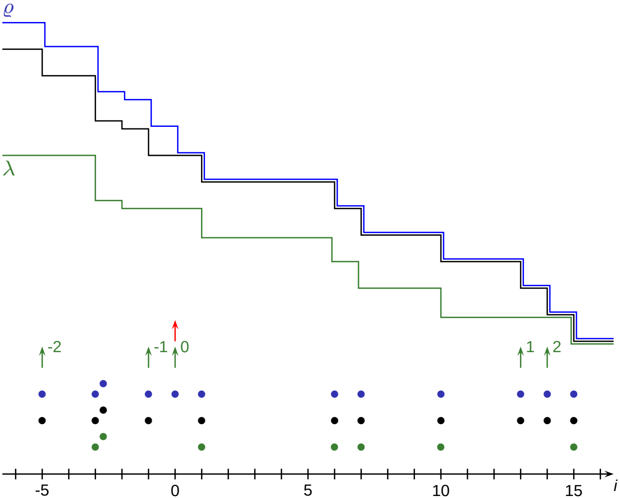

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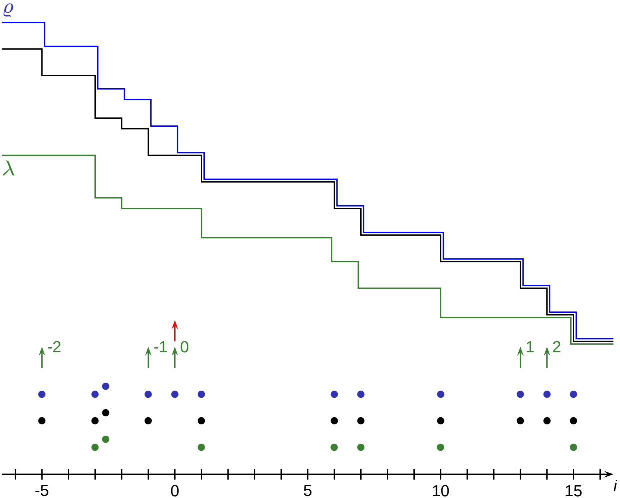

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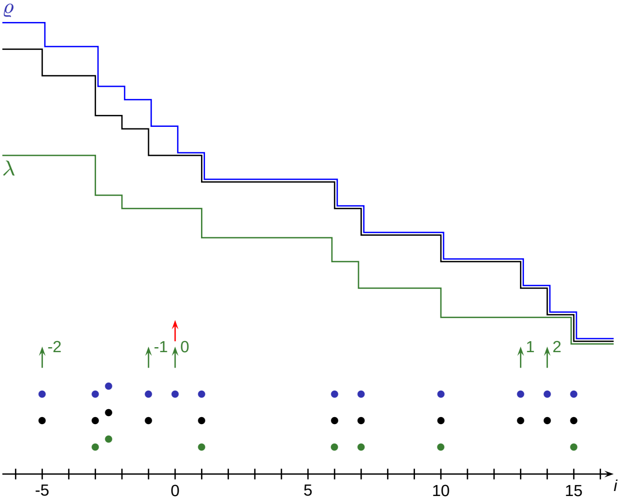

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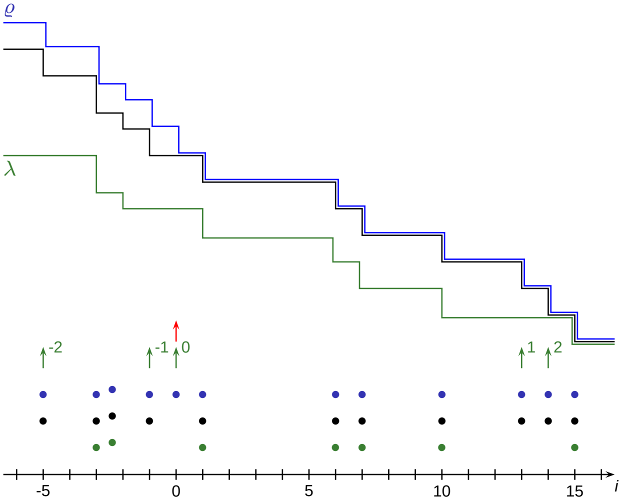

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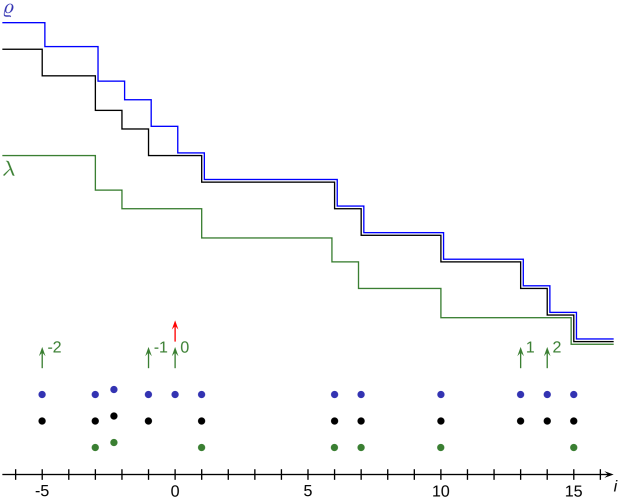

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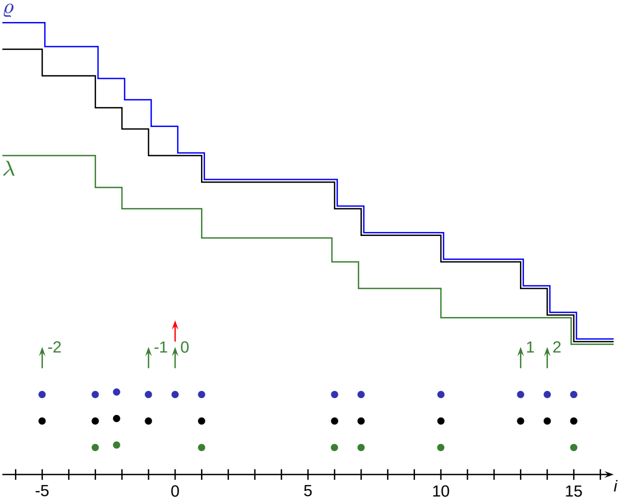

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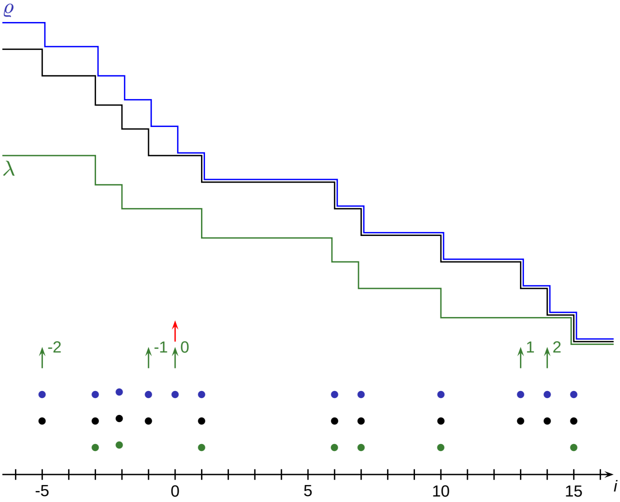

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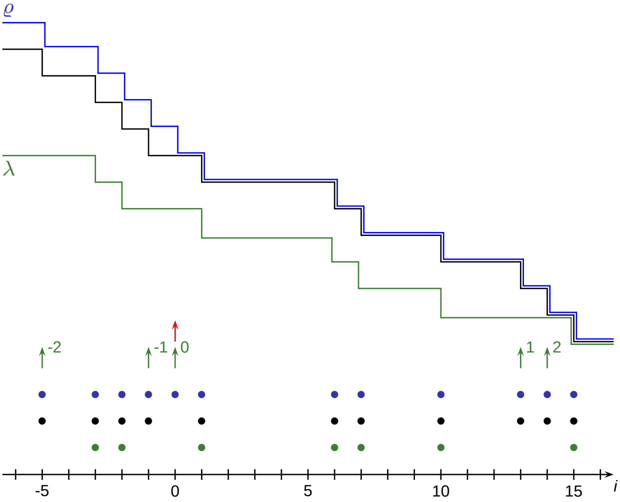

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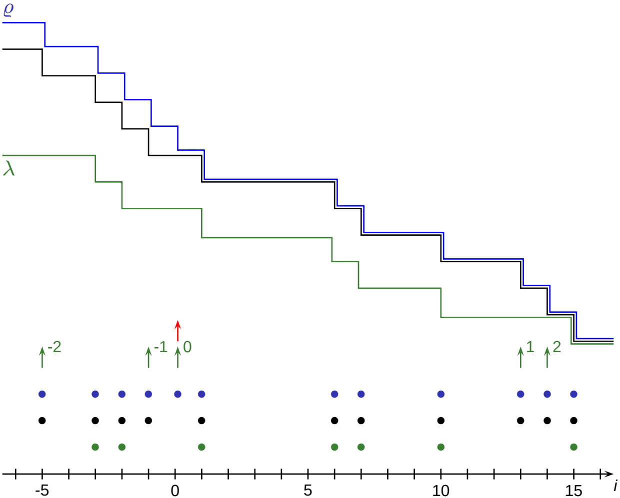

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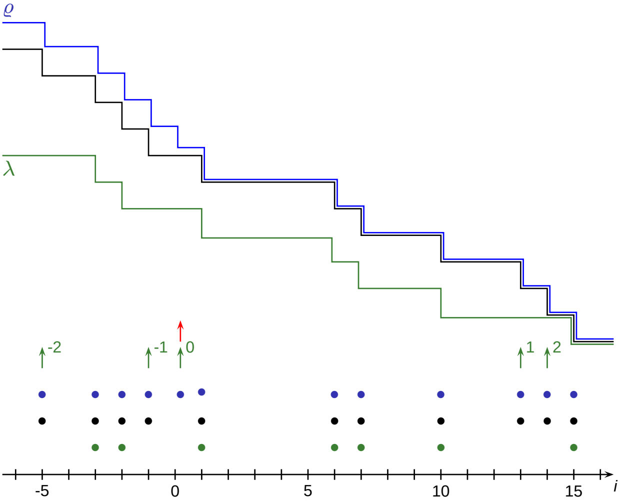

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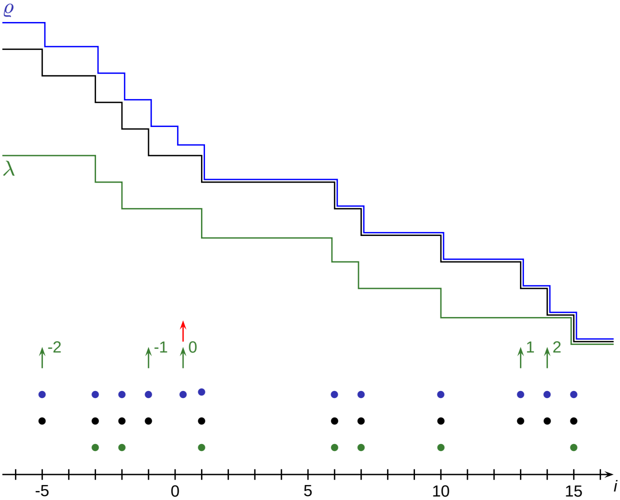

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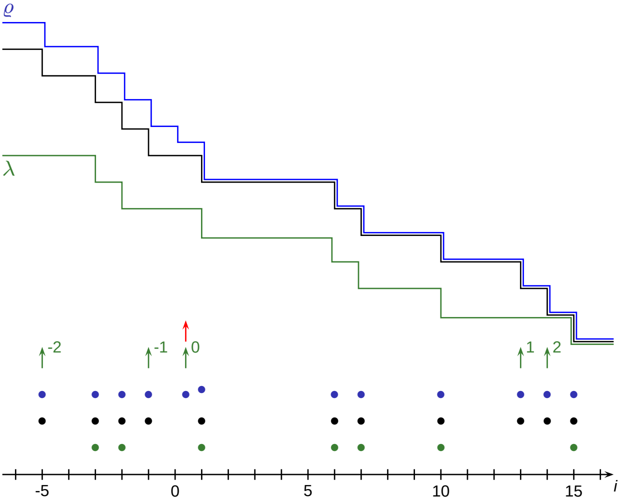

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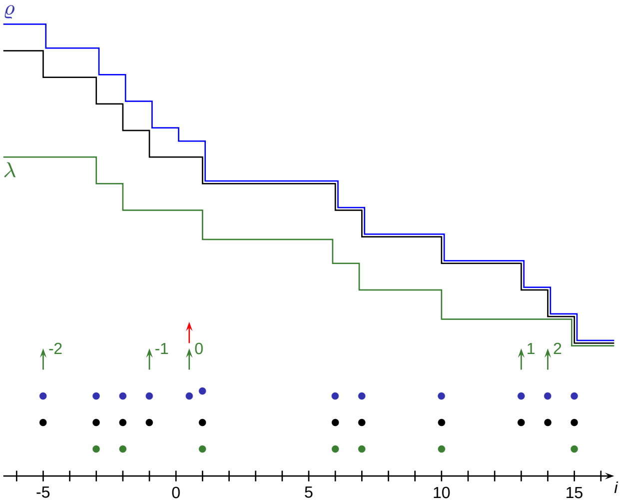

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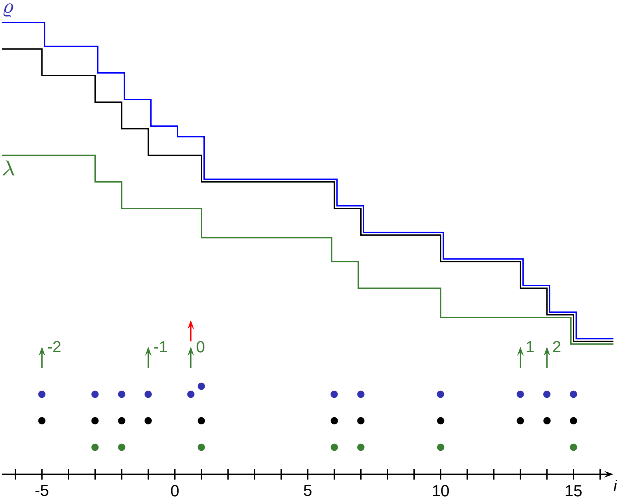

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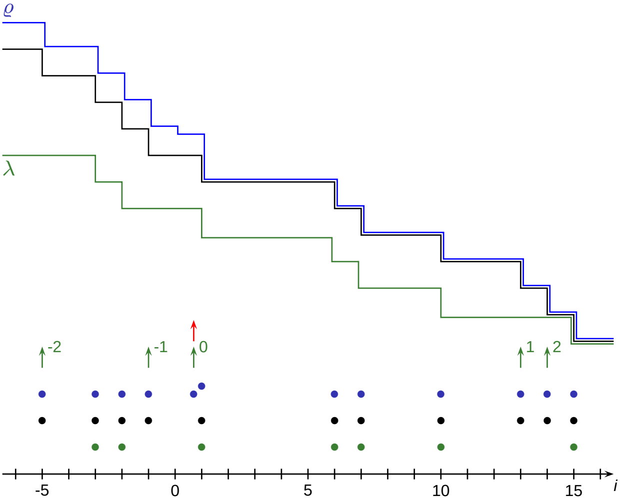

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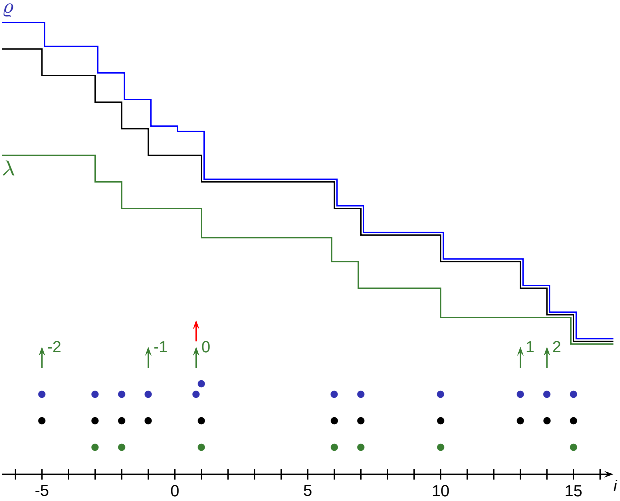

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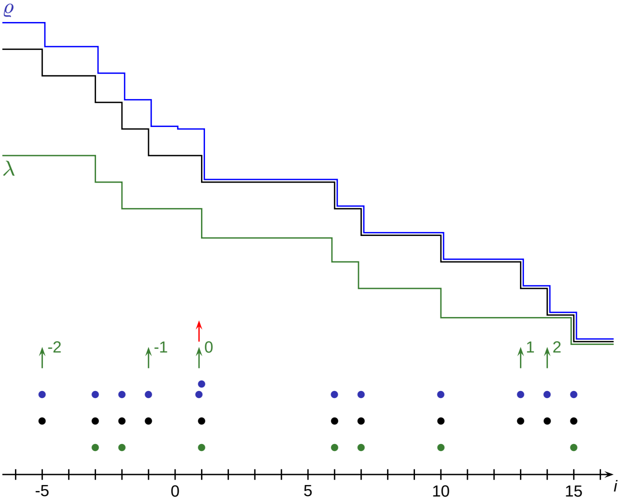

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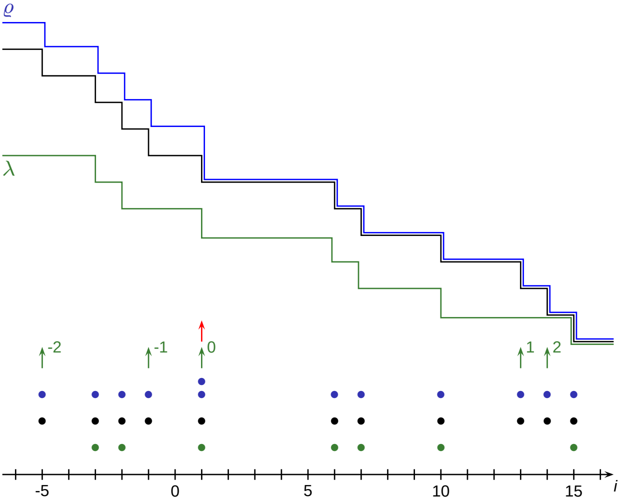

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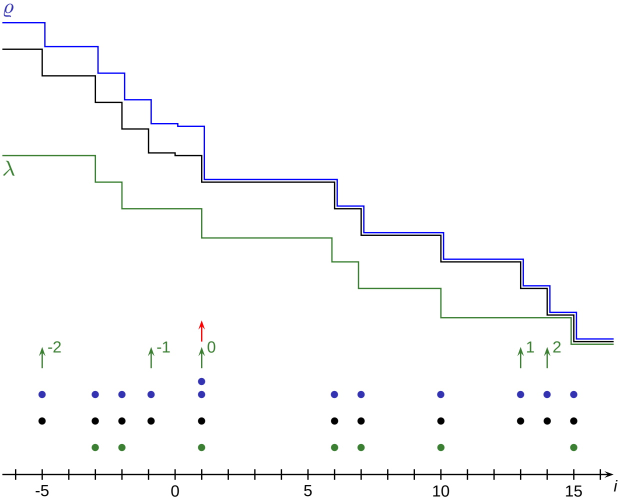

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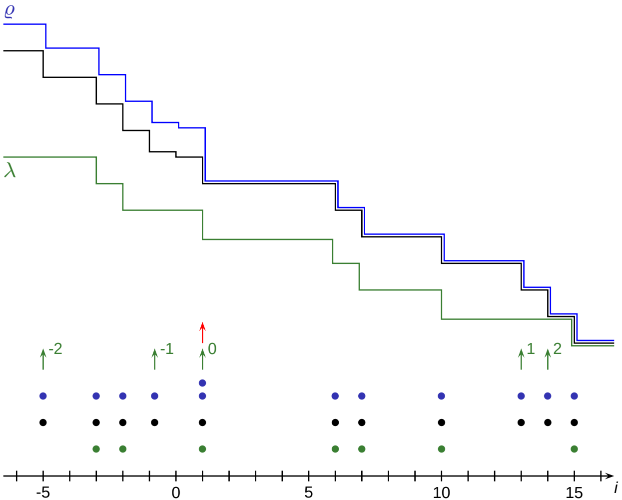

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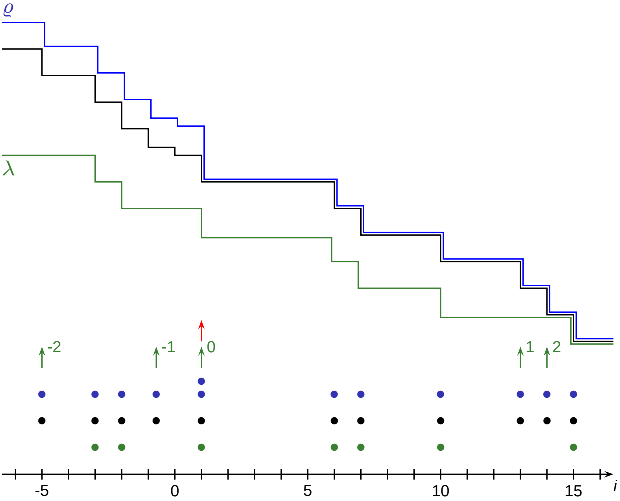

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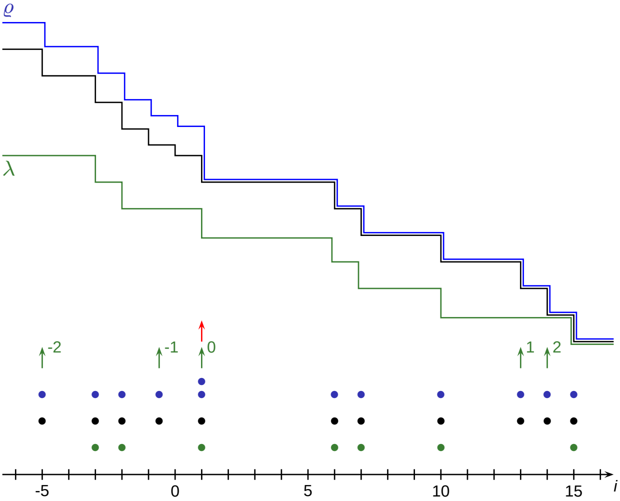

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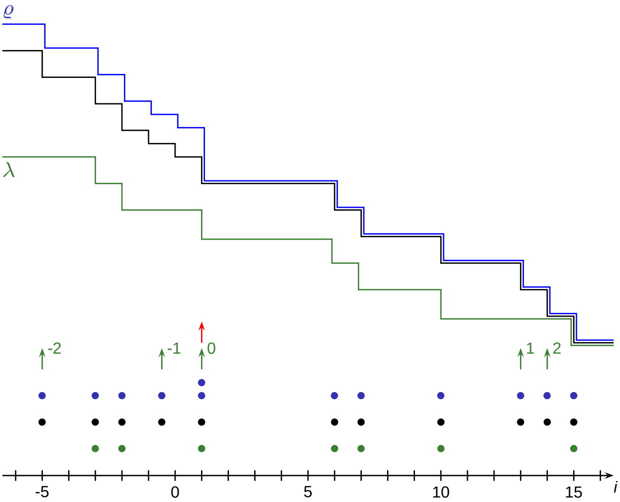

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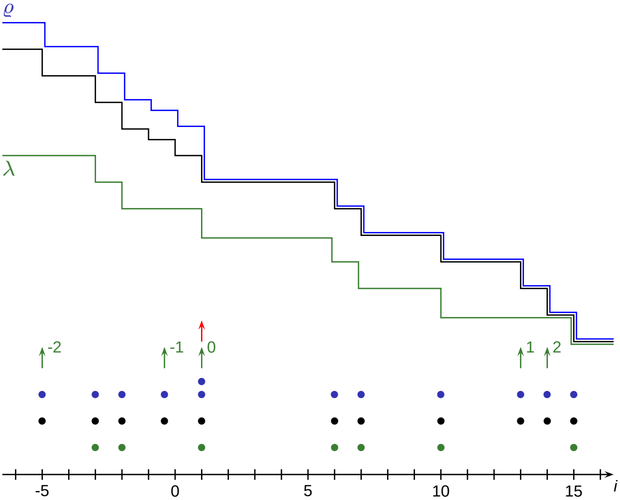

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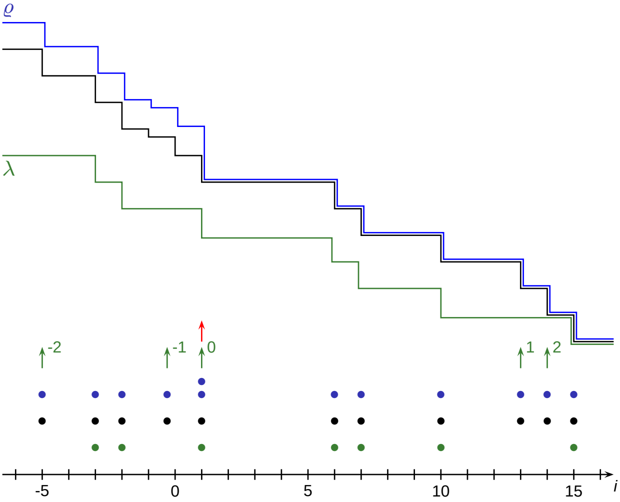

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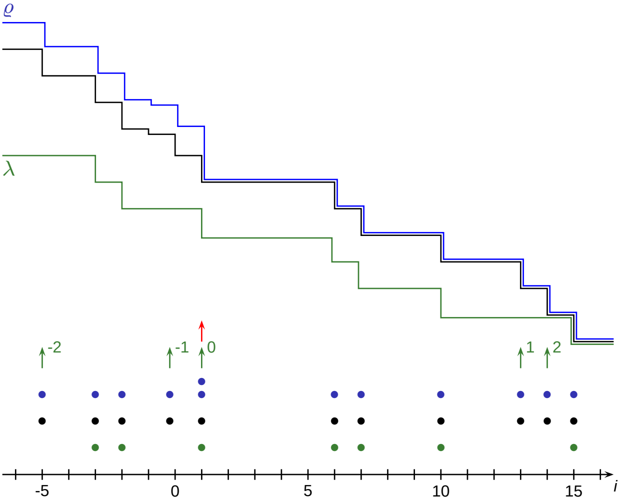

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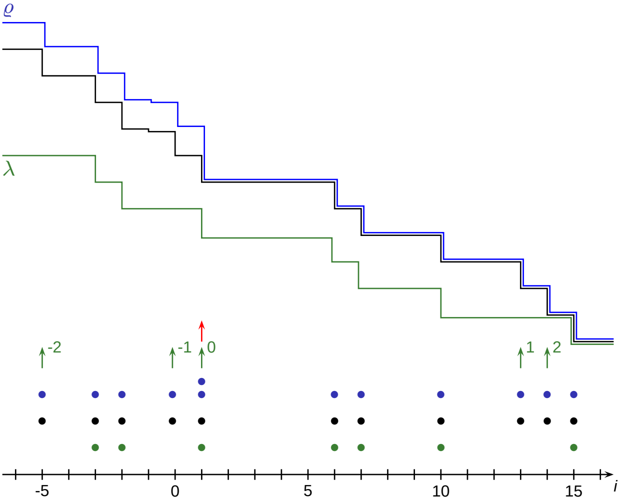

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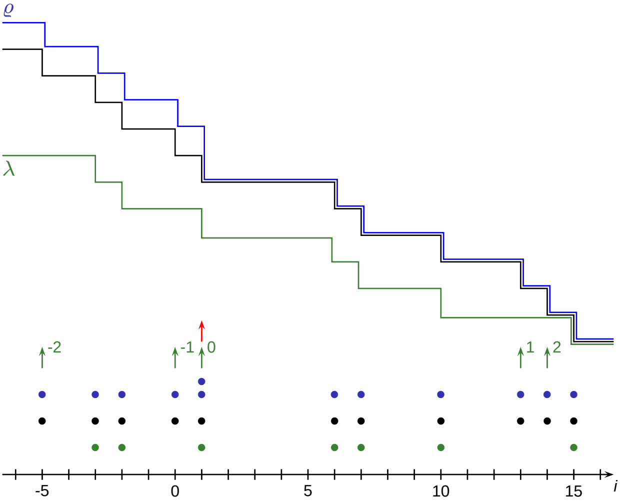

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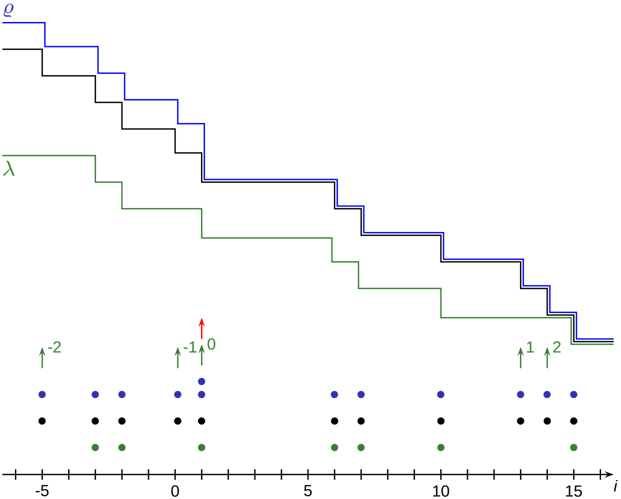

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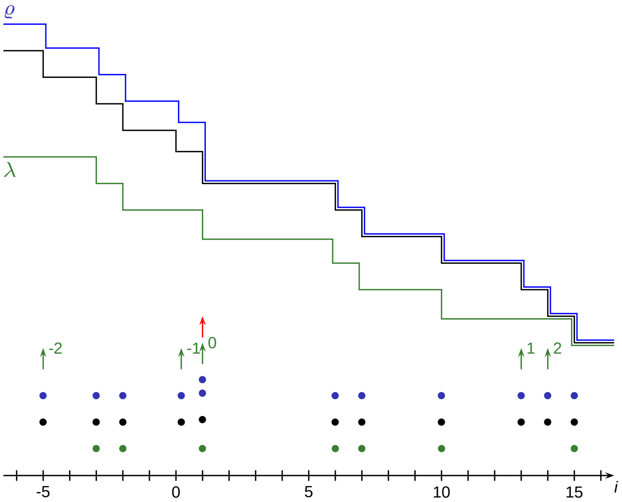

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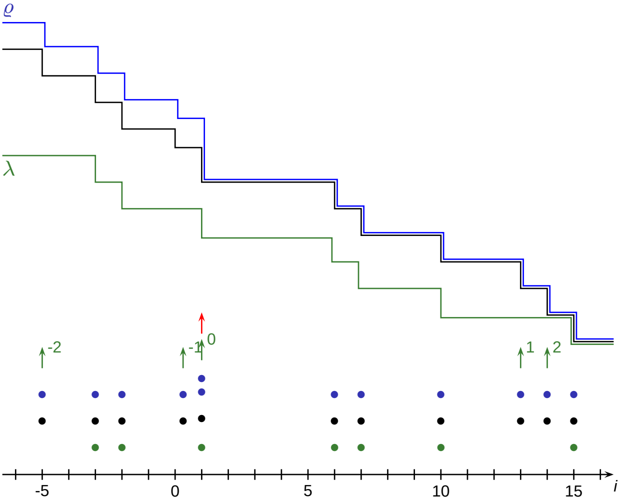

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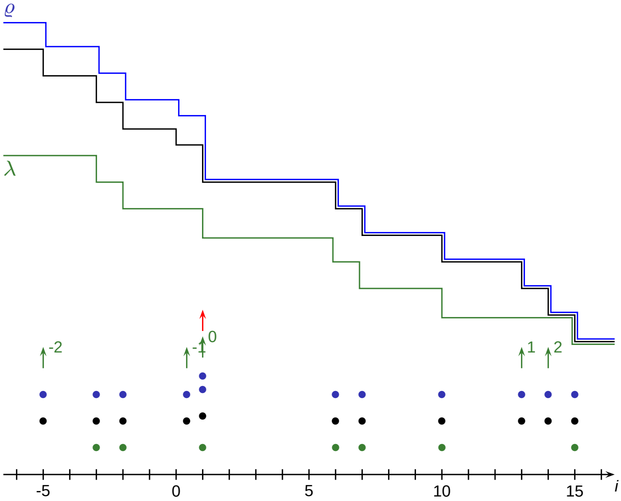

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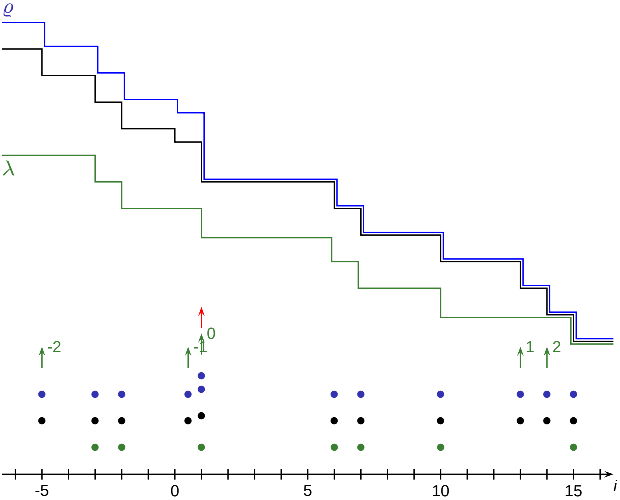

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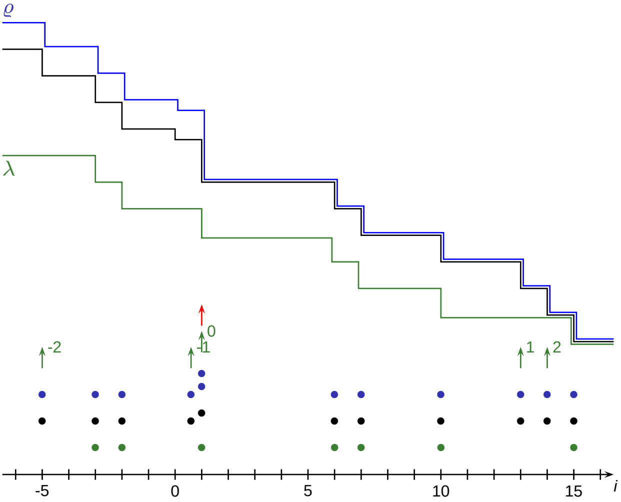

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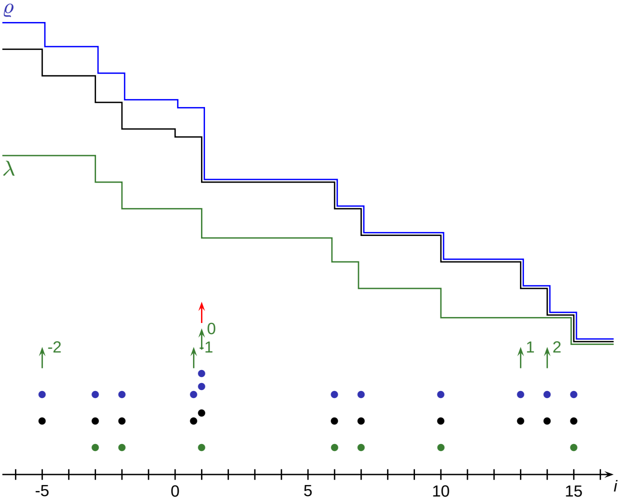

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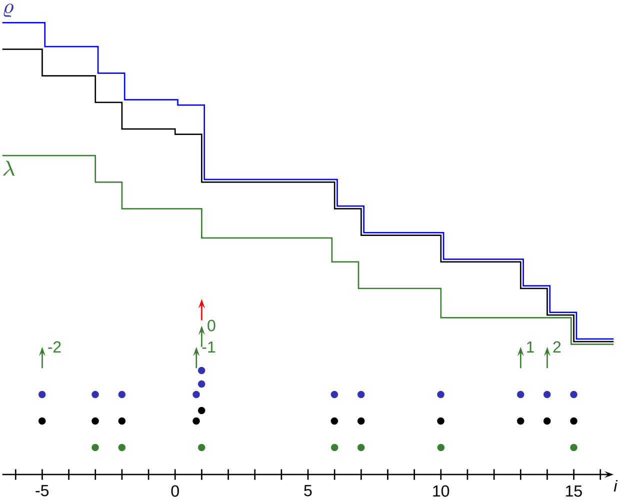

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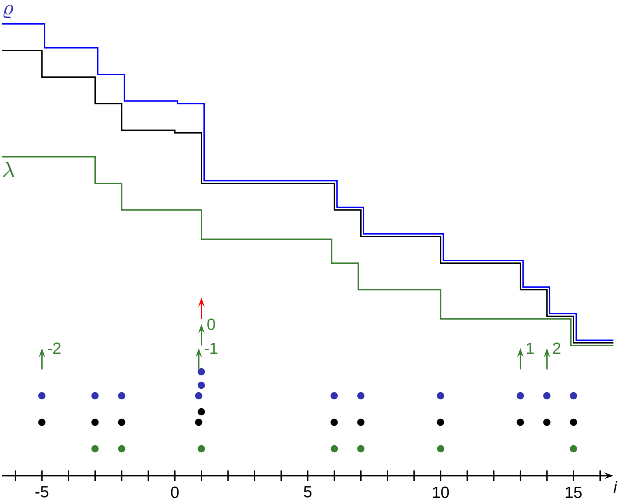

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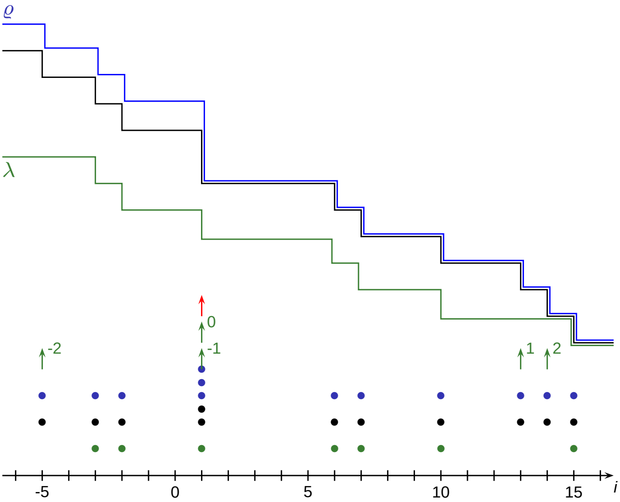

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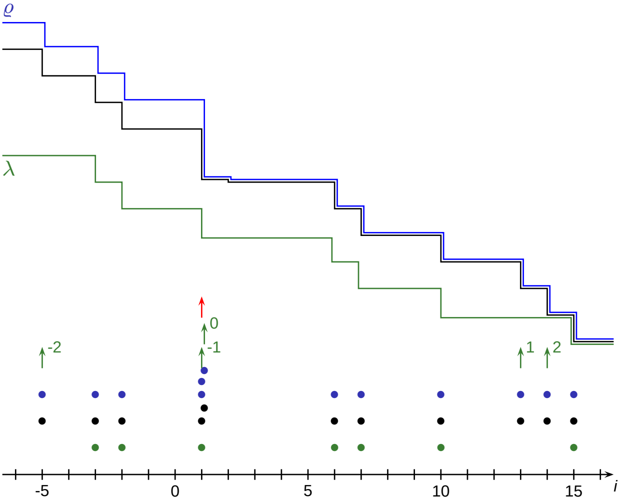

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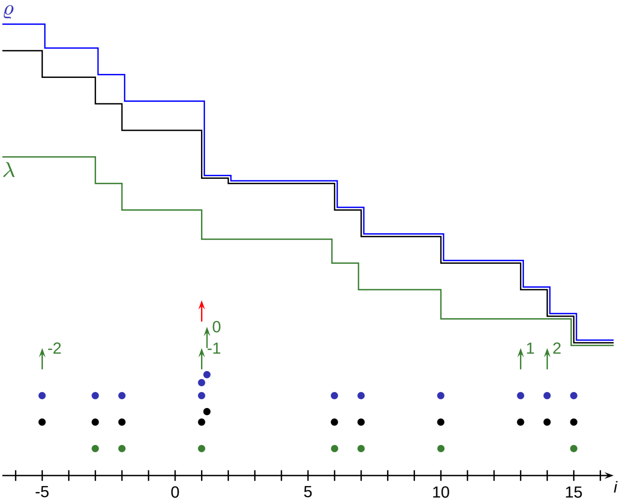

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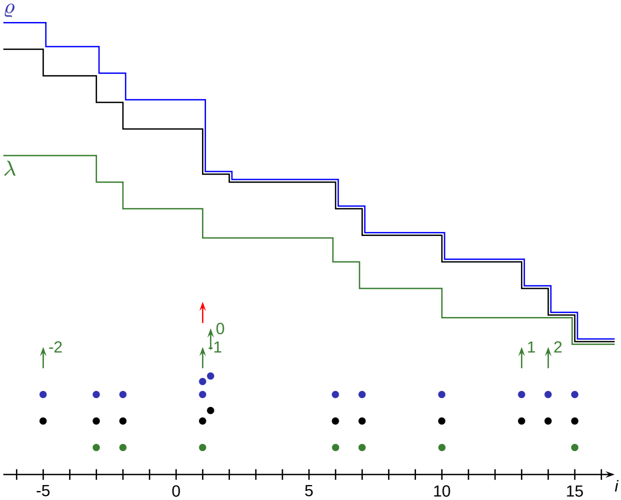

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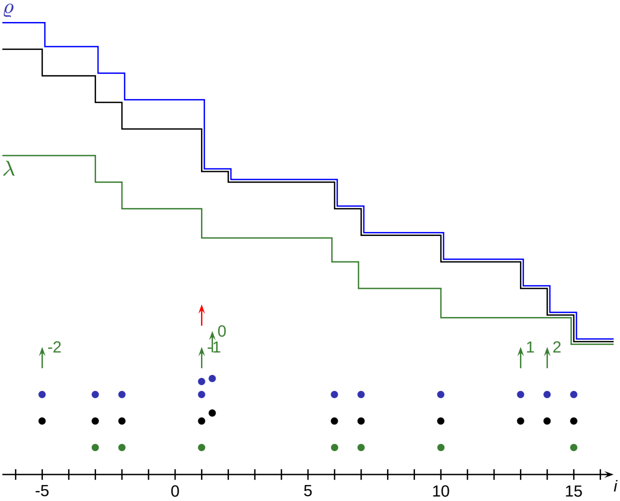

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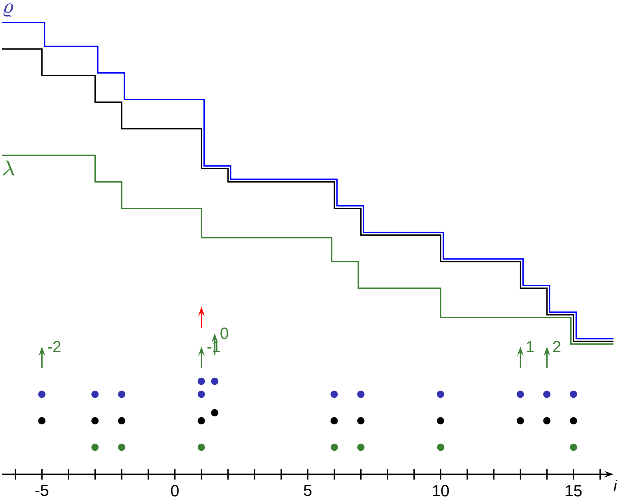

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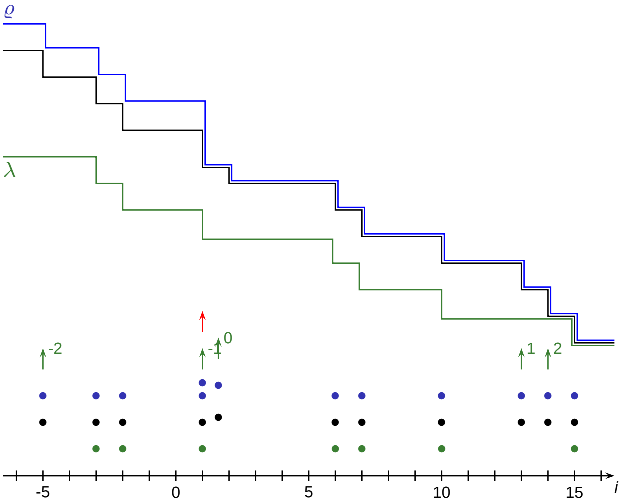

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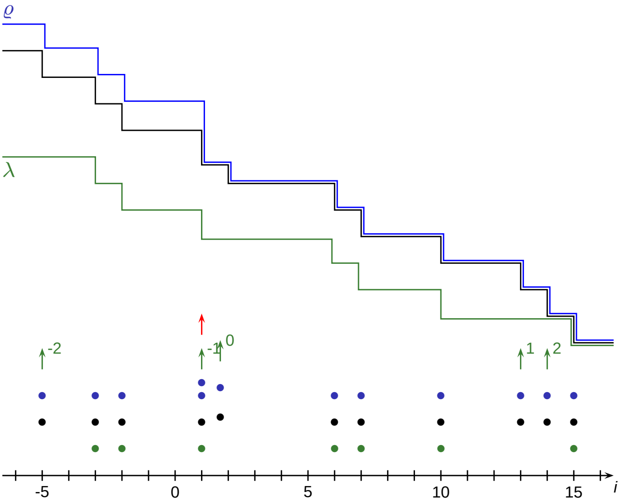

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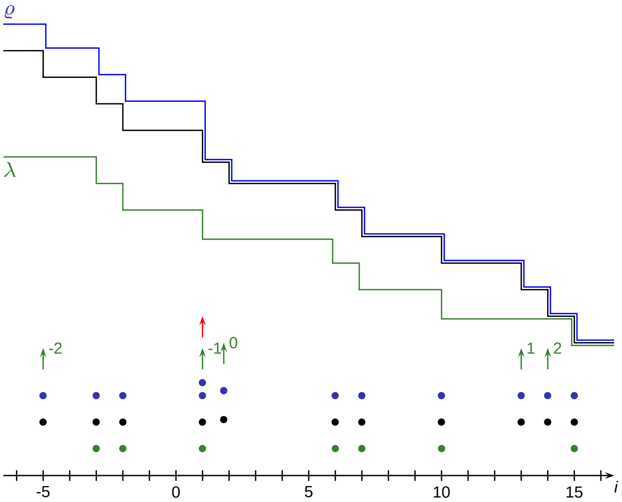

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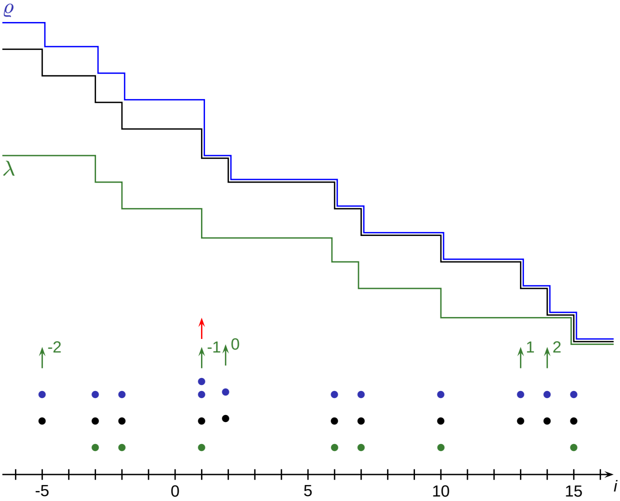

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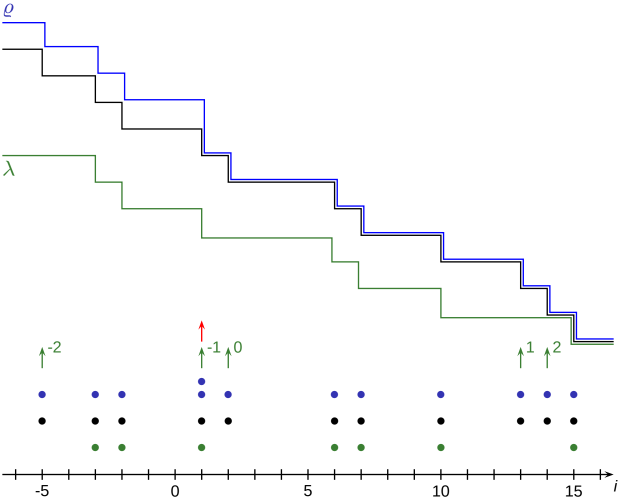

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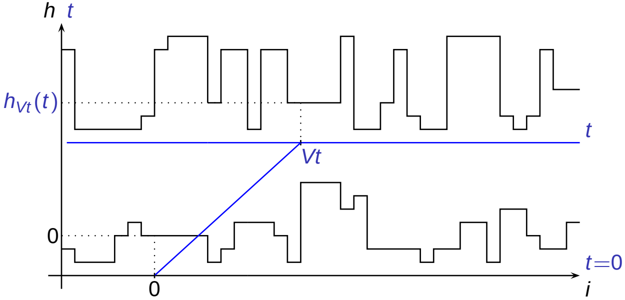

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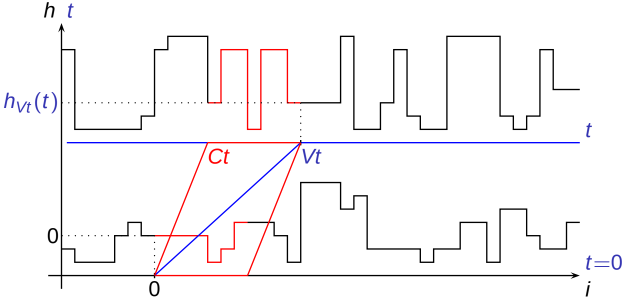

Initial fluctuations are transported along the characteristics on this scale.

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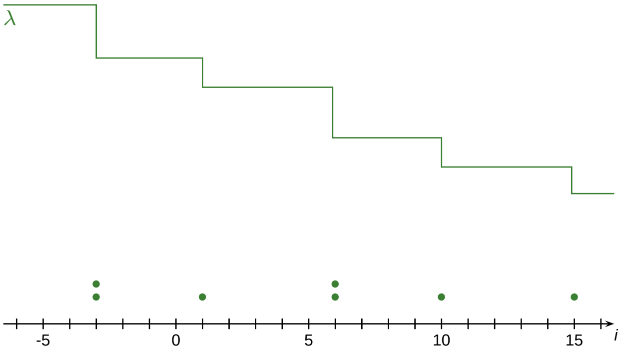
There are limit distribution results for TASEP e.g. by Johansson 2000, Prähofer and Spohn 2001, Ferrari and Spohn 2006.
Their methods give limit distributions as well, but are very model-dependent: they rewrite the model as a determinantal process, and perform asymptotic analysis of the determinants.

Proof: many second class particles



Second class particle current: difference in growth.

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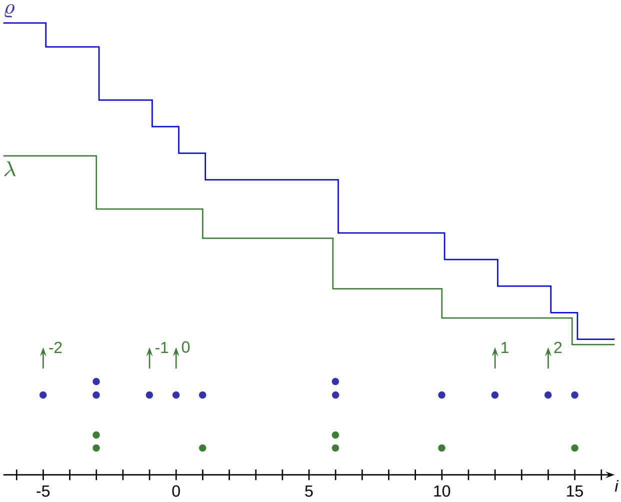

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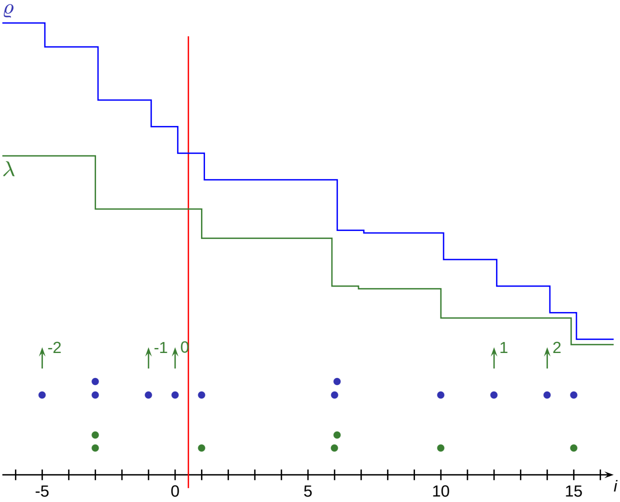

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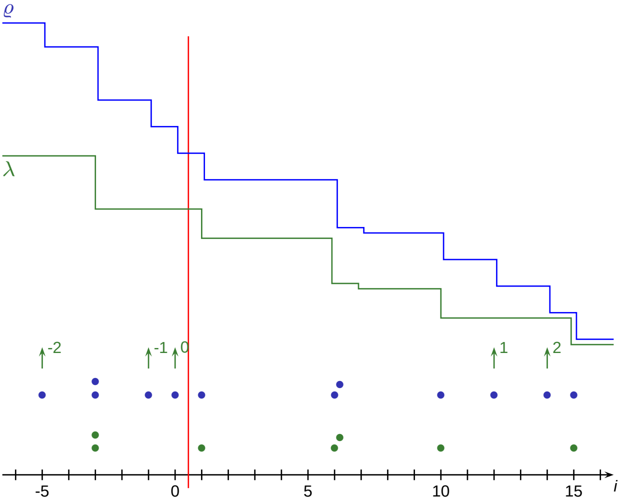

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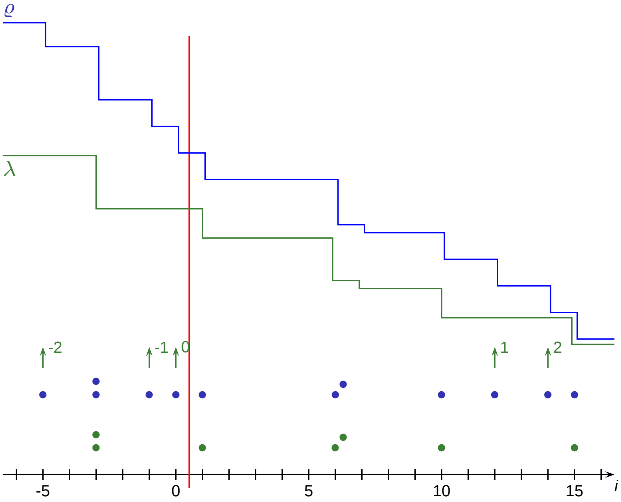

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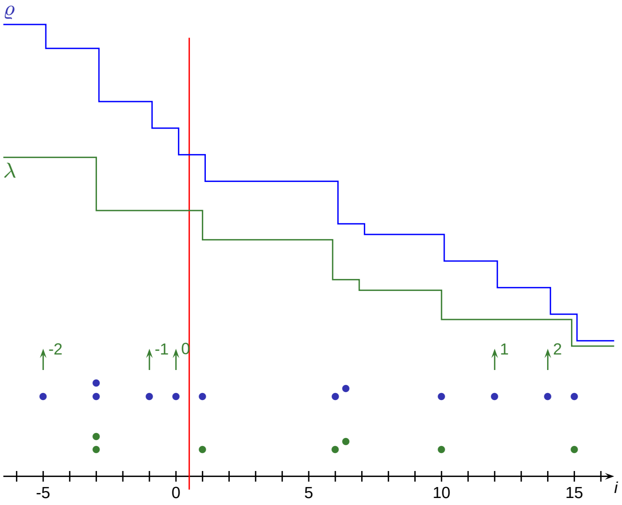

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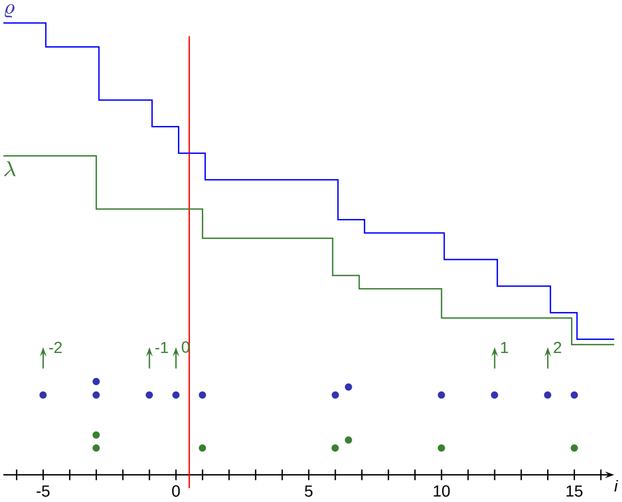

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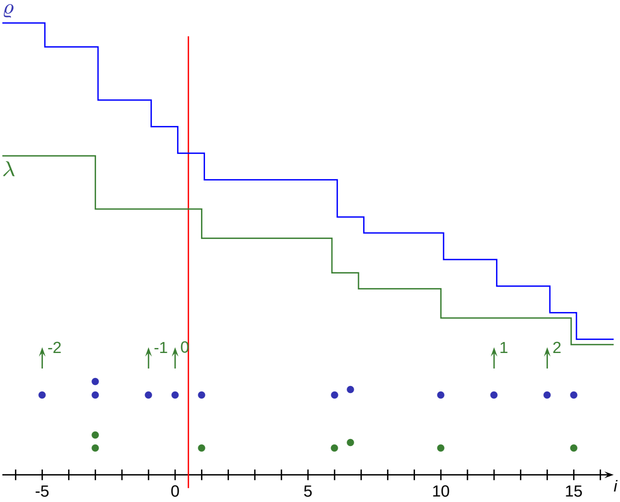

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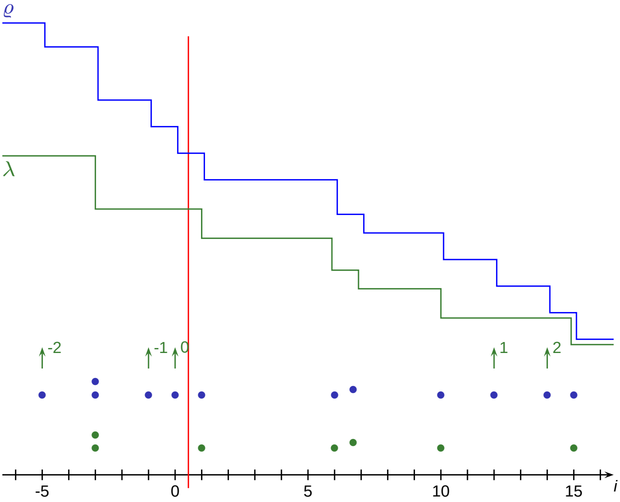

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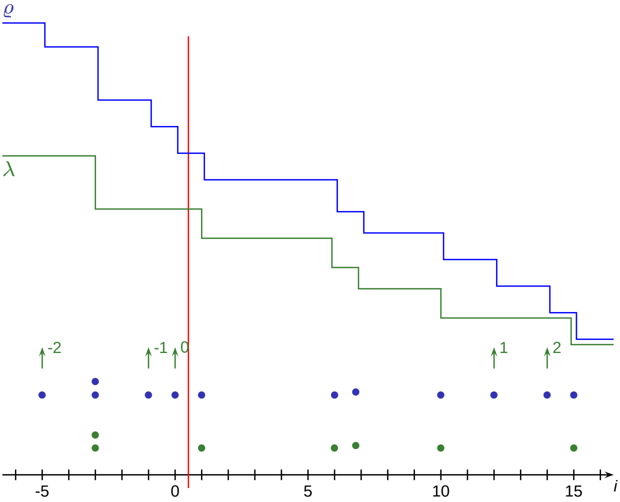

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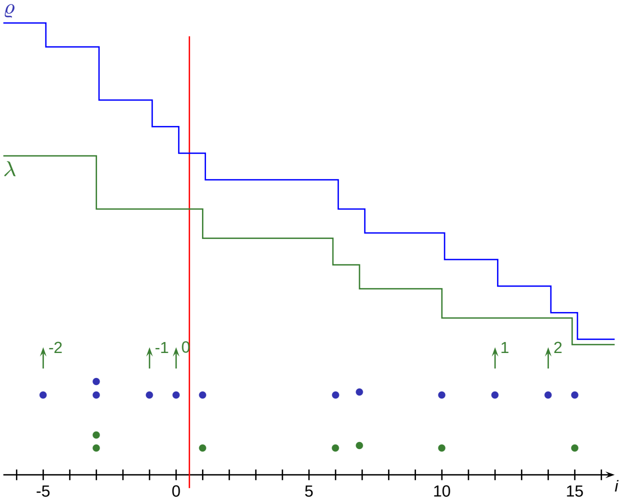

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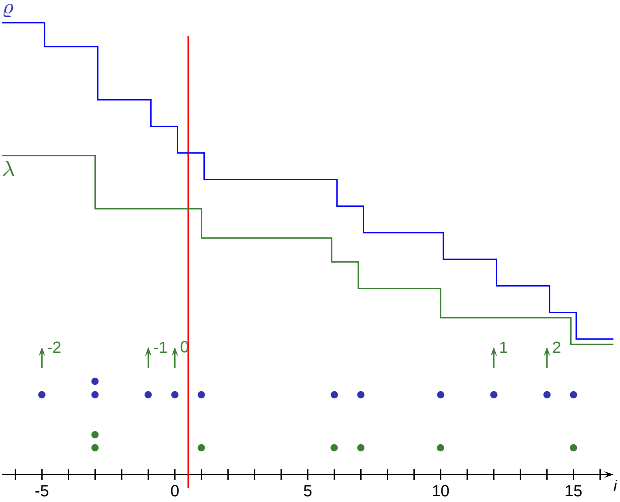
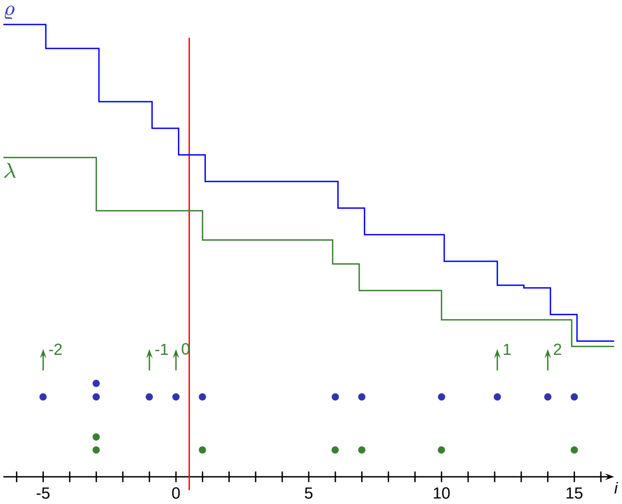


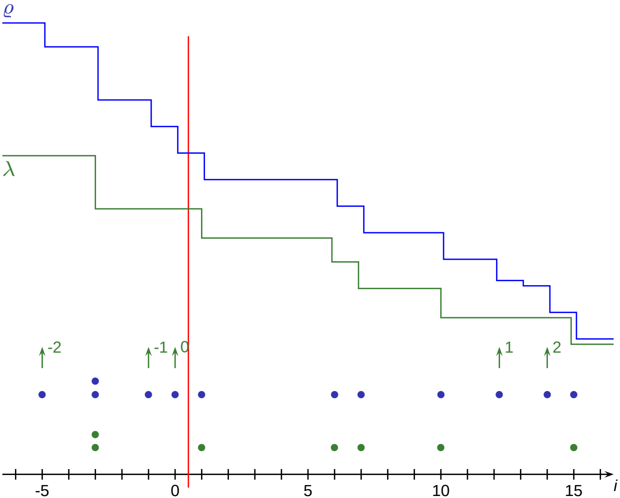
Second class particle current: difference in growth.

Proof: many second class particles



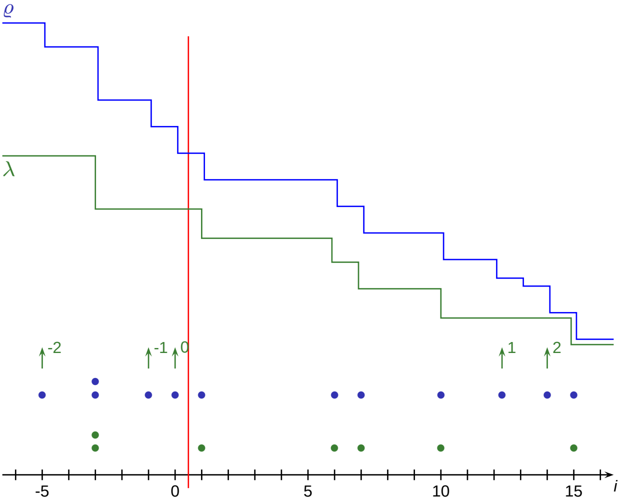
Second class particle current: difference in growth.

Proof: many second class particles



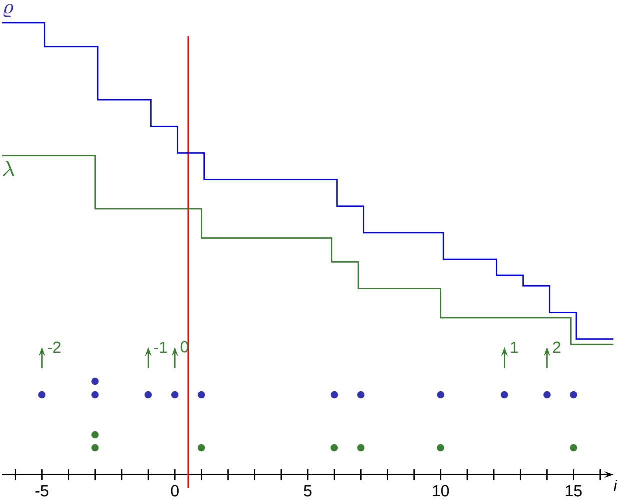
Second class particle current: difference in growth.

Proof: many second class particles



Second class particle current: difference in growth.

Proof: many second class particles



Second class particle current: difference in growth.

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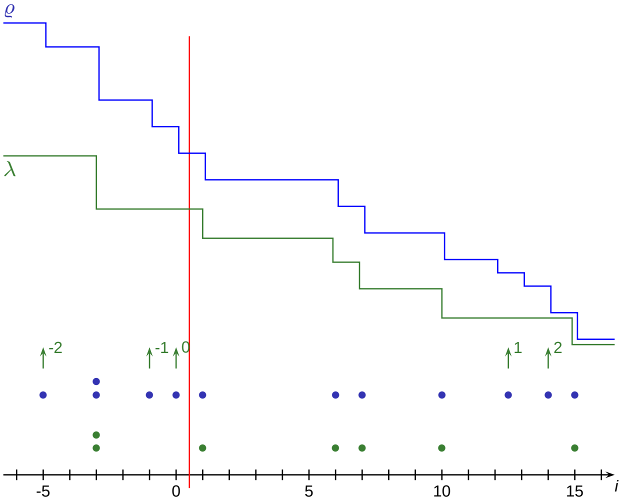

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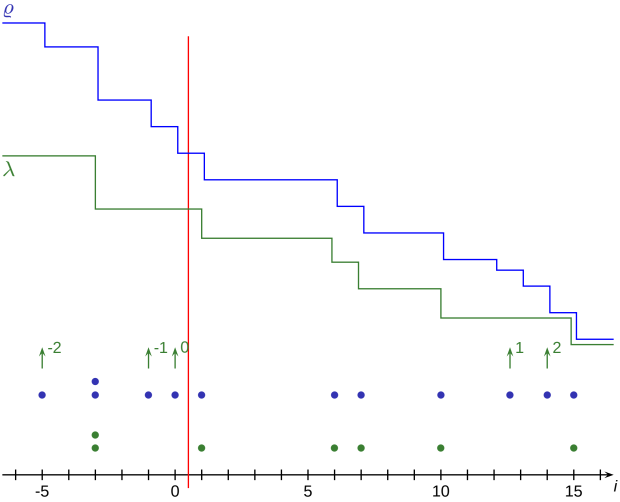

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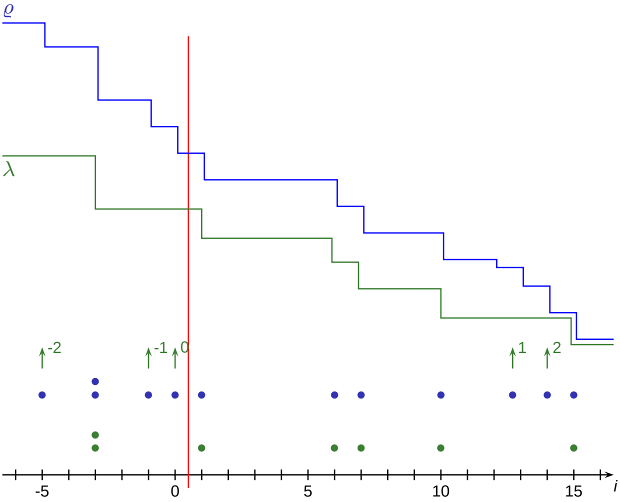

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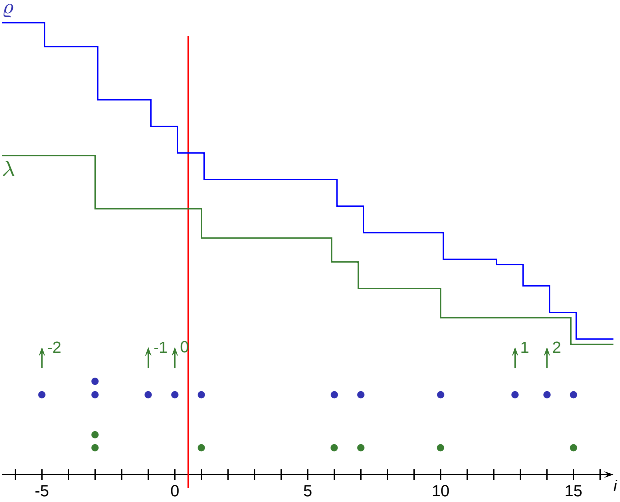

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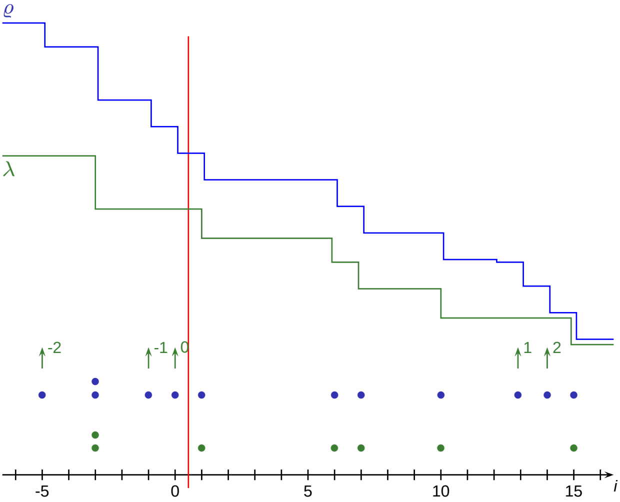

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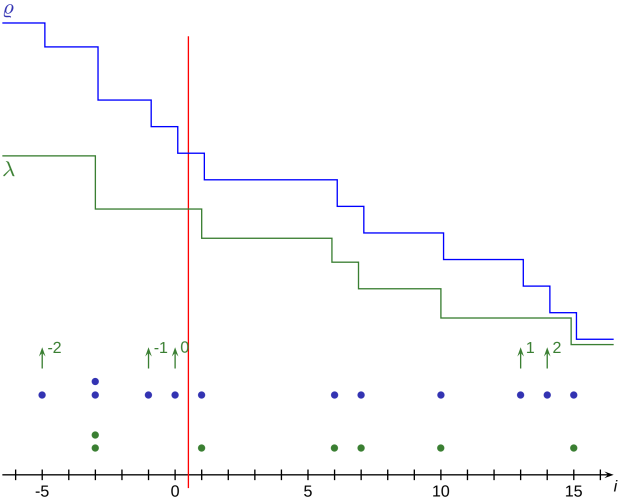

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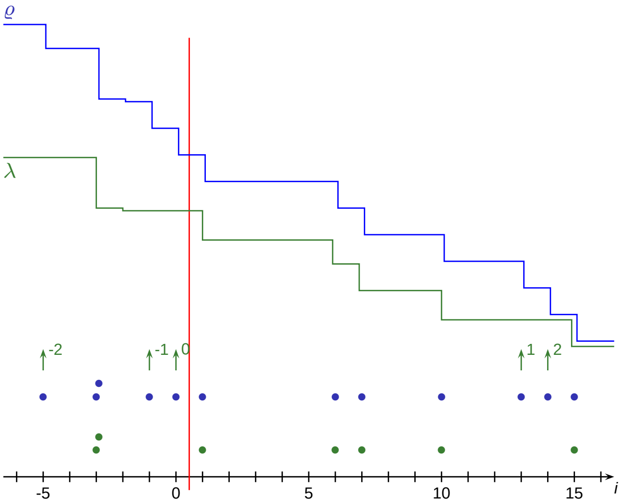

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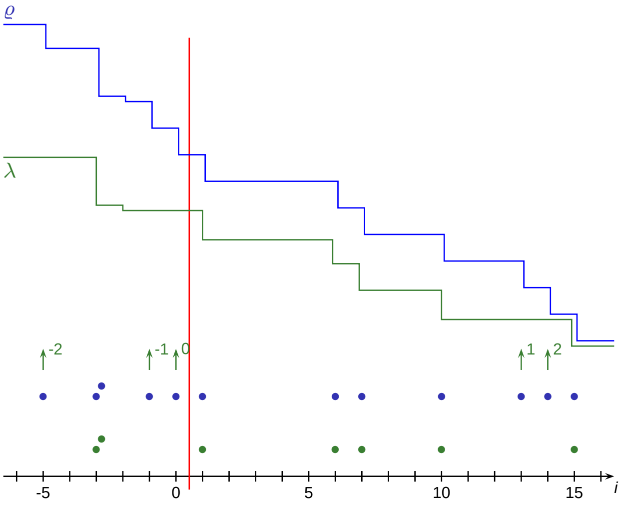

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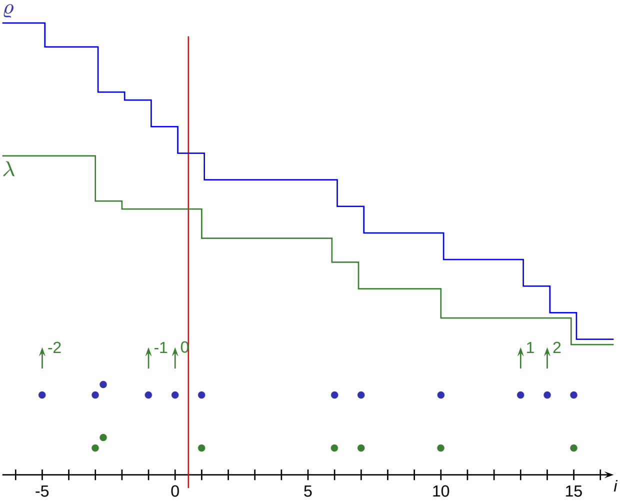

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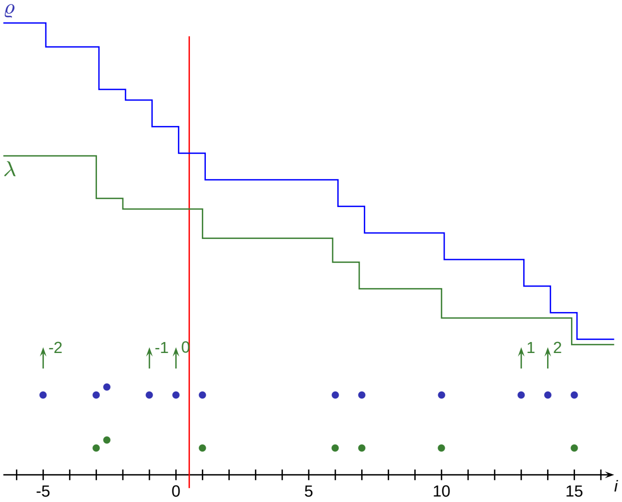

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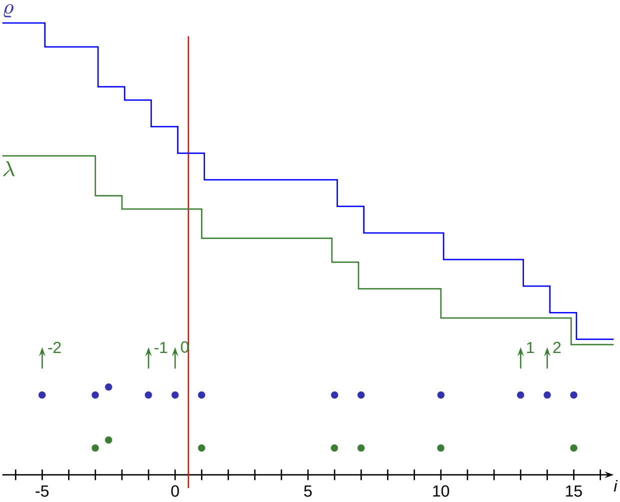

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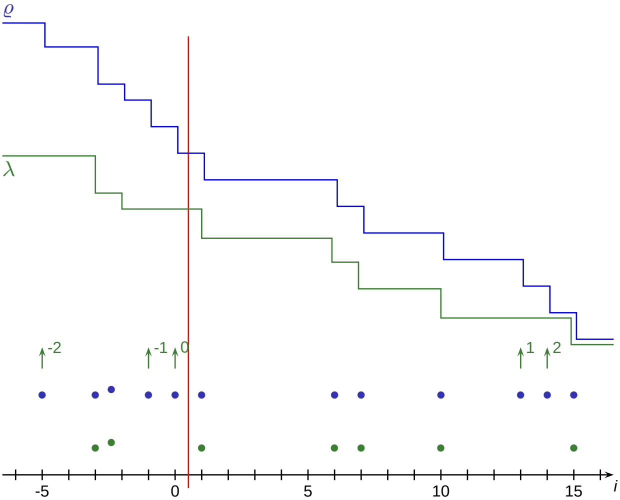

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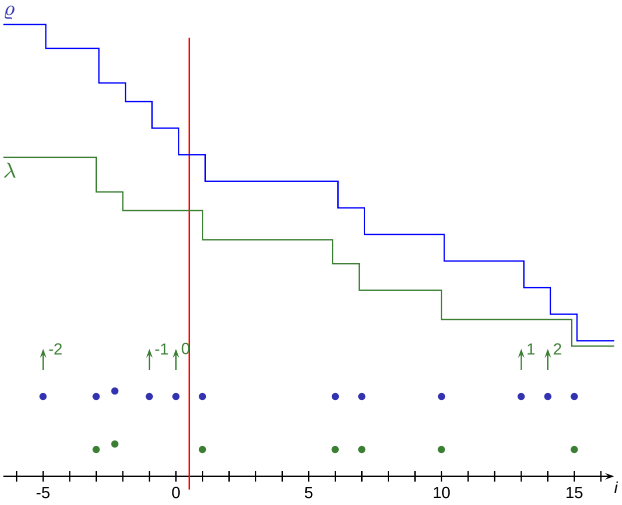

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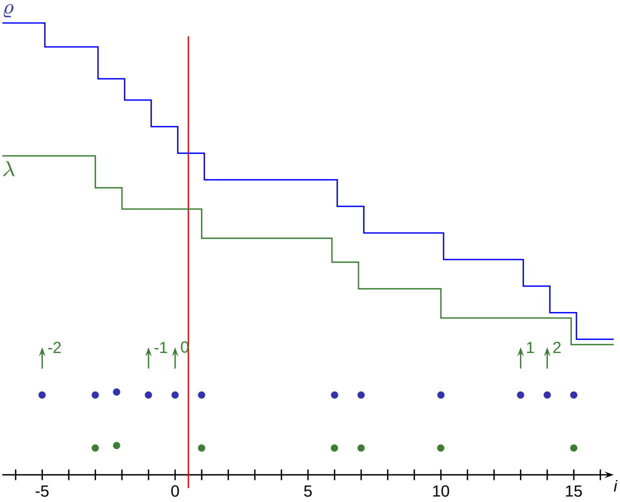

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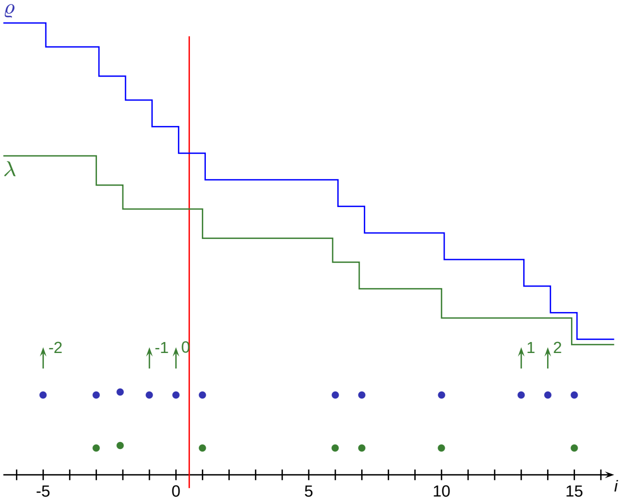

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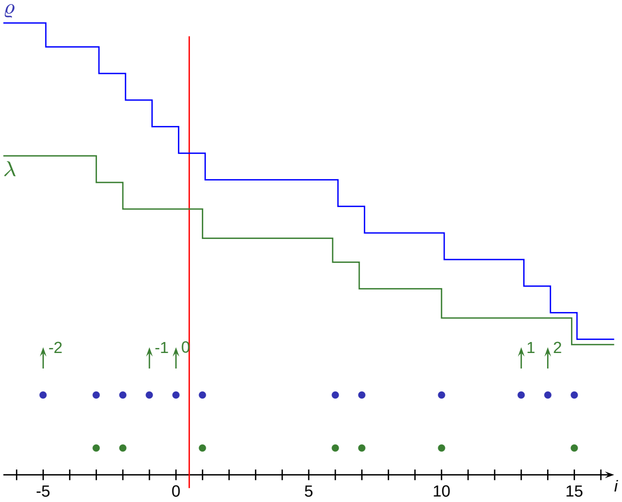

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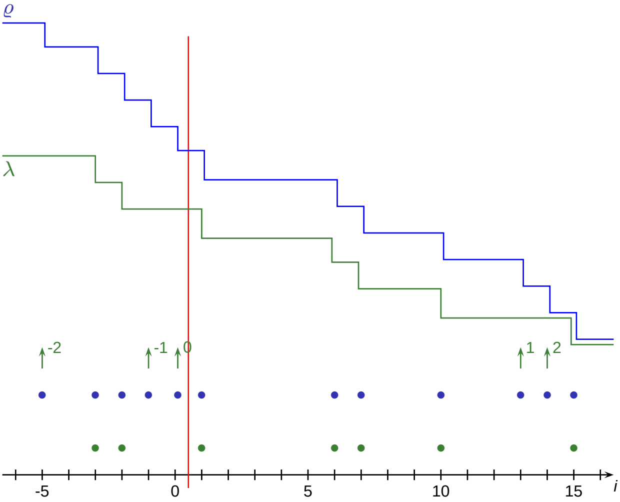

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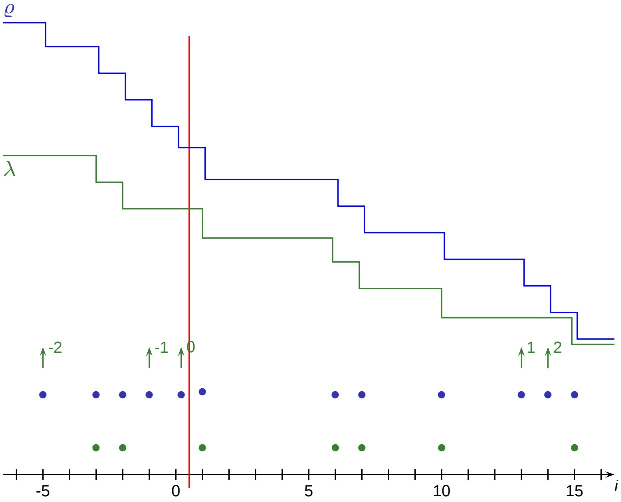

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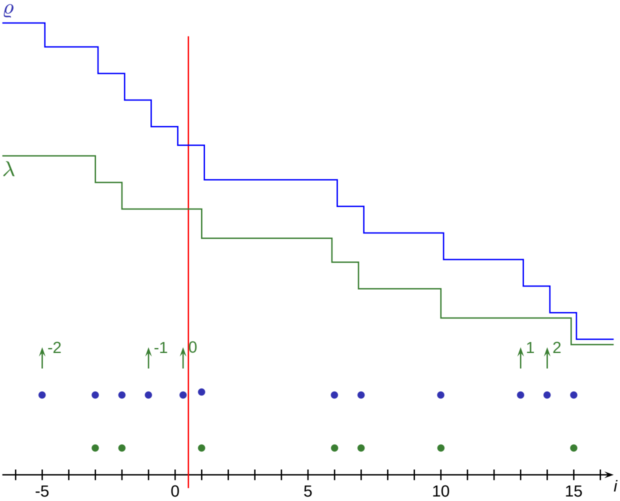

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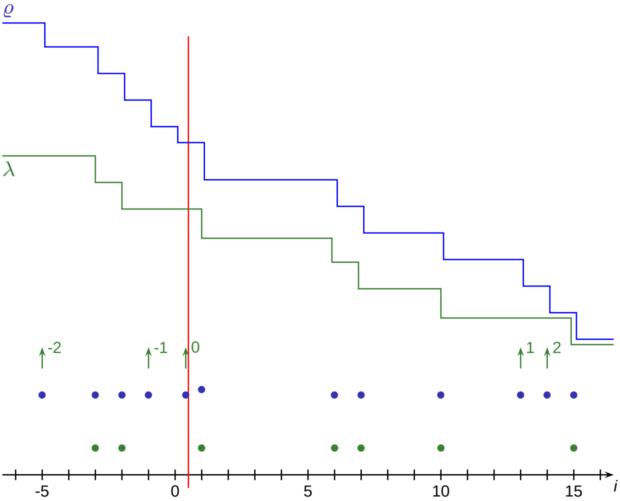

Second class particle current: difference in growth.

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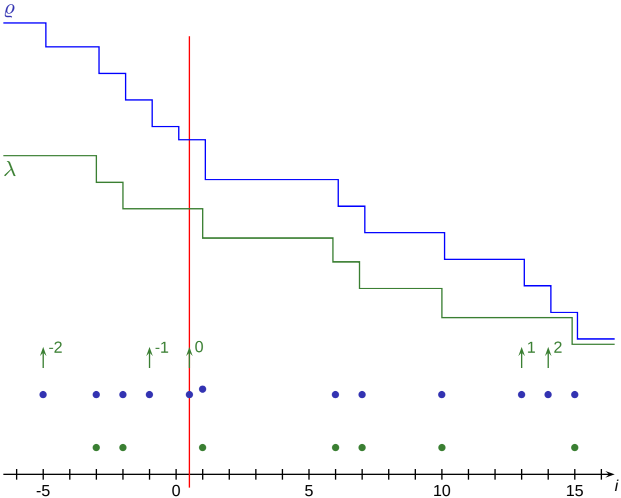

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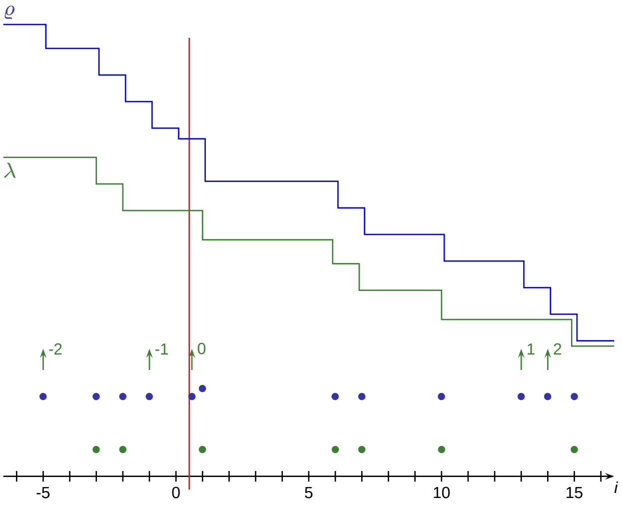

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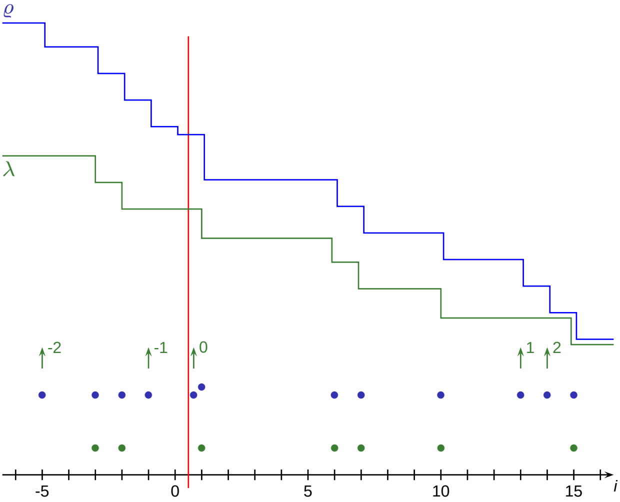

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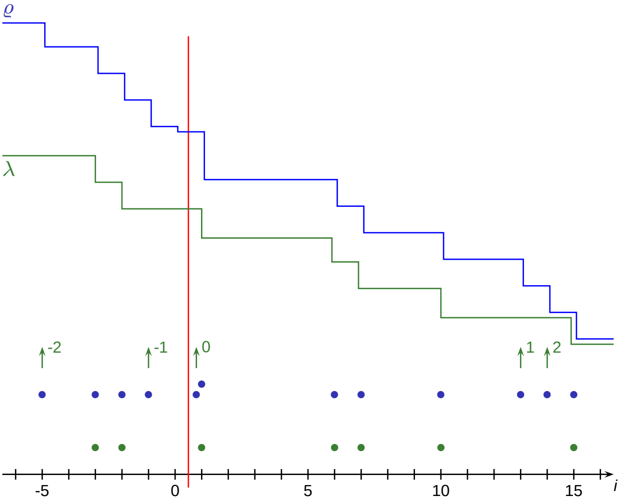

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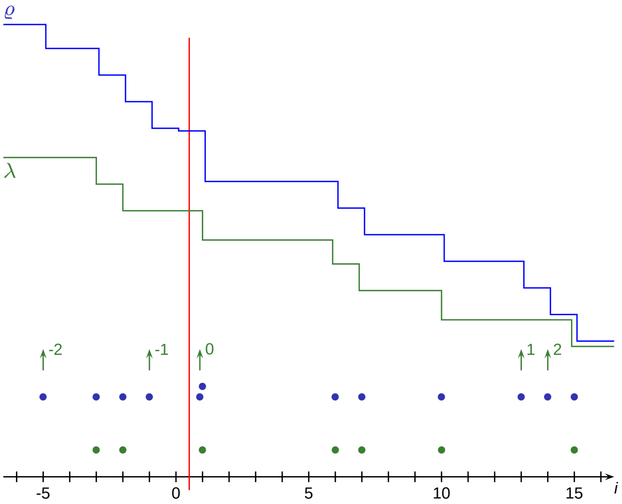

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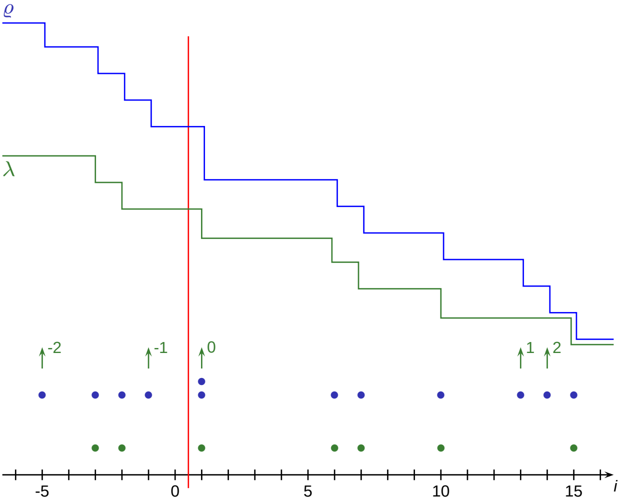

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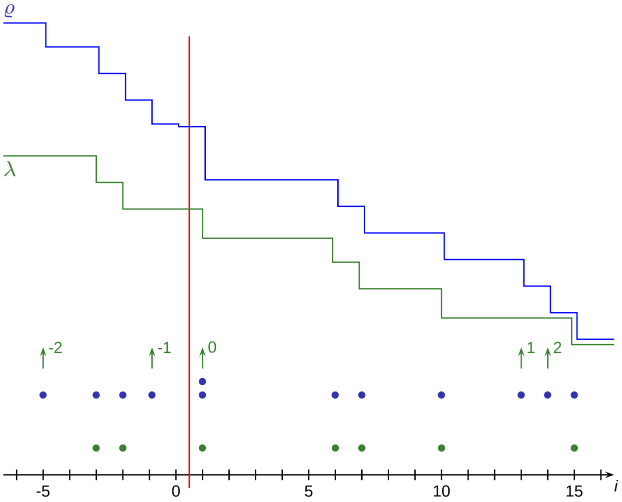

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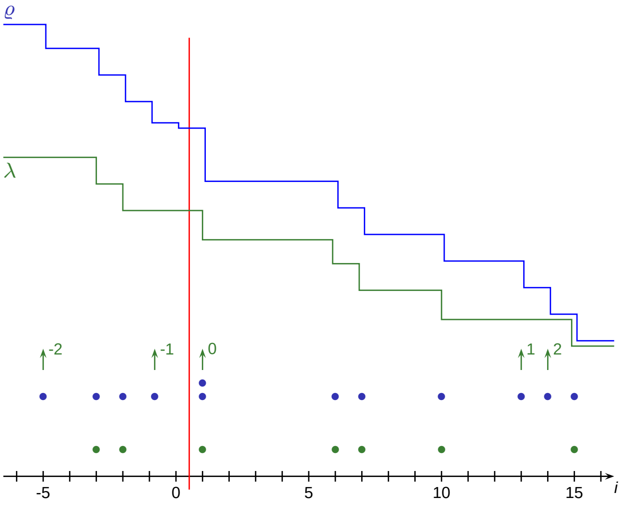

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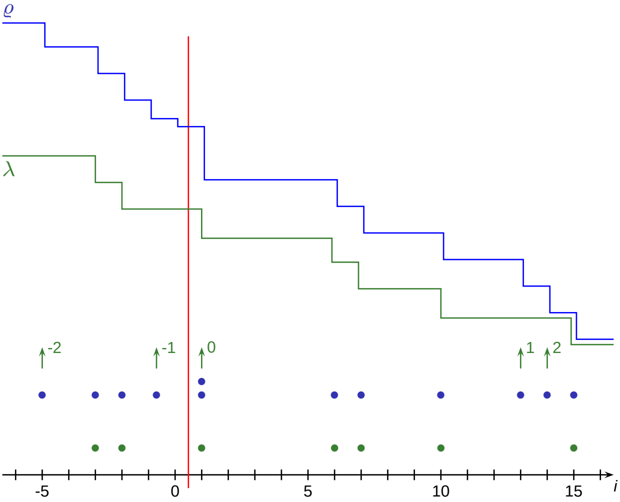

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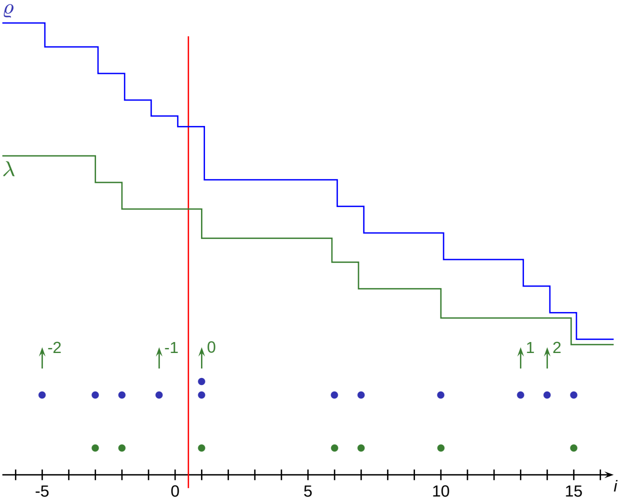

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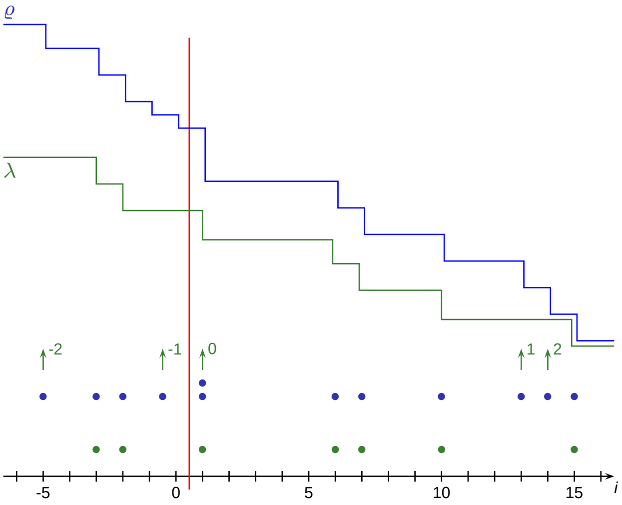

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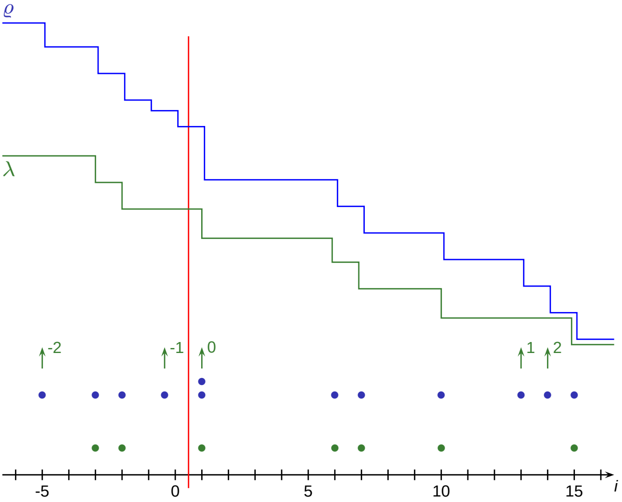

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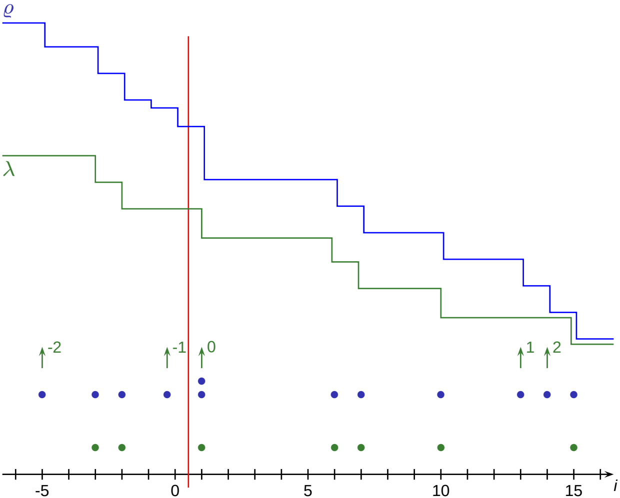

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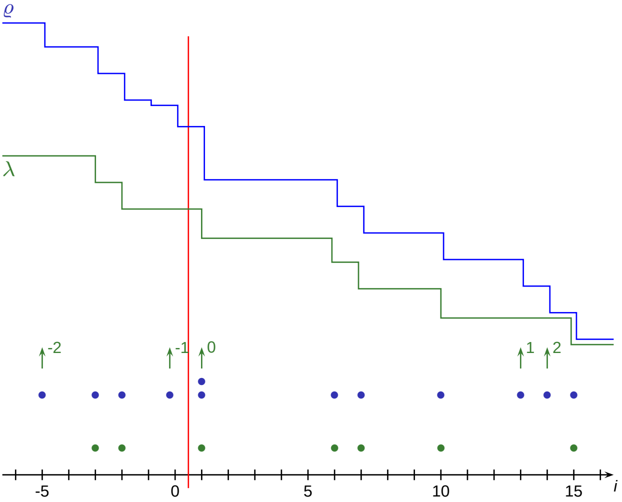

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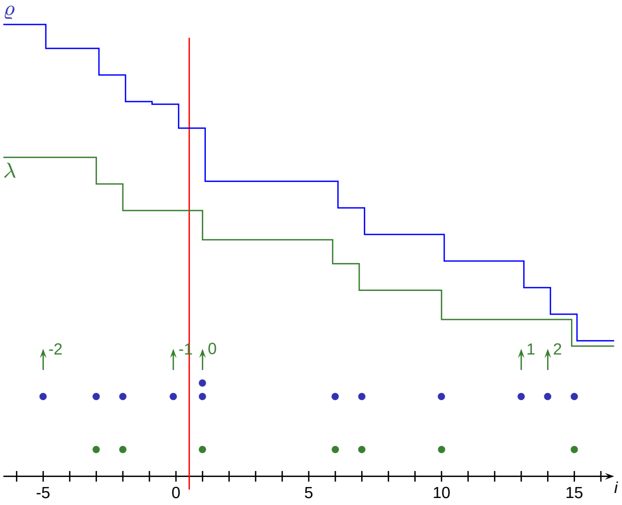

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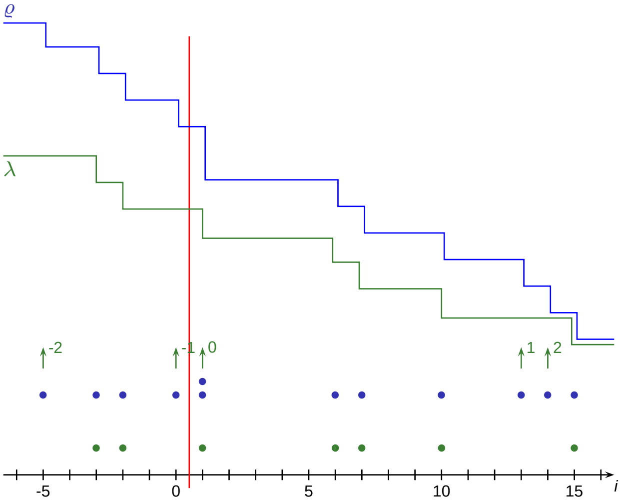

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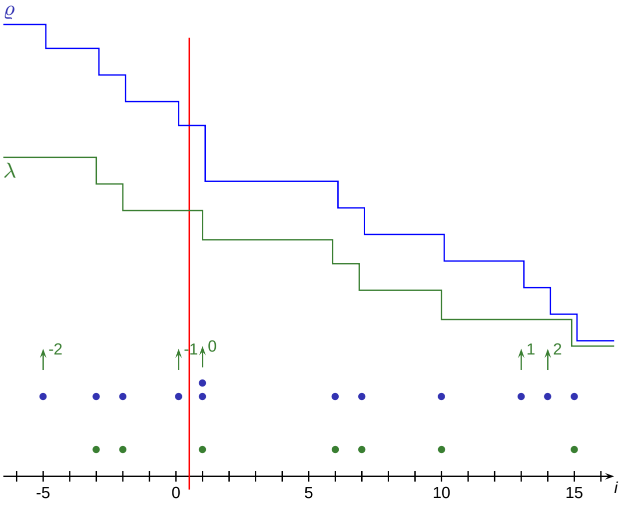

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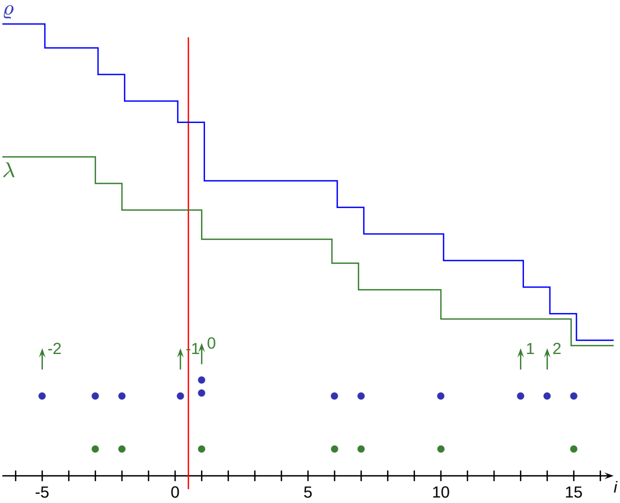

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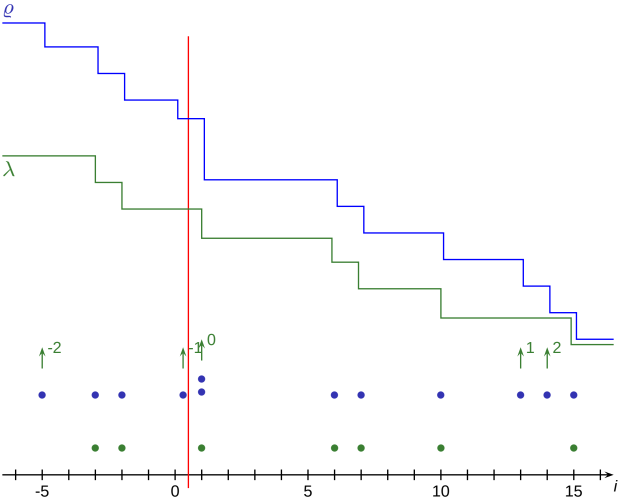

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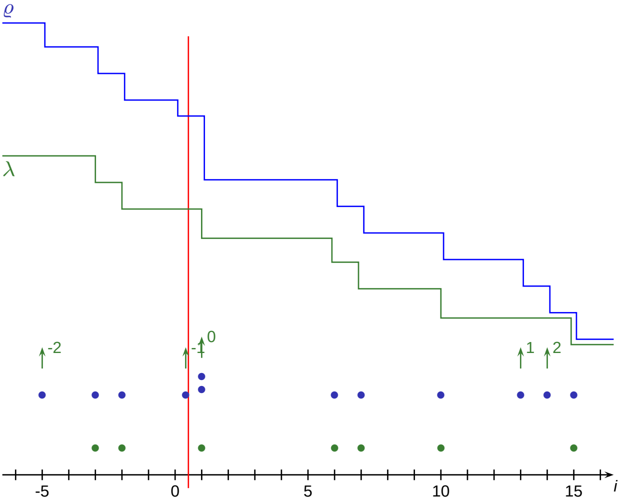

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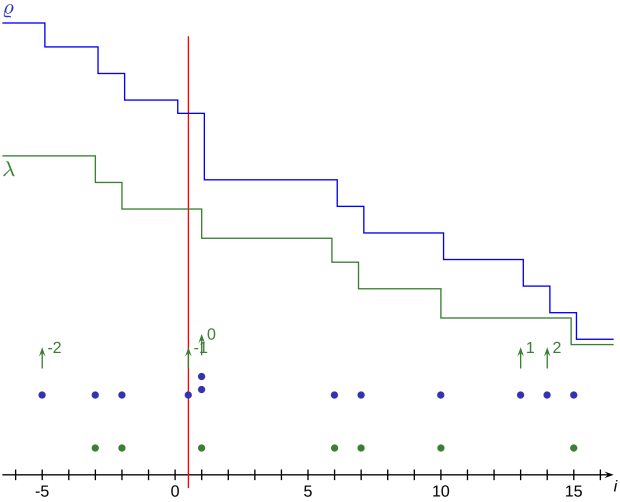

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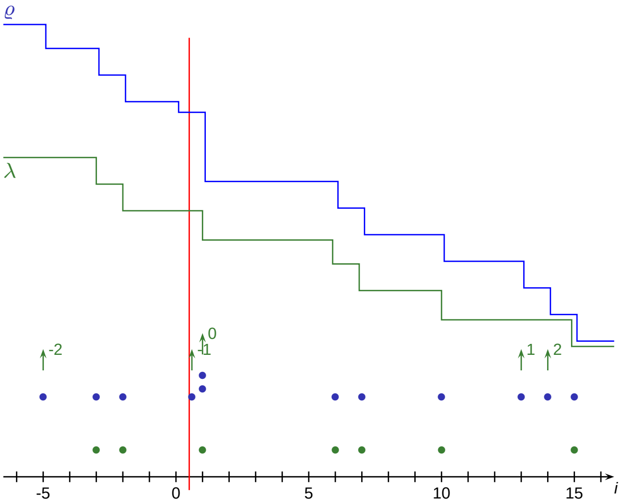

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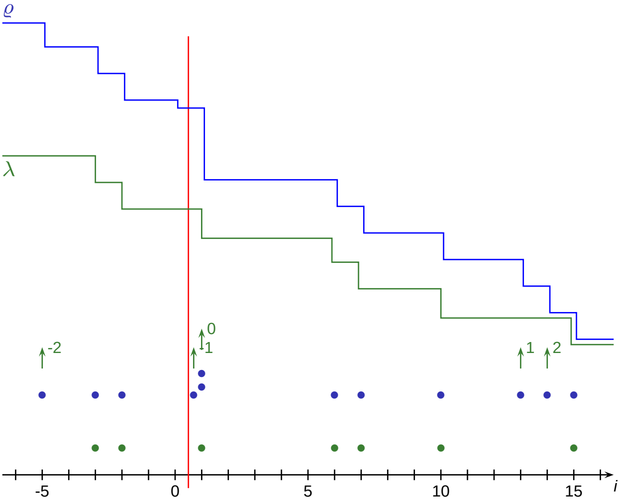

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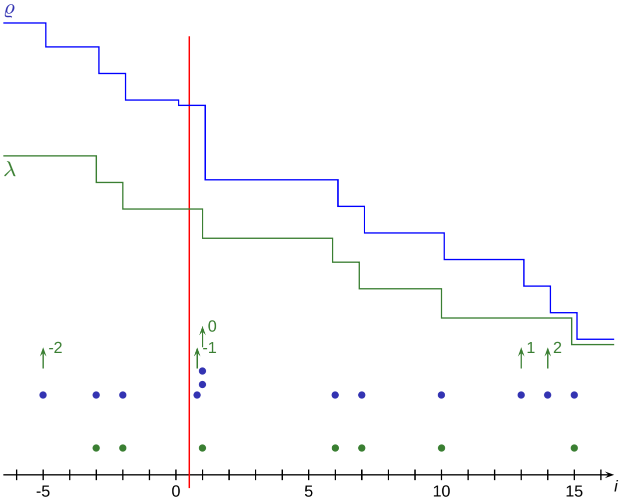

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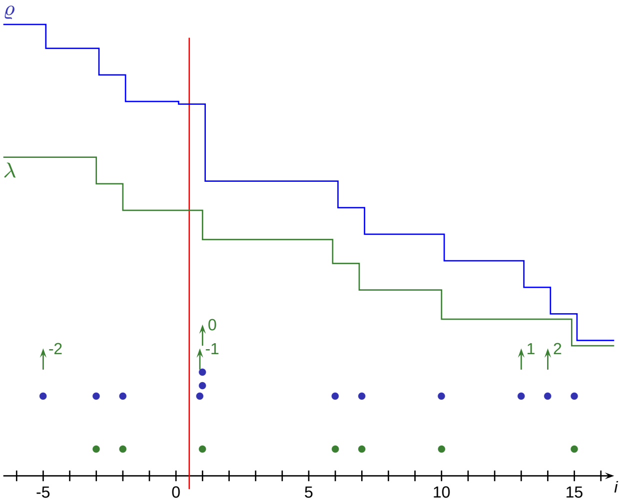

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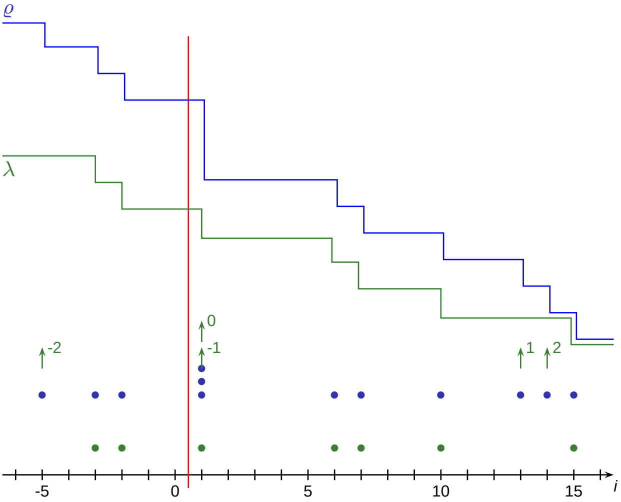

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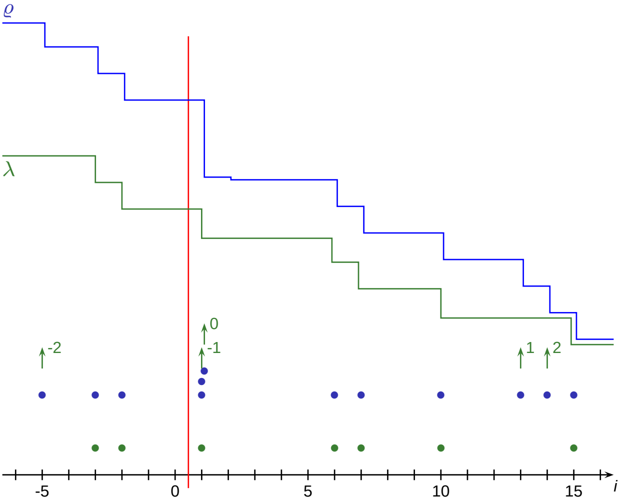

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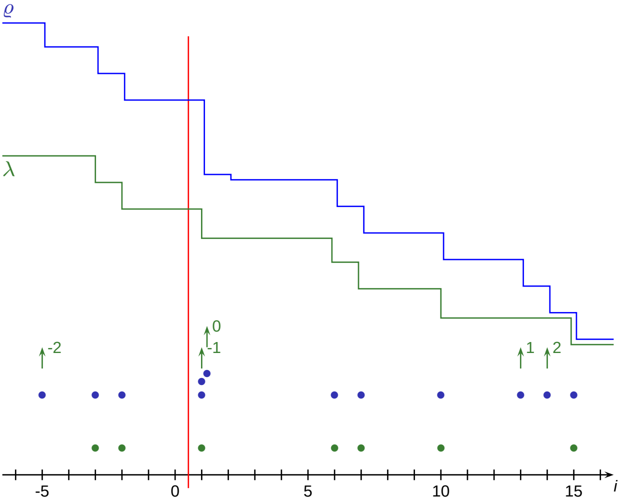

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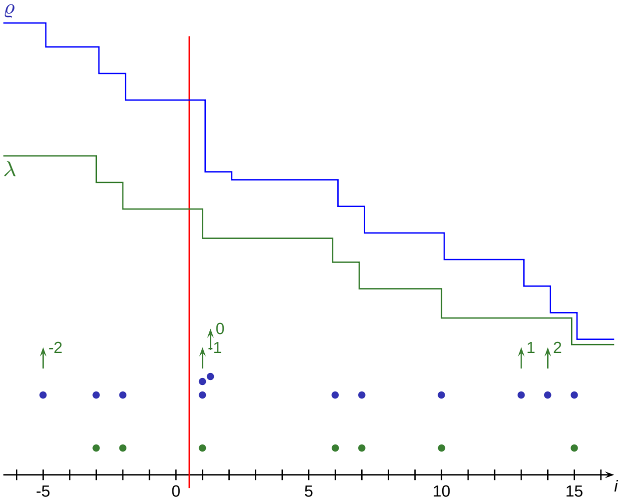

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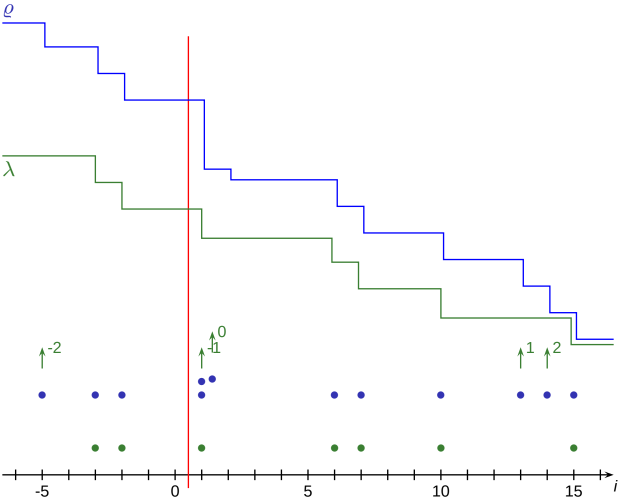

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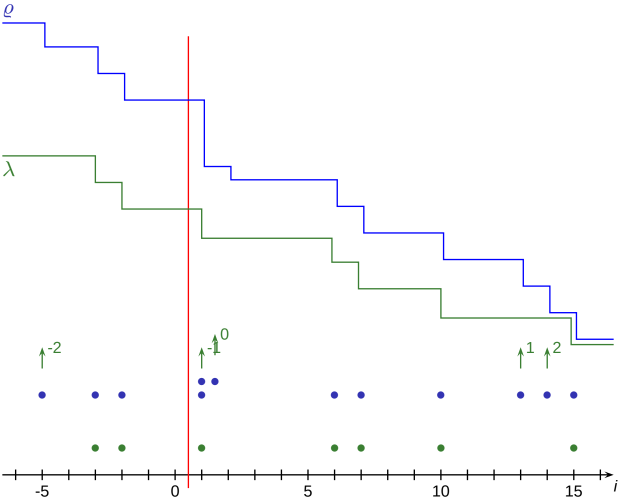

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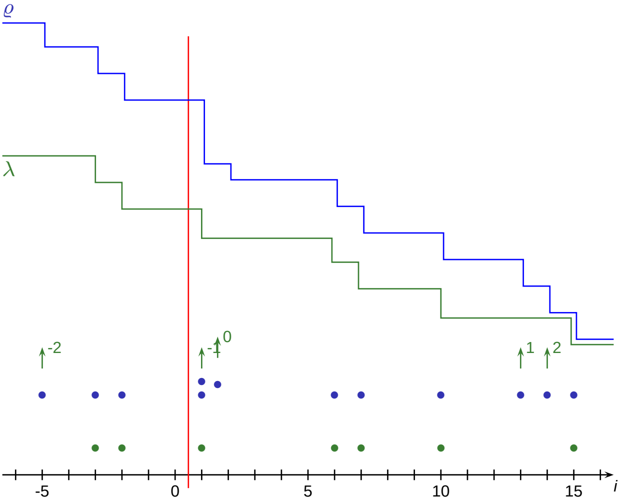

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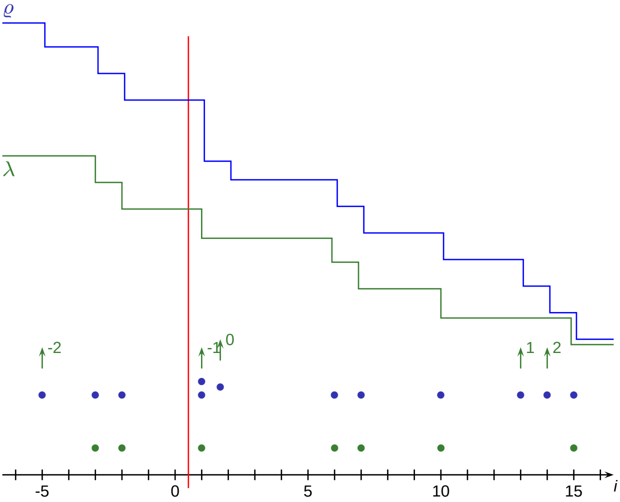

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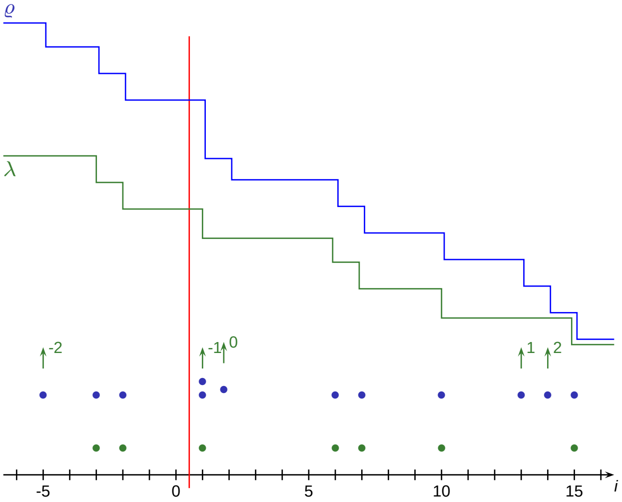

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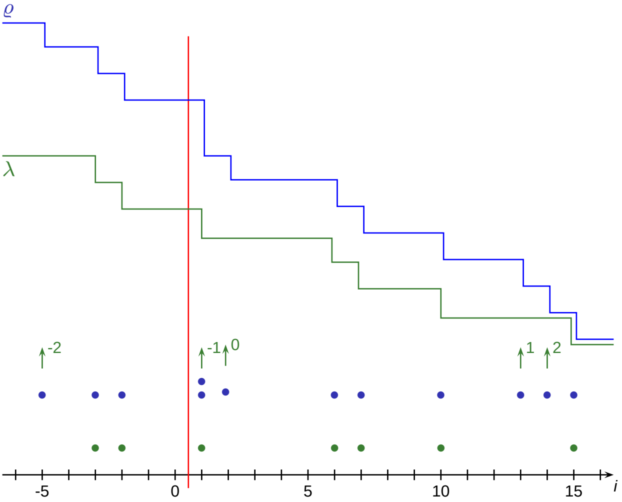

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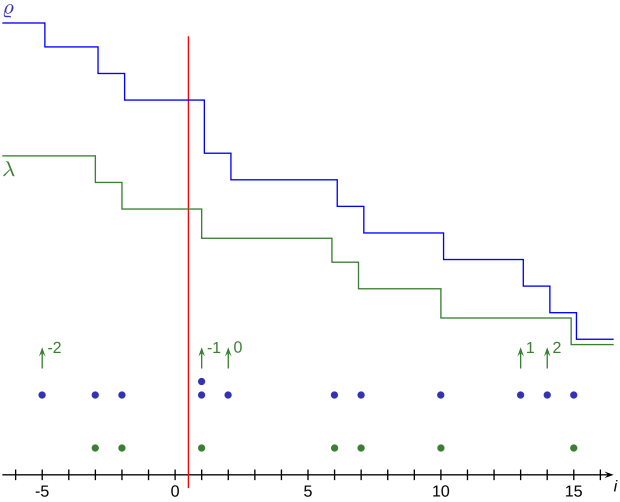

Second class particle current: difference in growth.

Upper bound (concave case)

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Centering $h_{Ct}(t) - h_{Ct}(t)$ brings in a second-order Taylor-expansion of $H(\varrho)$.

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Centering $h_{Ct}(t) - h_{Ct}(t)$ brings in a second-order Taylor-expansion of $H(\varrho)$. This is another point where concavity of the flux matters.

Upper bound (concave case)

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The computations result in (remember $\mathbf{E}(Q(t)) = Ct$)

$$\mathbf{P}\{Q(t) - Ct \geq u\} \leq c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{Ct}(t)).$$

Upper bound

Theorem (B. - Seppäläinen; also ideas from B. Tóth, H. Spohn and M. Prähofer)

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Hence proceed with

$$\begin{aligned}\mathbf{P}\{Q(t) - Ct \geq u\} &\leq c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{Ct}(t))\\ &= c \cdot \frac{t^2}{u^4} \cdot \mathbf{E}|Q(t) - C \cdot t|.\end{aligned}$$

Upper bound

With

$$\widetilde{Q}(t) := Q(t) - Ct \qquad \text{and} \qquad E := \mathbf{E}|\widetilde{Q}(t)|,$$

we have (with a similar lower deviation bound)

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Claim: this already implies the $t^{2/3}$ upper bound:

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$$\begin{aligned} \mathbf{Var}(h_{Ct}(t)) &\overset{\text{Thm}}{=} \text{const.} \cdot \mathbf{E}|Q(t) - Ct| \\ &= \text{const.} \cdot E \le c \cdot t^{2/3}. \end{aligned}$$

■

Lower bound

In the upper bound, the relevant orders were

$$u \text{ (deviation of } Q(t)) \sim t^{2/3}, \qquad \varrho - \lambda \sim t^{-1/3}.$$

The lower bound works with similar arguments: compare models of which the densities differ by $t^{-1/3}$, and use connections between $Q(t), X(t)$ and heights.

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The critical feature in both the upper bound and lower bound was microscopic convexity/concavity: $Q(t) \geq X(t)$ (convex) or $Q(t) \leq X(t)$ (concave).

Microscopic convexity/concavity

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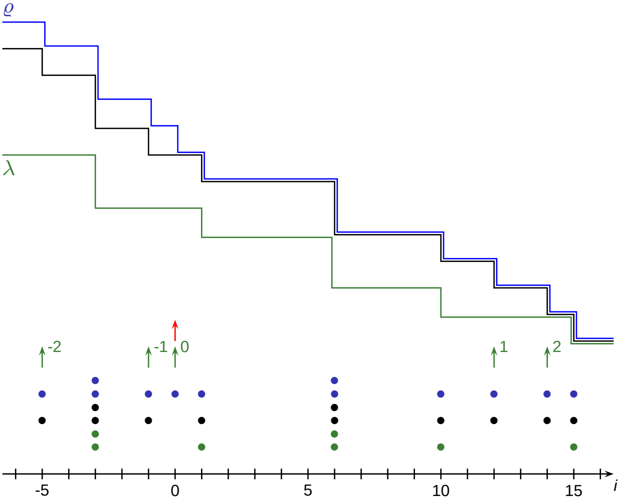
The critical feature: microscopic concavity

$Q(t) \leq X(t)+$tight error



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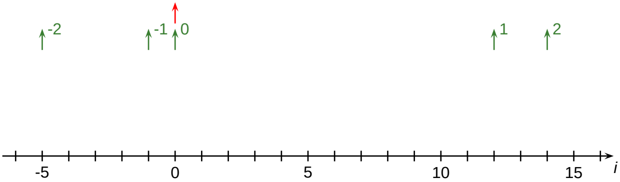

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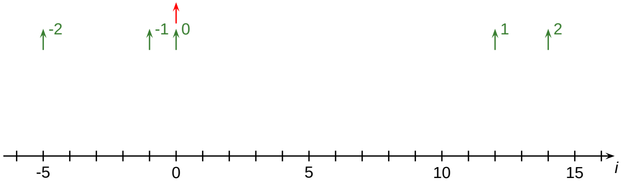

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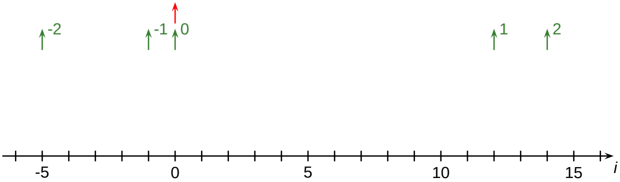

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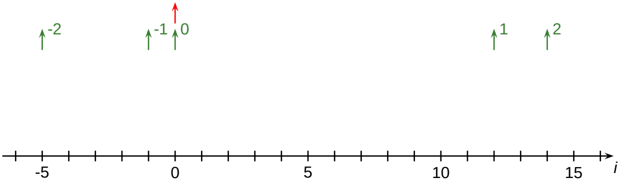

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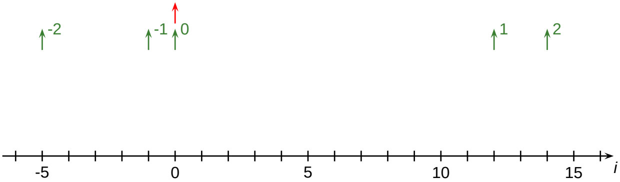

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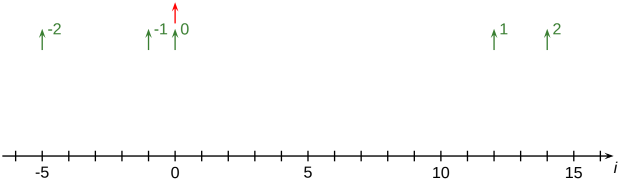

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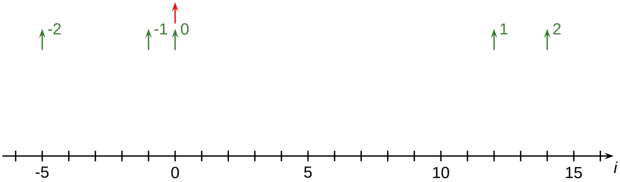

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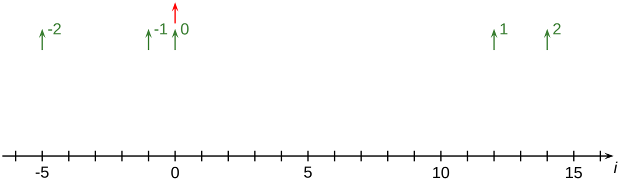

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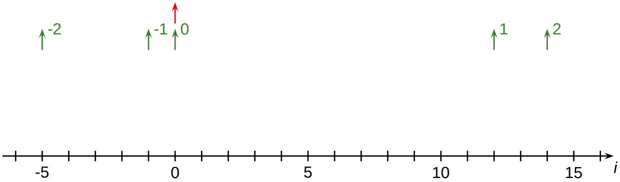

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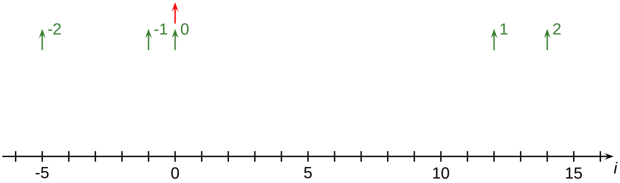

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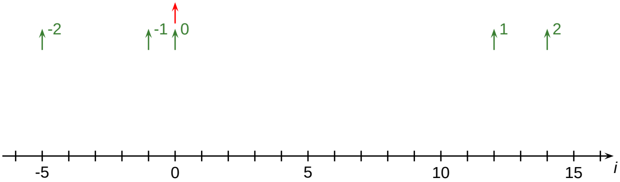

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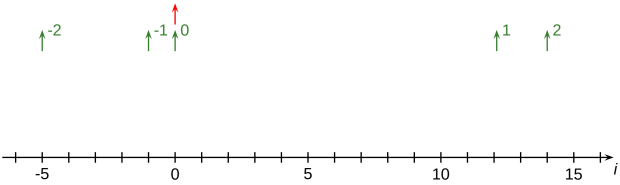

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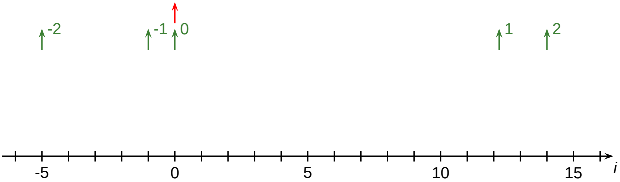

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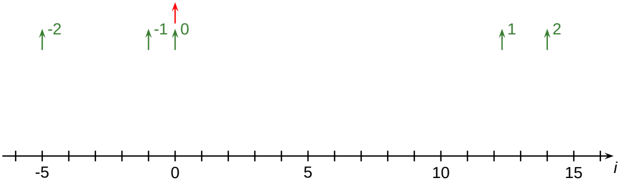

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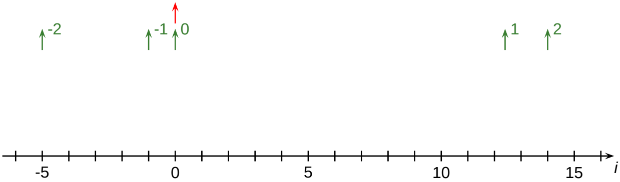

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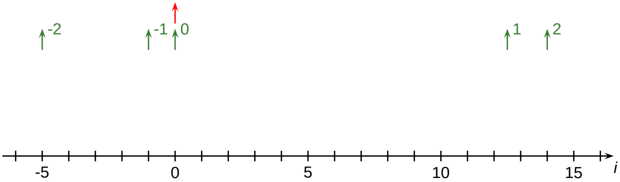

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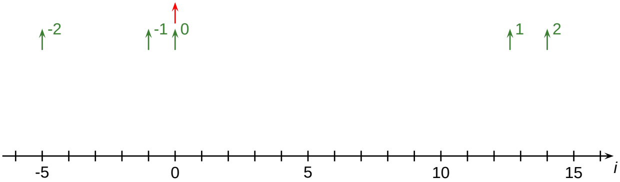

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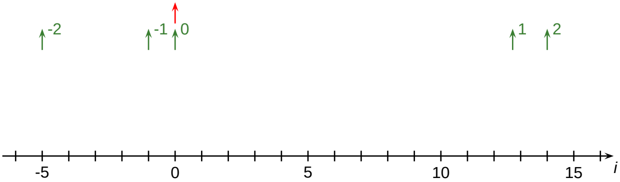

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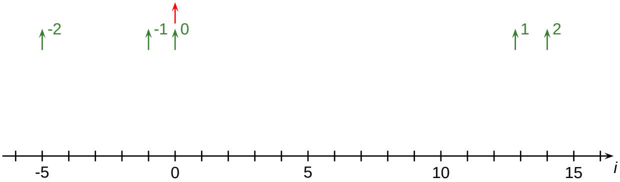

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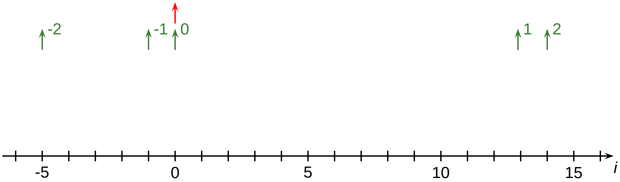

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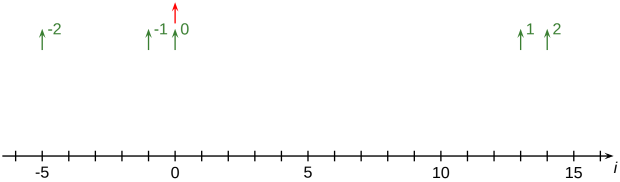

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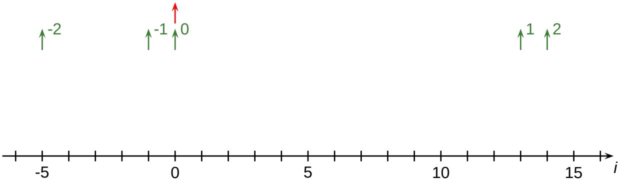

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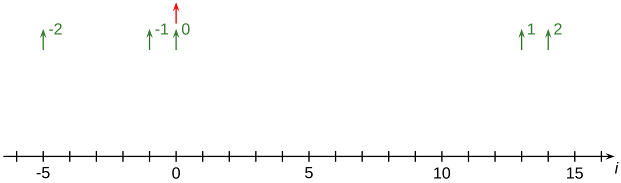

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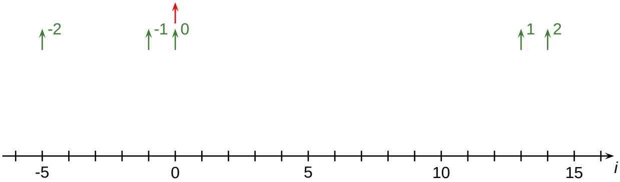

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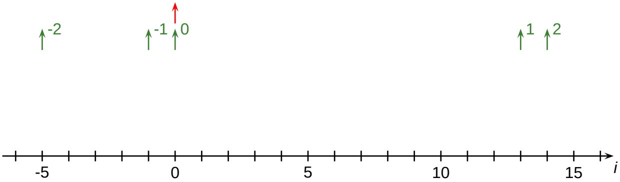

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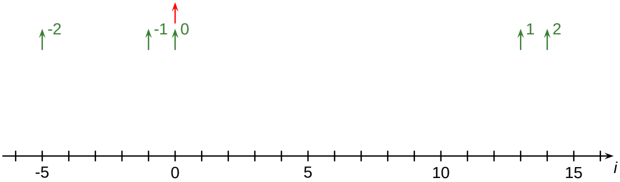

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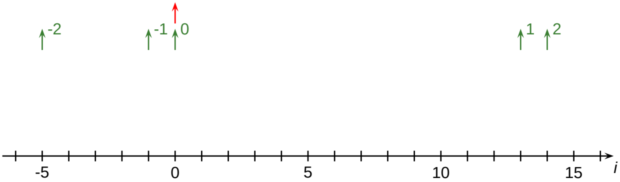

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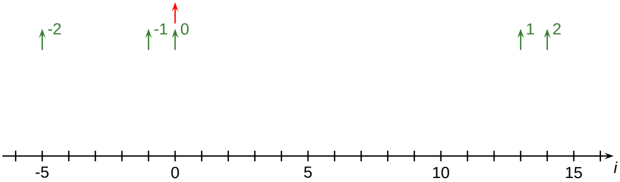

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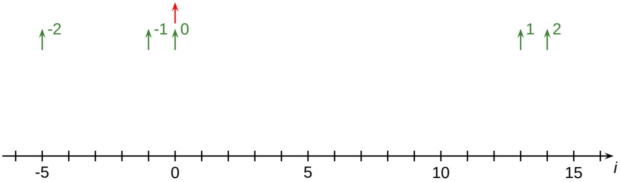

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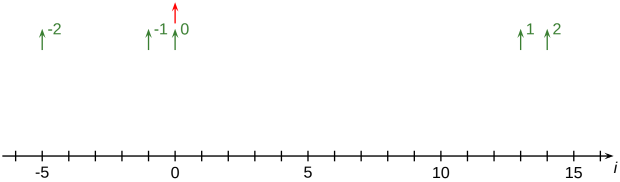

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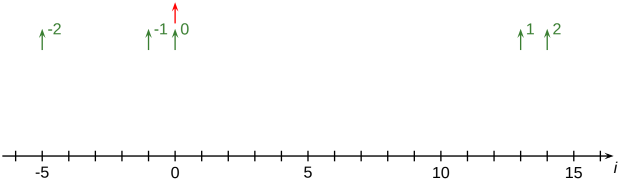

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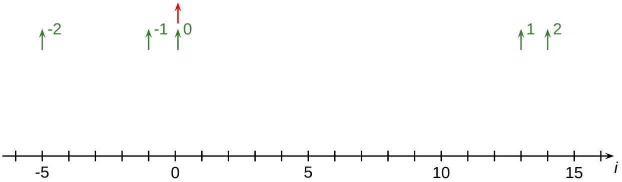

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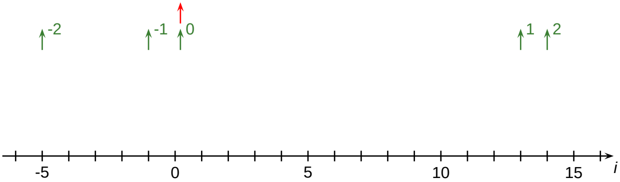

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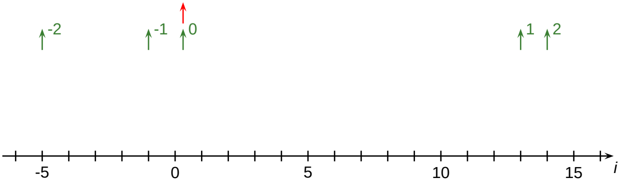

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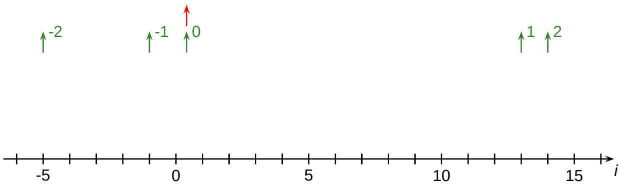

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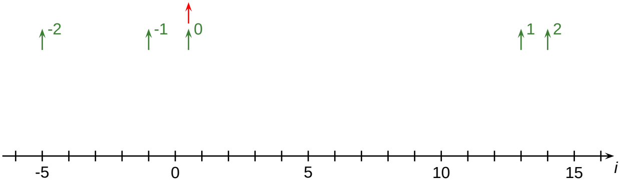

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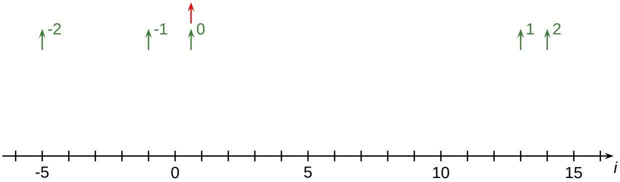

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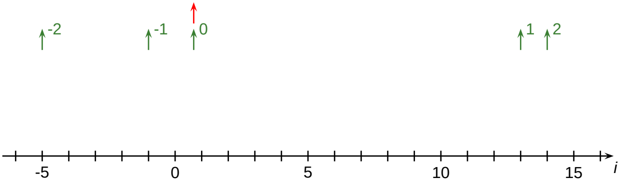

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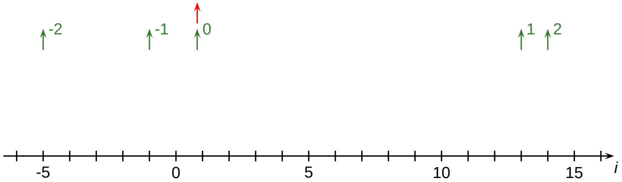

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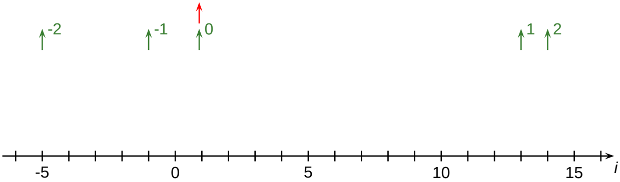

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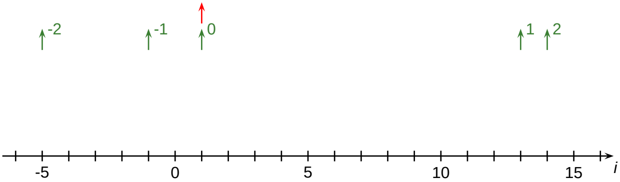

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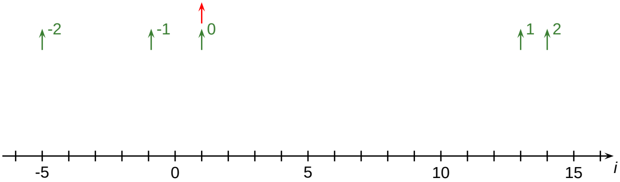

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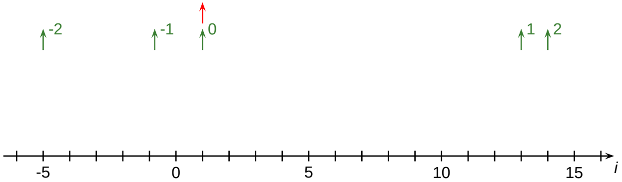

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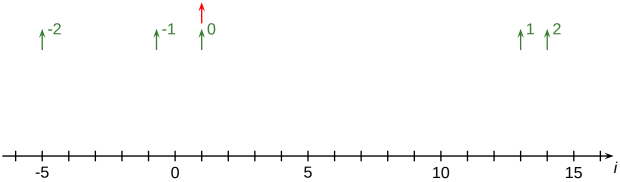

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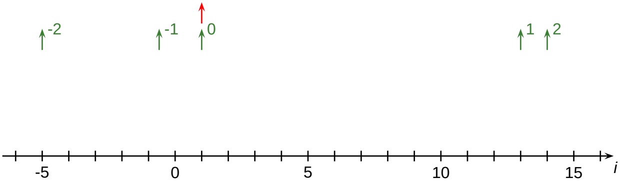

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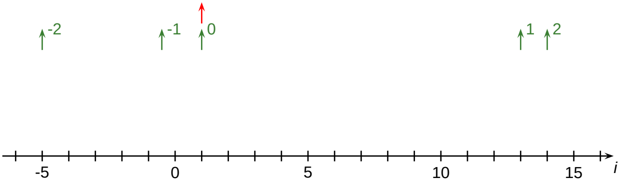

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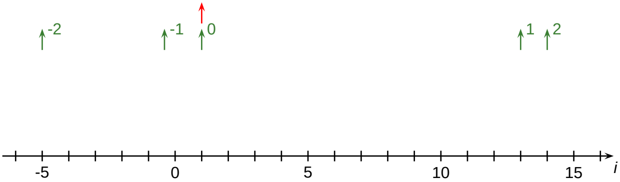

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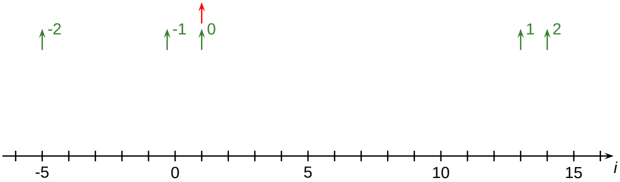

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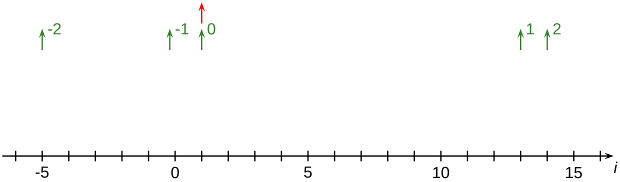

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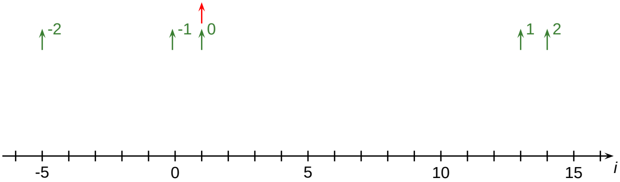

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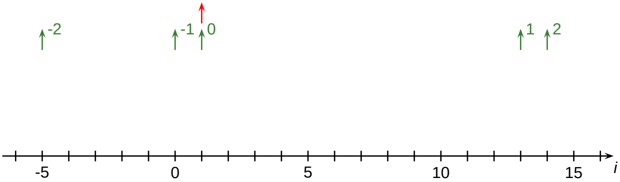

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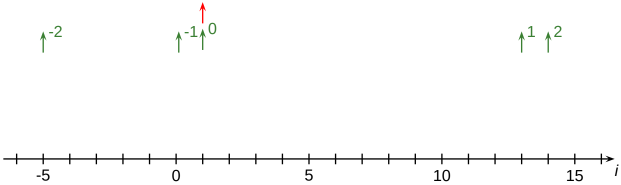

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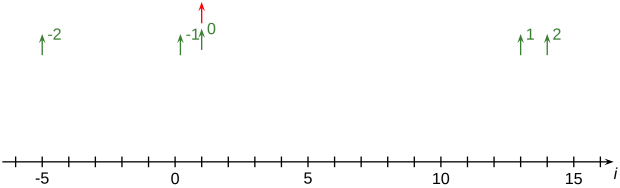

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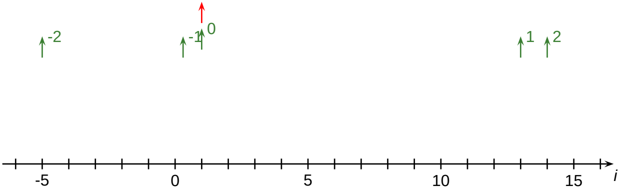

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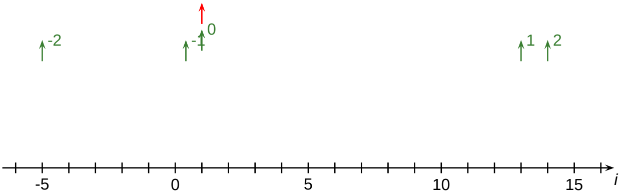

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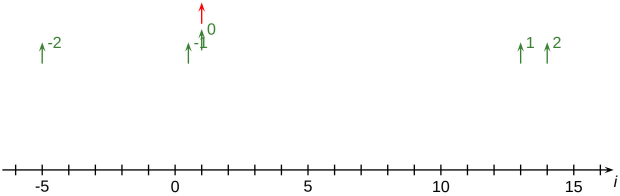

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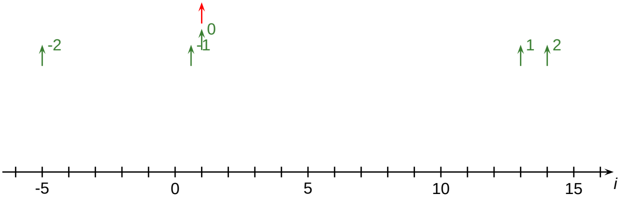

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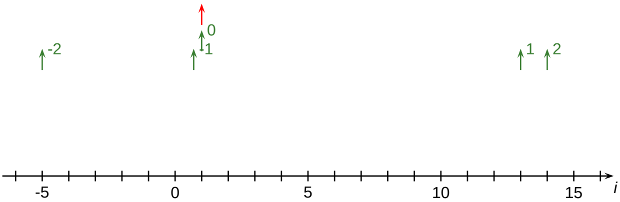

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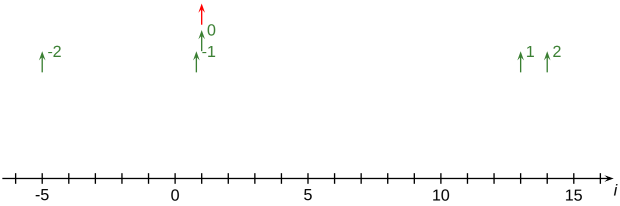

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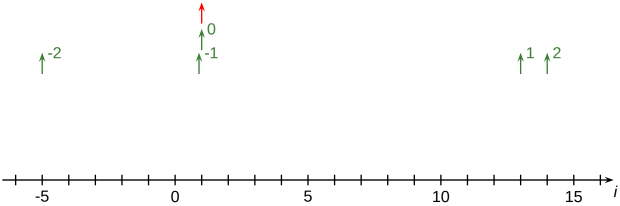

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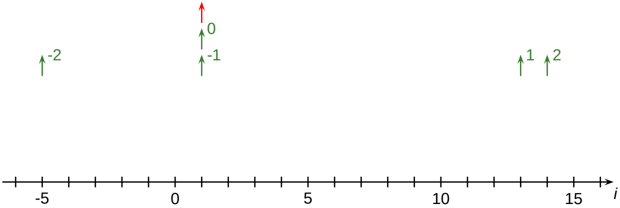

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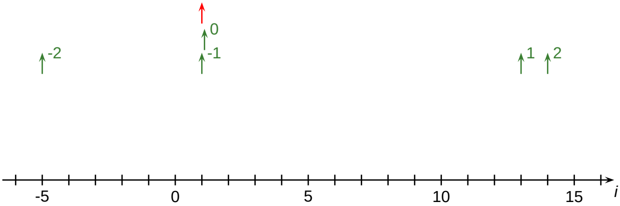

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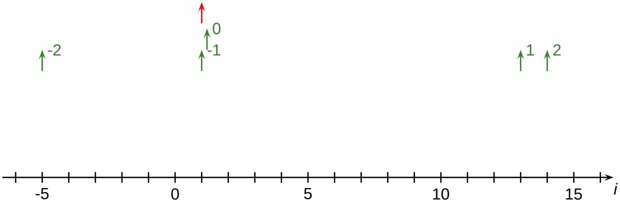

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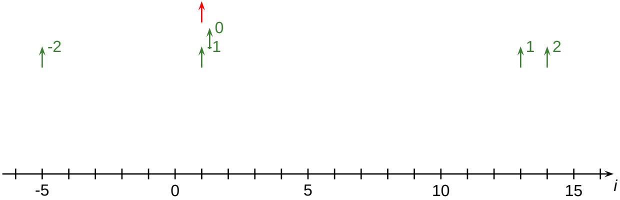

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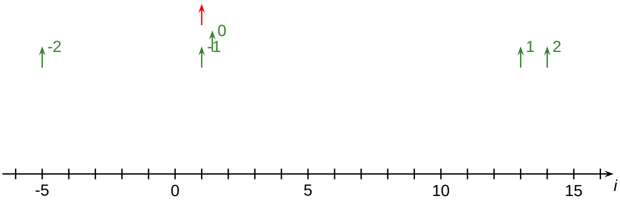

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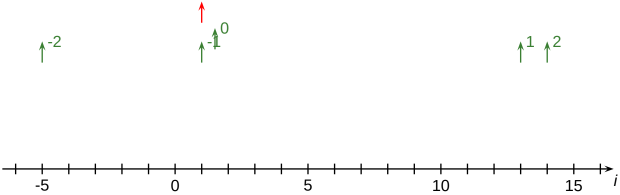

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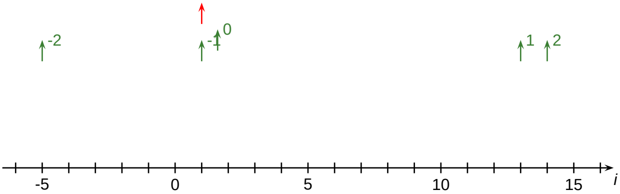

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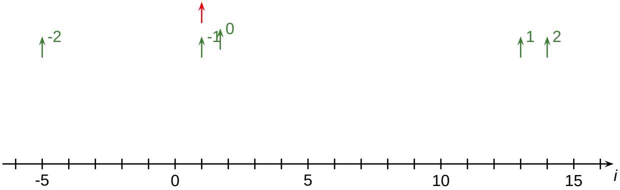

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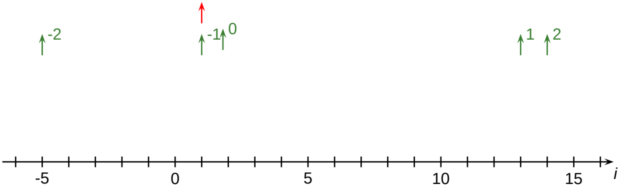

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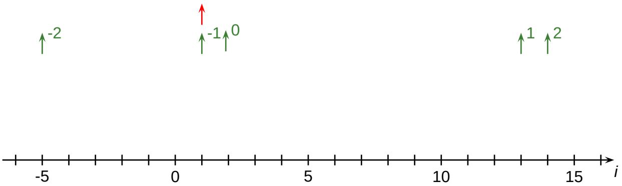

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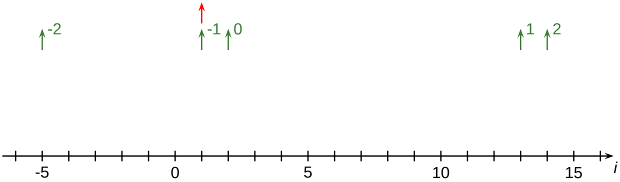

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This is the form of microscopic concavity we currently use: $m_Q(t)$ is dominated by a time-independent distribution with finite variance.

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Once this is done, we could proceed with less and less convex/concave models to see how $t^{1/3}$ scaling turns to $t^{1/4}$ for linear models (random average process, linear rate AZRP)...

Thank you.