

Road layout in the KPZ class

joint with Riddhipratim Basu, Sudeshna Bhattacharjee, Karambir Das, David Harper

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University of Bristol

Groningen Probability Seminar
22nd October 2024.

A naive Poisson model

Last passage percolation

Our model

Questions

Answers

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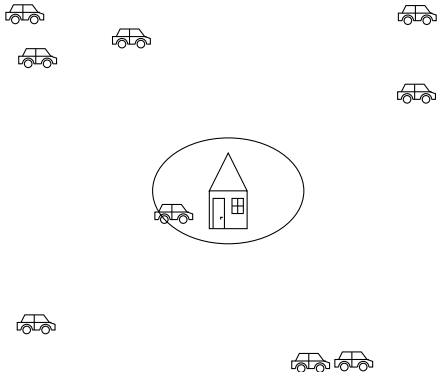
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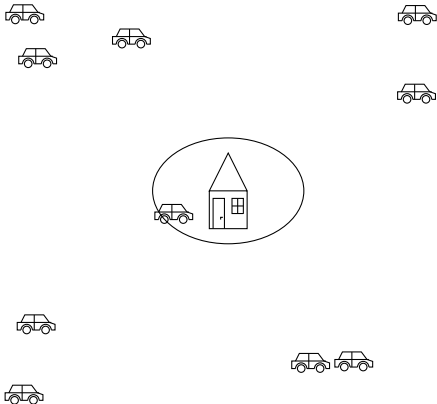
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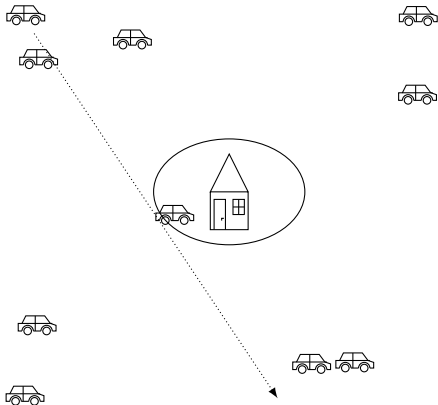
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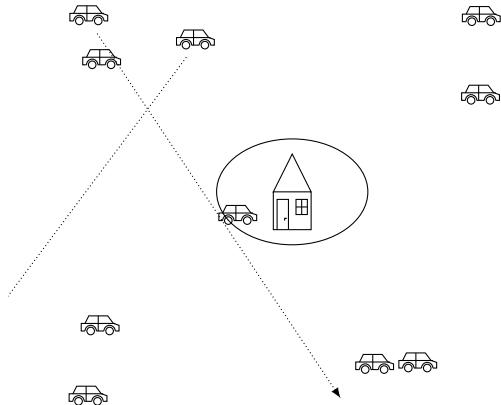
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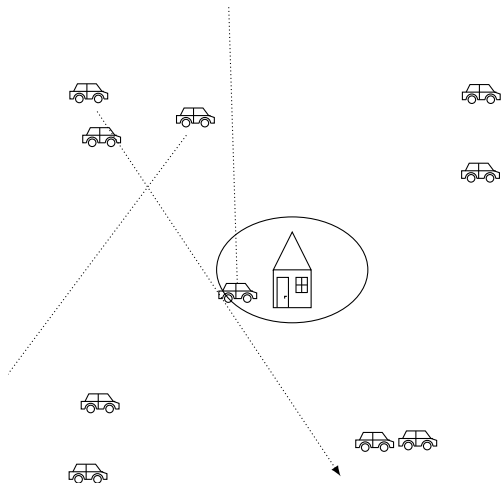
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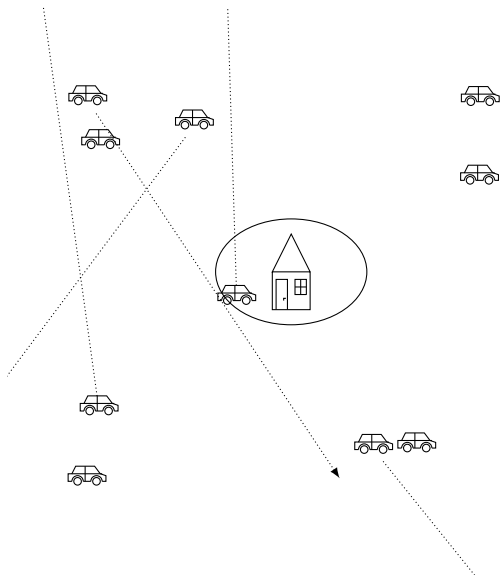
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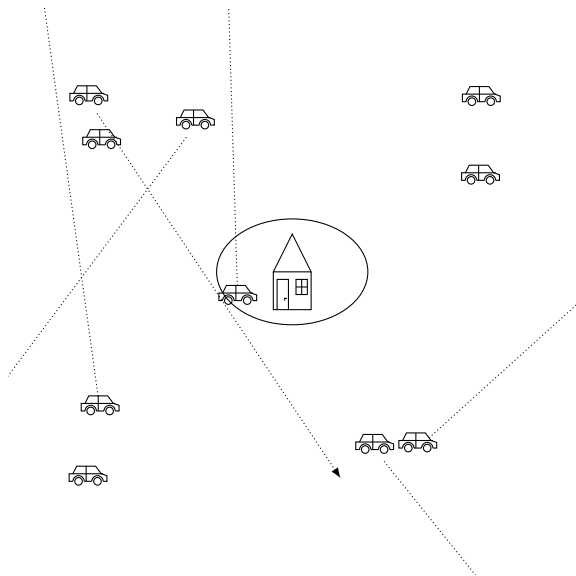
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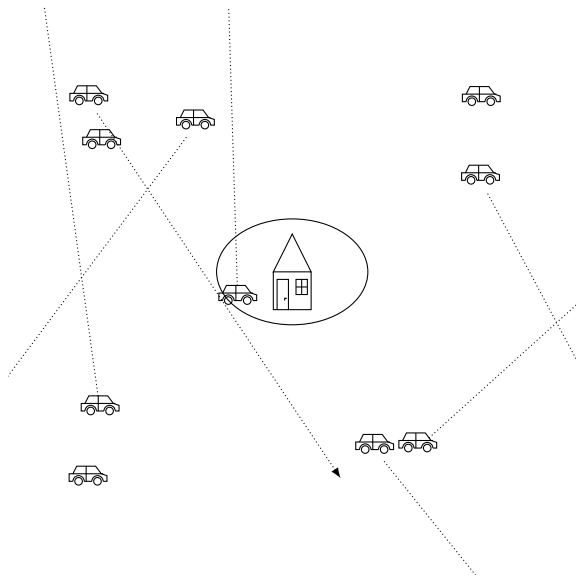
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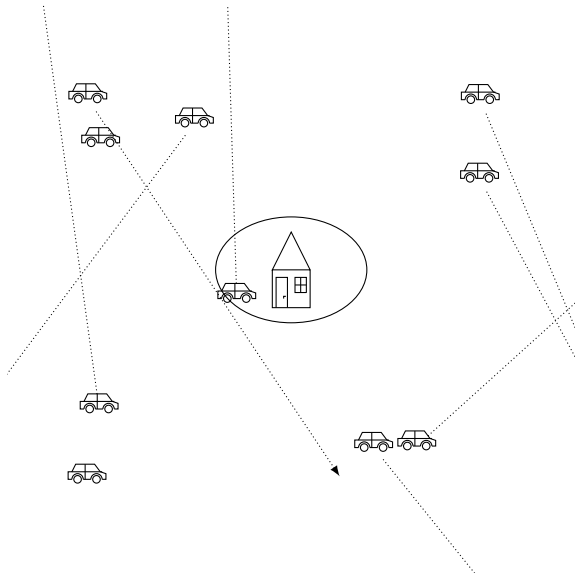
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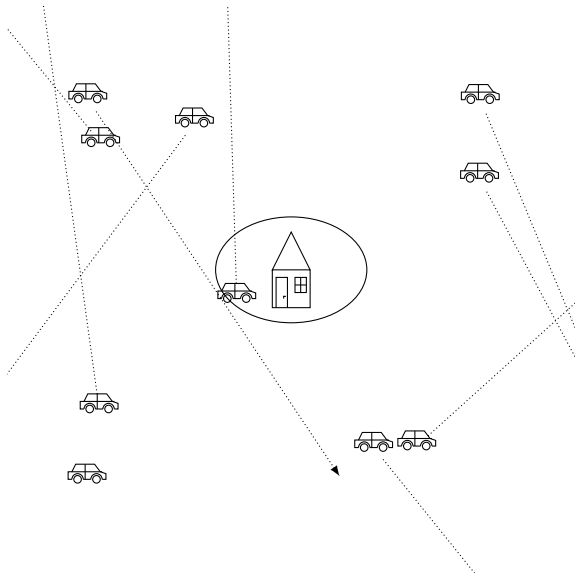
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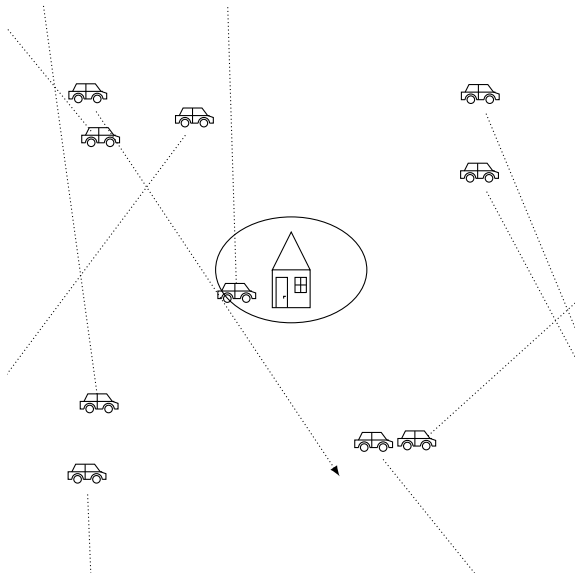
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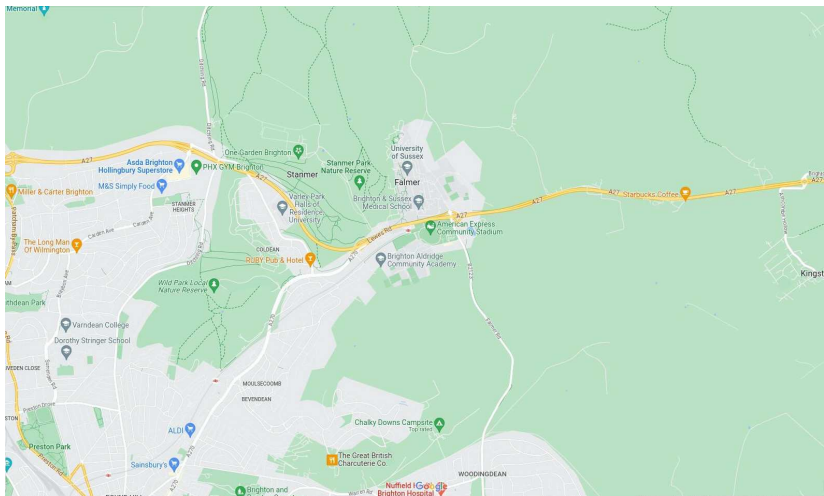
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- ▶ Unfortunately $D \gg r \dots$

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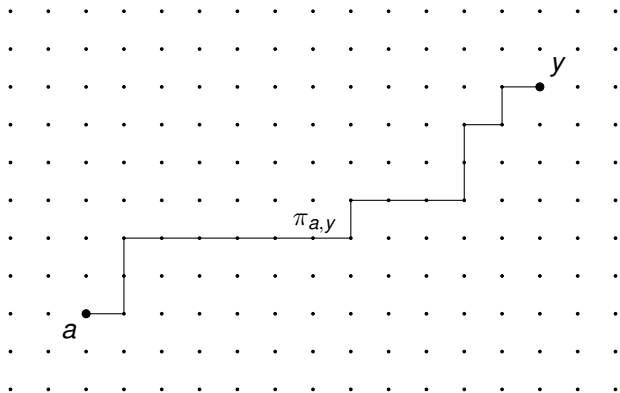
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- ▶ More tools in Exponential last passage percolation (LPP). Should behave similarly to FPP.

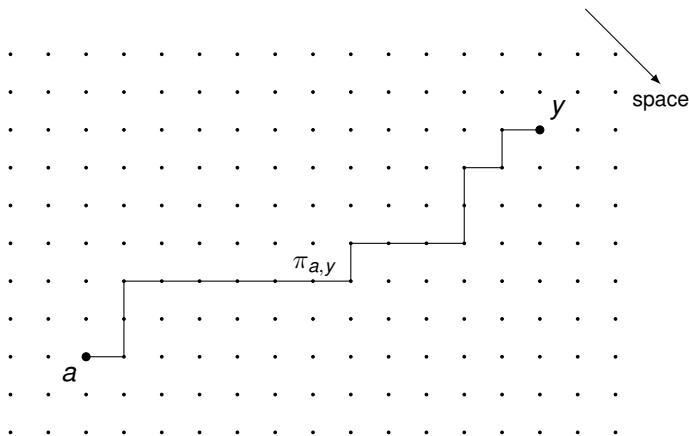
Last passage percolation

- ▶ Place ω_z i.i.d. $\text{Exp}(1)$ for $z \in \mathbb{Z}^2$.
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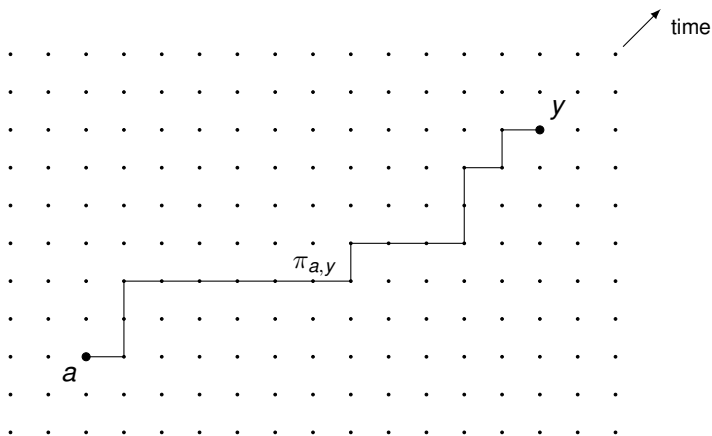
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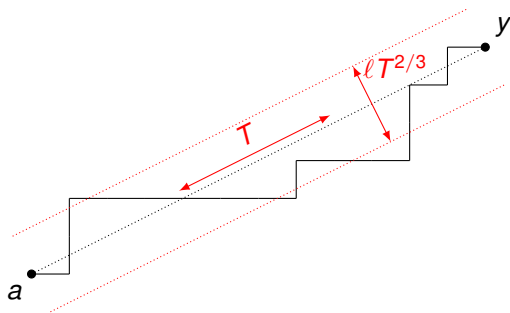


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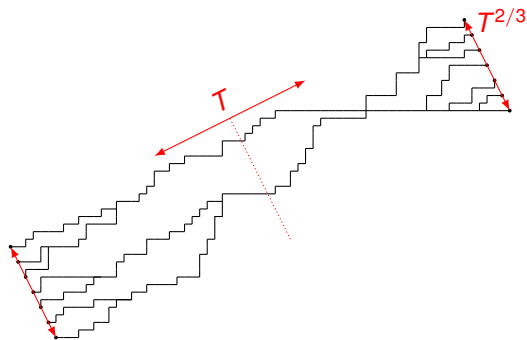
Last passage percolation: properties



$\mathbb{P}\{\text{geodesic exits width } lT^{2/3}\} \leq \text{const} \cdot e^{-Cl^3}$ [Basu, Sarkar, Sly '19;
Busani, Ferrari '22]

(KPZ transversal fluctuations).

Last passage percolation: properties



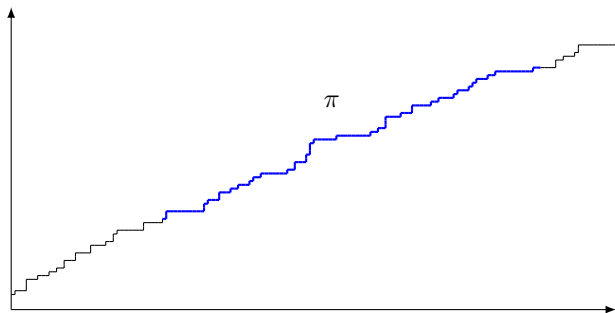
$$\mathbb{P}\{\text{more than } \ell \text{ geodesics at mid-line}\} \leq \text{const} \cdot e^{-C\ell^{1/128}}$$

[Basu, Hoffman, Sly '22]

(Midpoint problem).

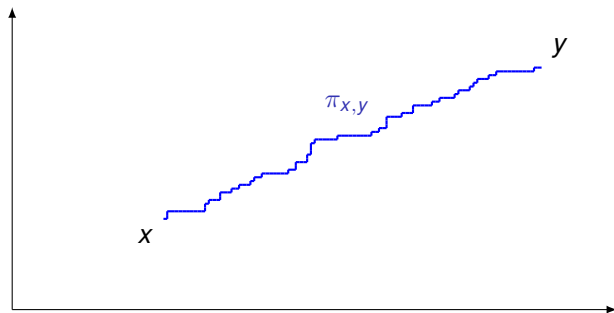
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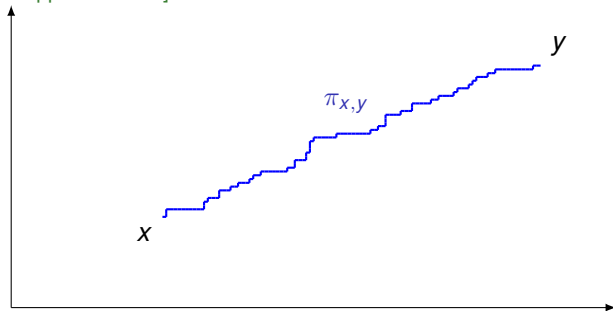
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For any fixed direction this a.s. exists and is unique. [Newman (et al) '96]; [Wüthrich '02]; [Ferrari, Pimentel '05]; [Coupier '11]; [Janjigian, Rassoul-Agha, Seppäläinen '23]



Our model

- ▶ Throw i.i.d. $\text{Exp}(1)$ weights on \mathbb{Z}^2 .

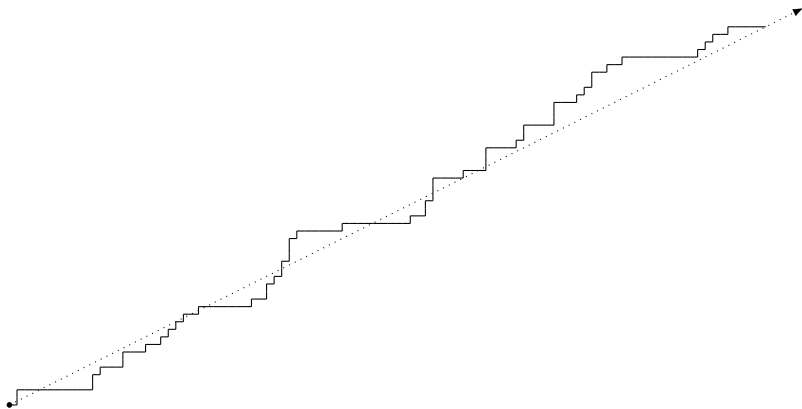
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- ▶ Throw i.i.d. $\text{Exp}(1)$ weights on \mathbb{Z}^2 .
- ▶ Give each point on \mathbb{Z}^2 $\text{Uniform}(\varepsilon, \frac{\pi}{2} - \varepsilon)$ independent angles. Cars start from everywhere, in random directions.

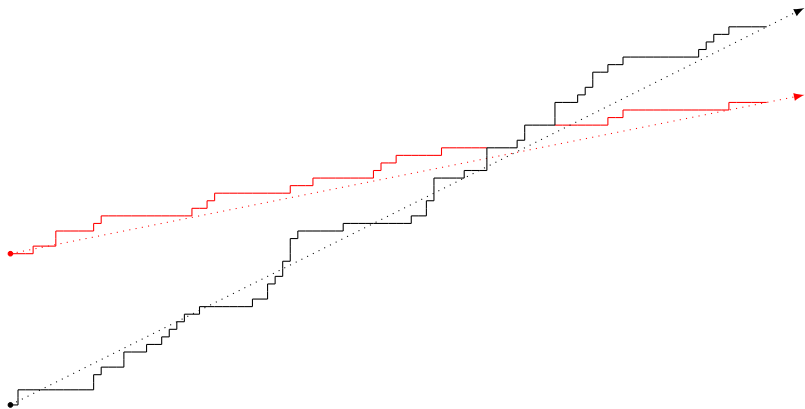
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- ▶ Draw the a.s. semi-infinite geodesic for each point to its chosen direction. Many of these will coalesce when the angles are close. That's our road map with traffic data on it. A road segment is *busy* when many geodesics use that edge.

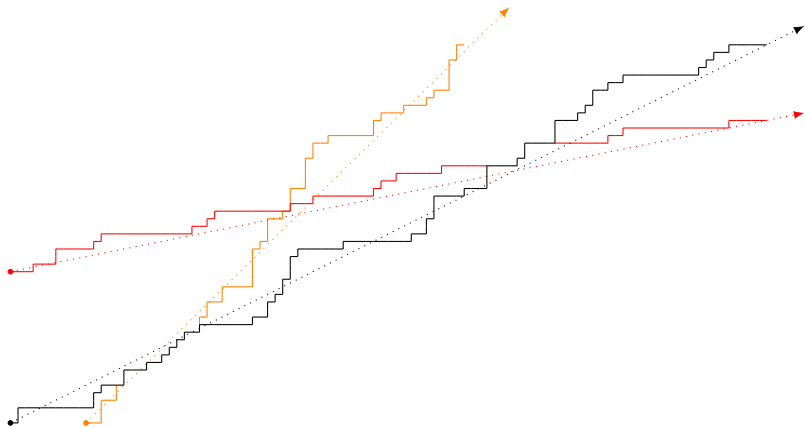
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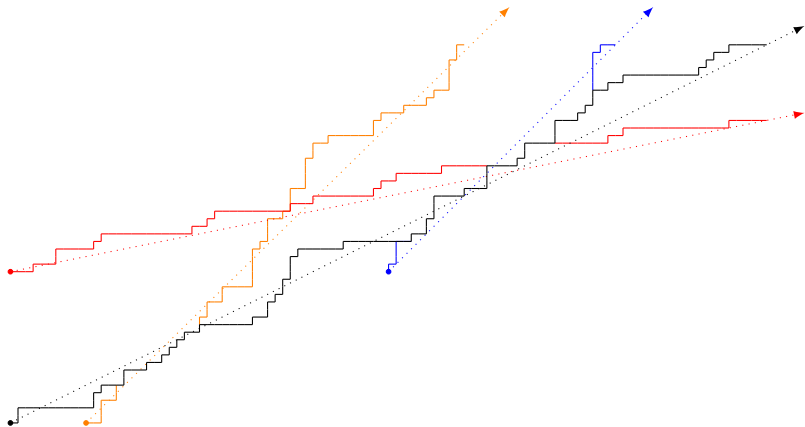
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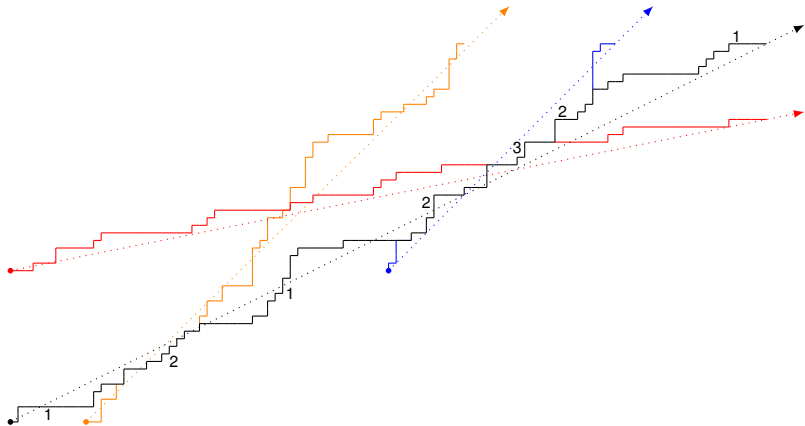
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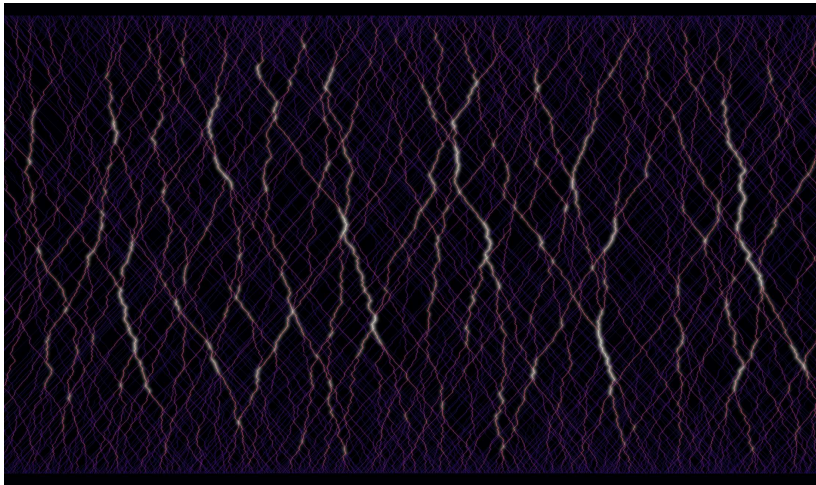
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Simulation by David Harper

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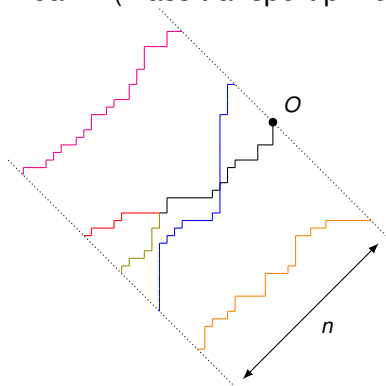
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- ▶ Is this actually a good model of real road networks out there?

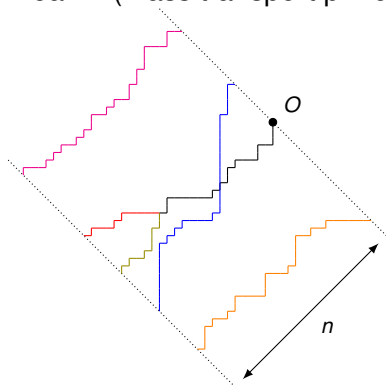
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From all layers: $N = \sum_{n=1}^{\infty} N_n$ is of infinite mean.

Answers

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$$cn^{-1/3} \leq \mathbb{P}\{\text{a car from distance } \geq n \text{ visits } O\} \leq Cn^{-1/3}.$$

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$$\frac{C}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

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With similar methods,

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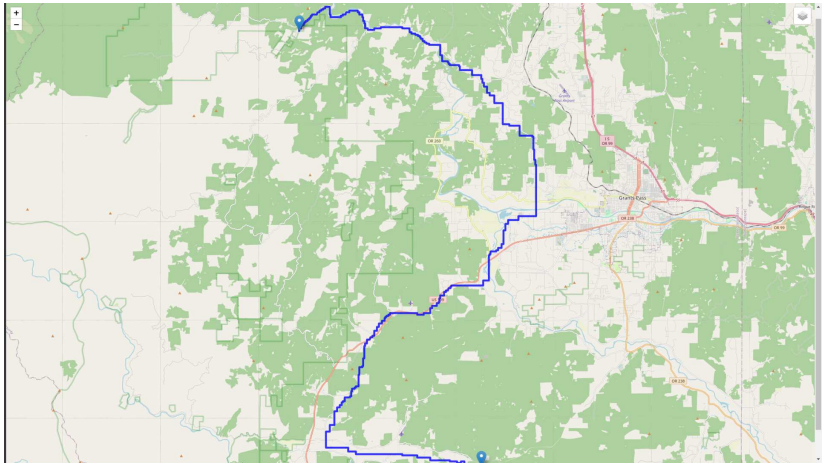
$$\mathbb{P}\left\{\text{no road with } \geq k^4 \text{ cars within distance } \frac{\delta k}{\log k}\right\} \geq 1 - C\delta.$$

We don't believe the log.

Theorem

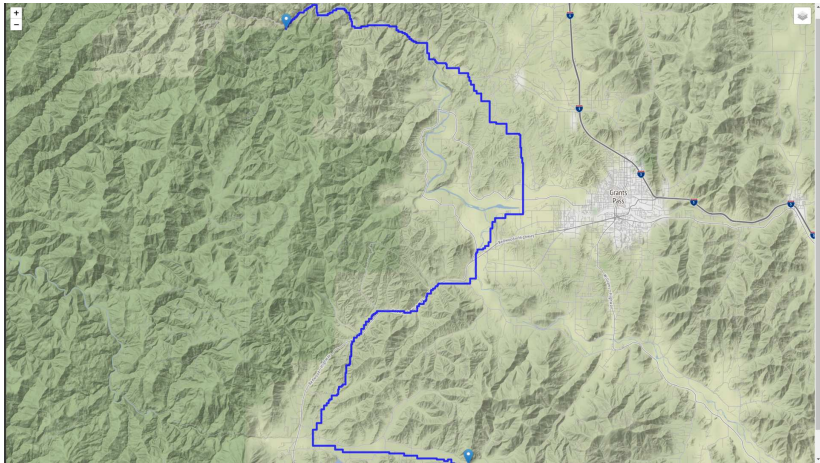
$$\mathbb{P}\{\text{yes, road with } \geq \text{const} \cdot k^4 \text{ cars within distance } k\} \geq c.$$

Is this all any good?



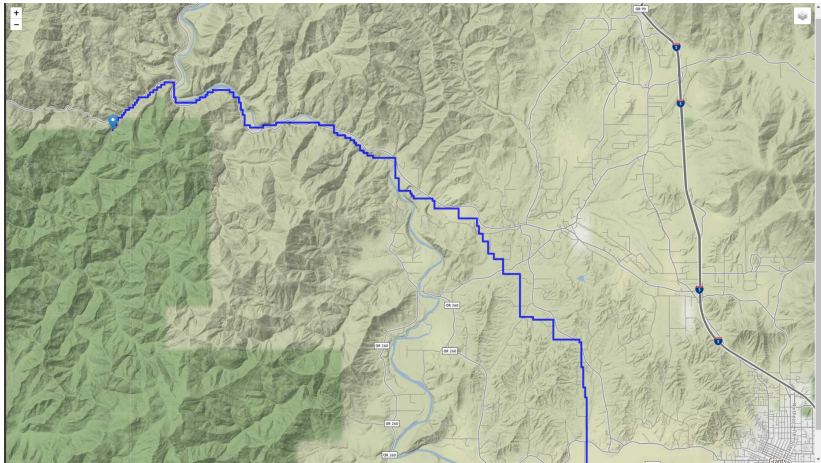
Simulation by David Harper

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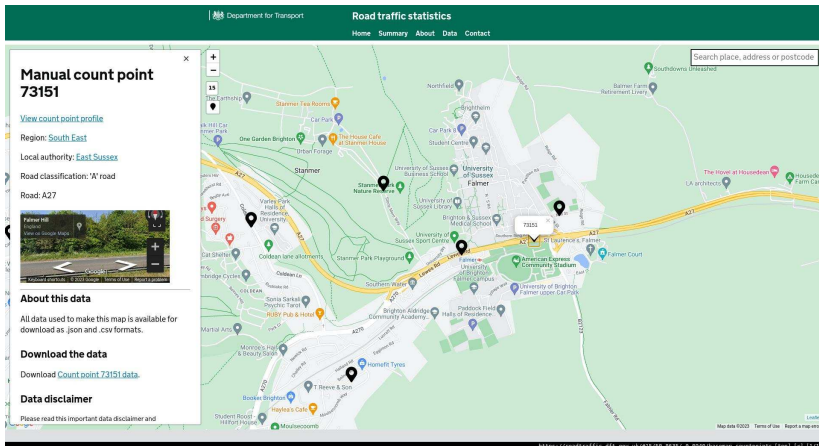
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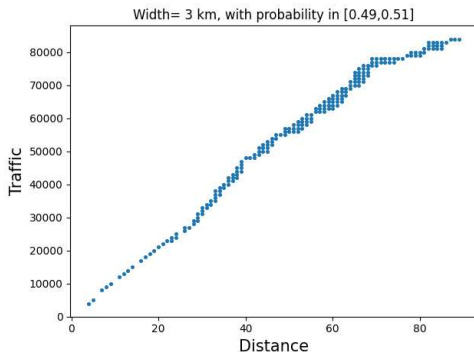


Is this all any good?



$\mathbb{P}\{\text{road with } \geq \ell \text{ cars within distance } k\} \dots ?$

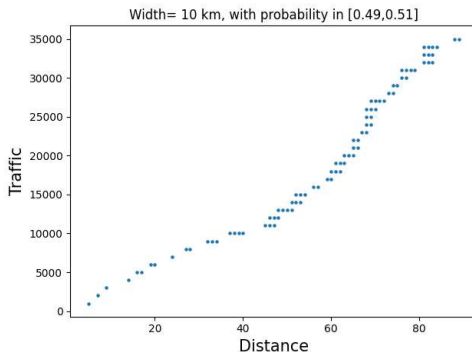
Is this all any good? On the South:



Between 49% and 51% of startpoints have at least this much traffic within the distance shown.

Thm: $\mathbb{P}\{\text{road with } \geq k^4 \text{ cars within distance } k\} \sim \mathcal{O}\left(\frac{1}{2}\right)$.

Is this all any good? On the North and the West:



Between 49% and 51% of startpoints have at least this much traffic within the distance shown.

Thm: $\mathbb{P}\{\text{road with } \geq k^4 \text{ cars within distance } k\} \sim \mathcal{O}\left(\frac{1}{2}\right)$.

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Thank you.