Ergodicity in particle systems

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University of Bristol

PAD RG, 4 June, 2025.

The models

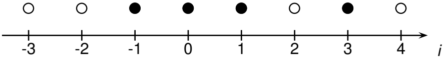
- Asymmetric simple exclusion process
- Zero range
- Bricklayers

Ergodicity

Tool: the second class particle

Proof

Asymmetric simple exclusion



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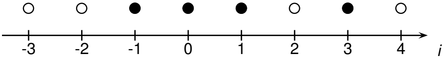

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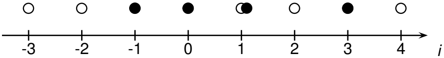

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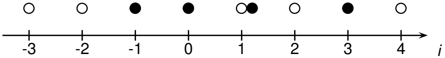

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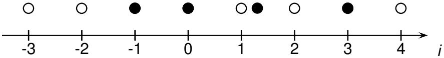

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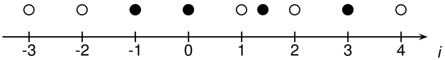

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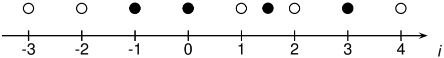

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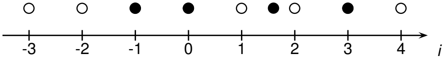

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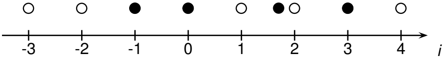

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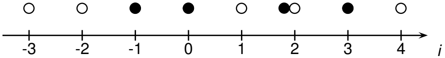

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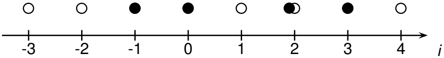

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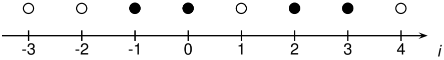

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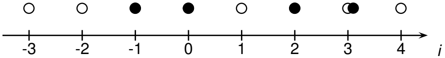

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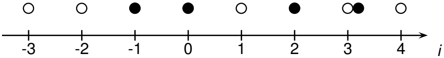

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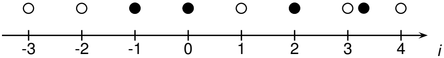

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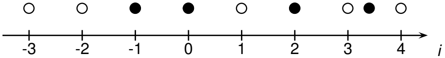

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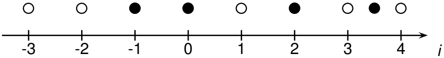

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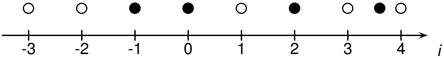

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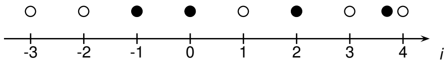

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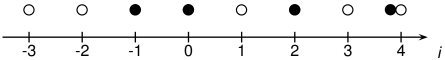

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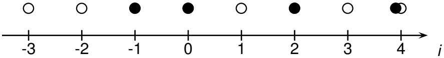

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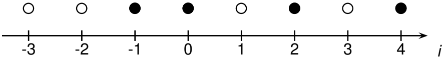

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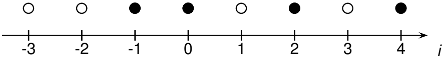

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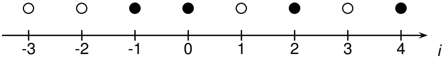

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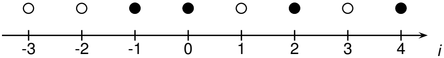

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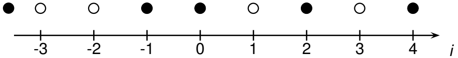

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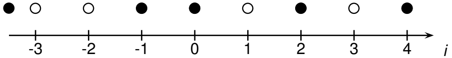

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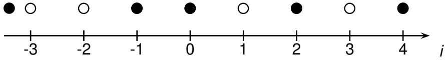

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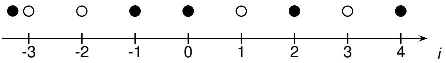

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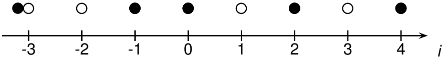

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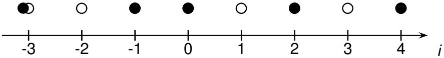

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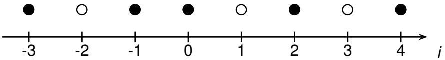

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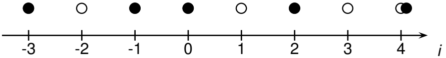

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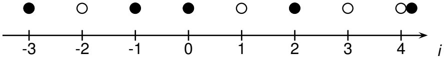

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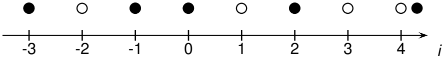

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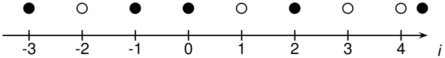

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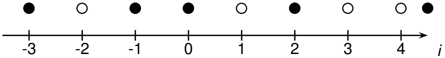

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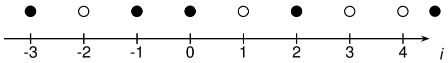

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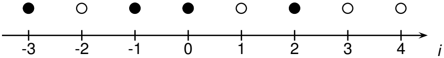

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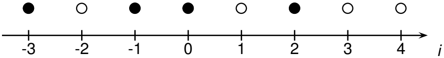

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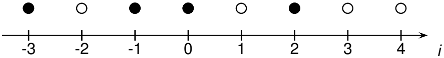

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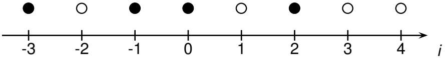

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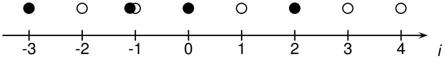

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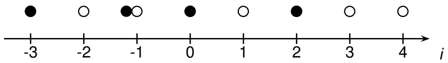

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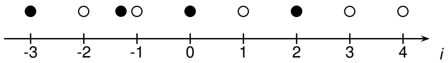

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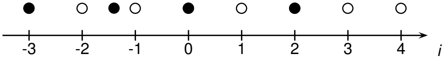

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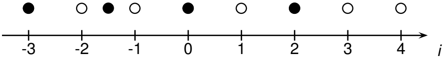

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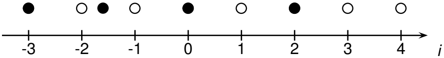

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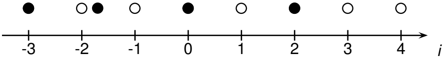

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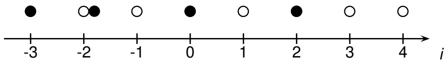

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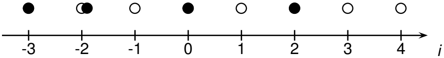

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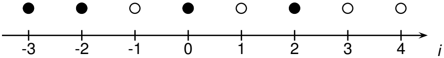

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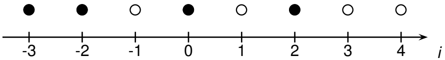
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

Asymmetric simple exclusion



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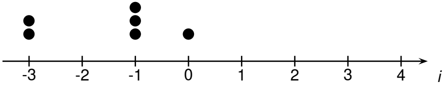
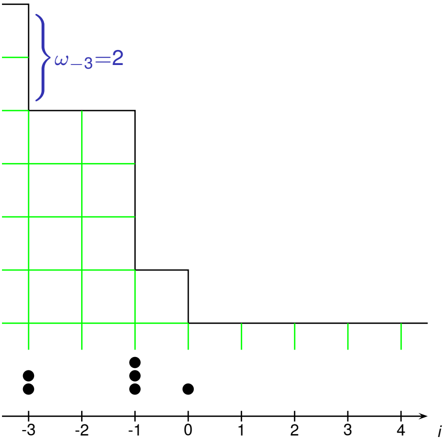
I.i.d. Bernoulli(ϱ) distribution is time-stationary for any ($0 \le \varrho \le 1$).

The asymmetric zero range process



The asymmetric zero range process



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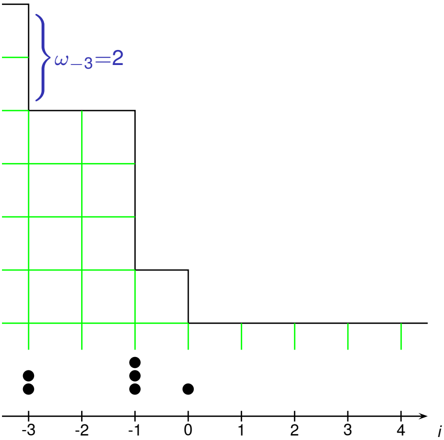

Particles jump
to the right with rate $p \cdot r(\omega_i)$ (r non-decreasing)
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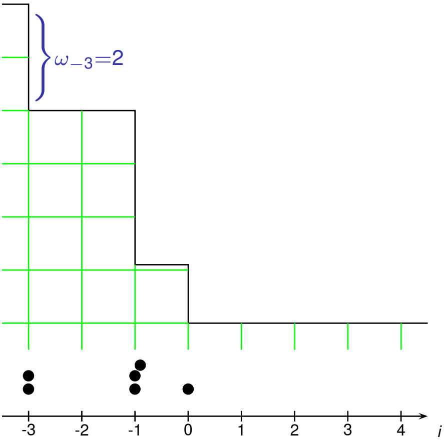
The asymmetric zero range process



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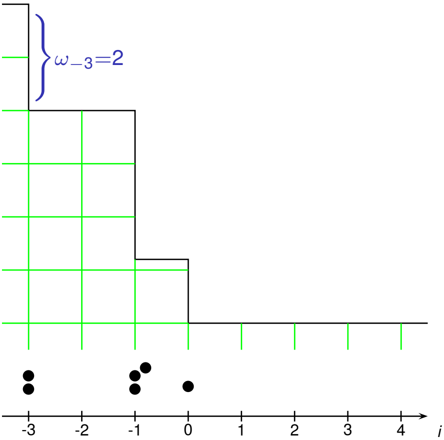

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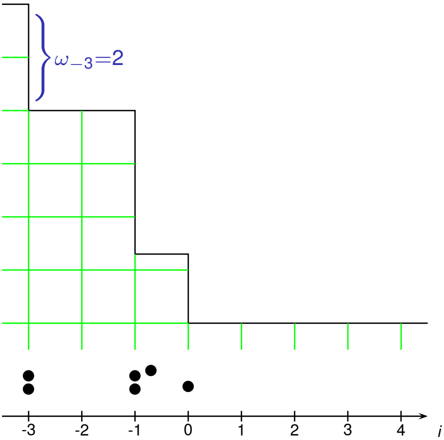

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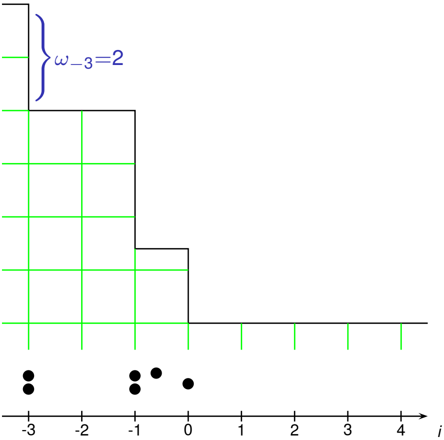

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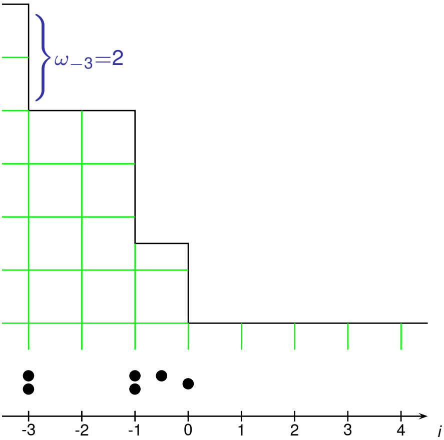

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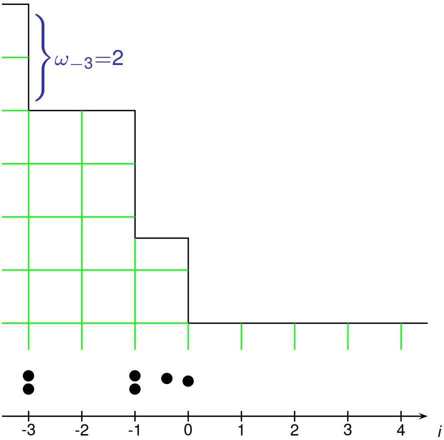

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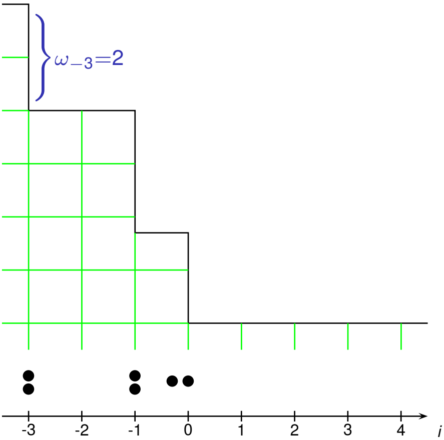

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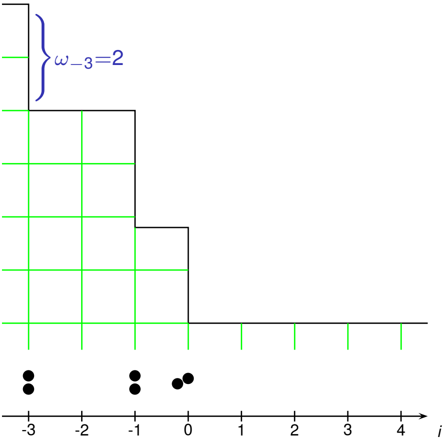

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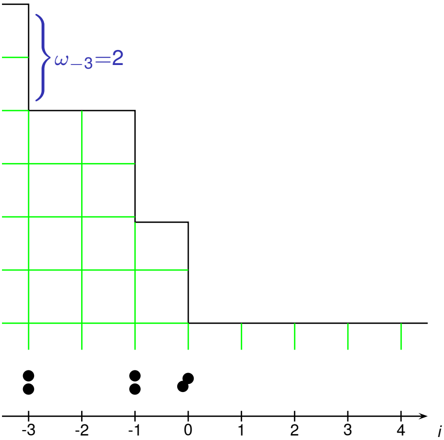

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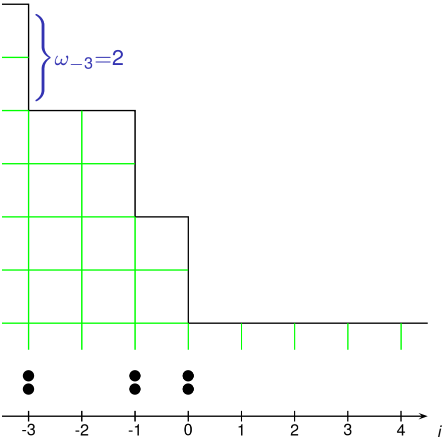

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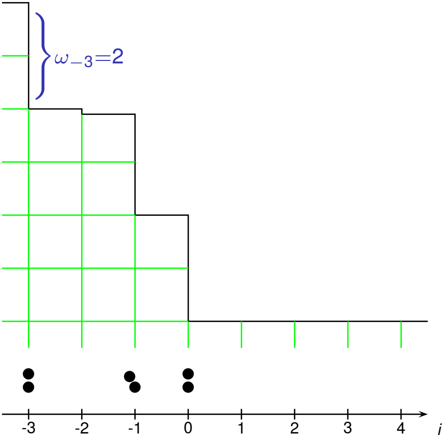

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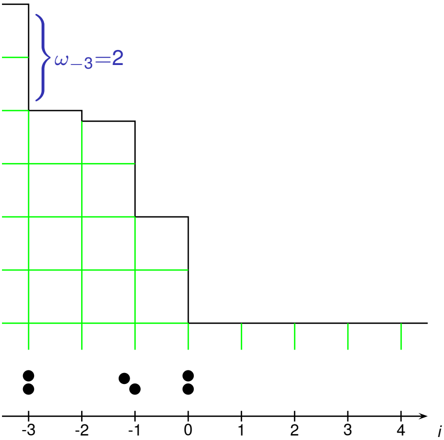

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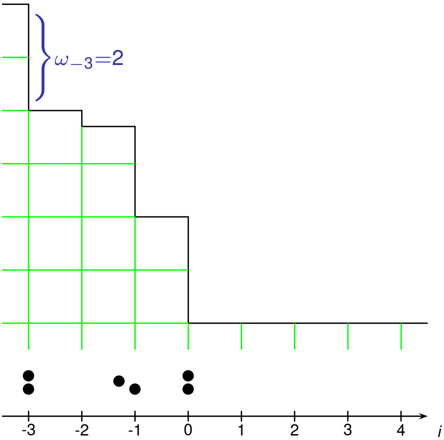

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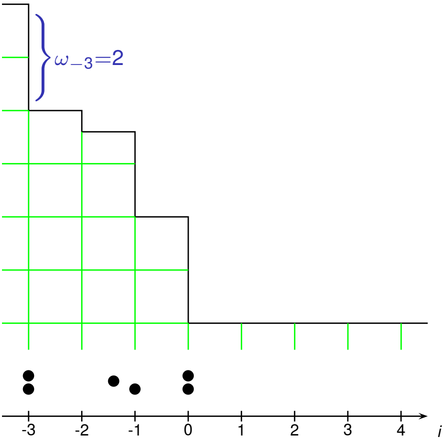

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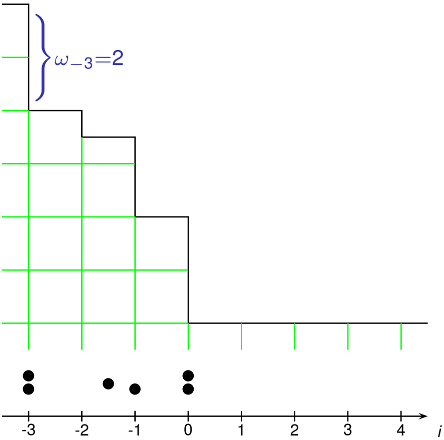

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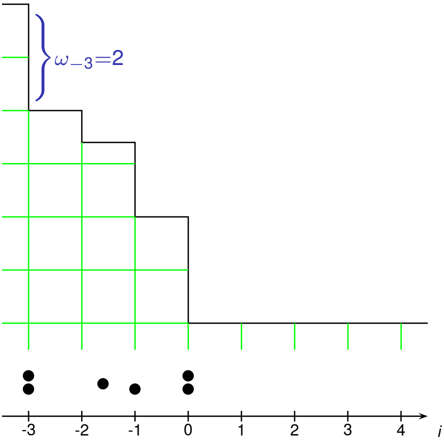

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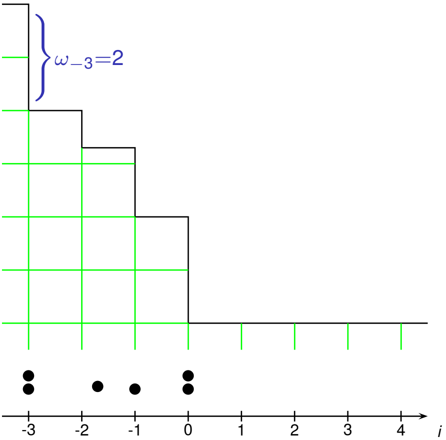

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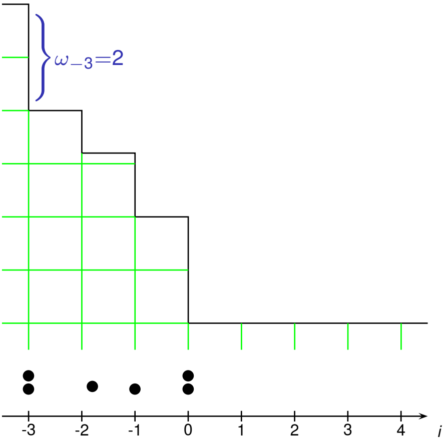

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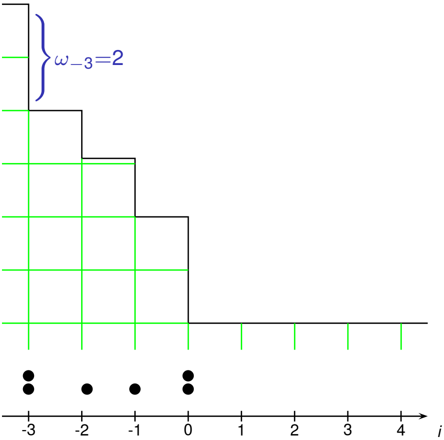

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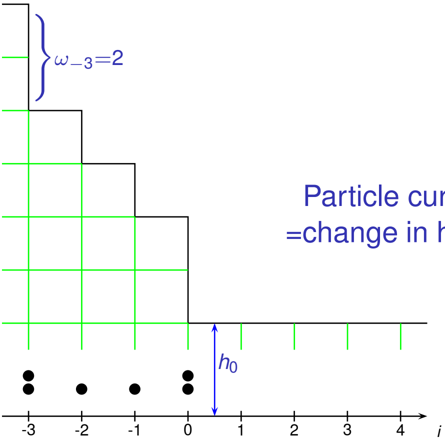

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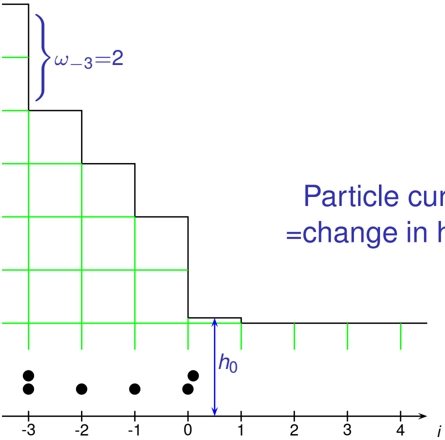

Particle current
=change in height

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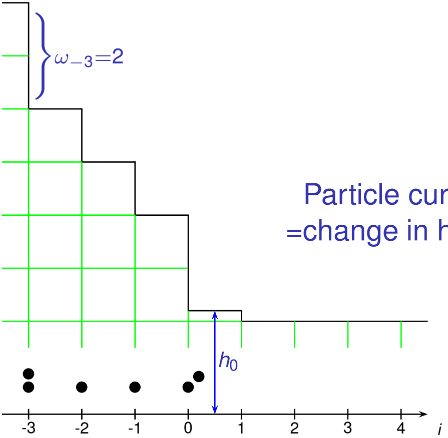

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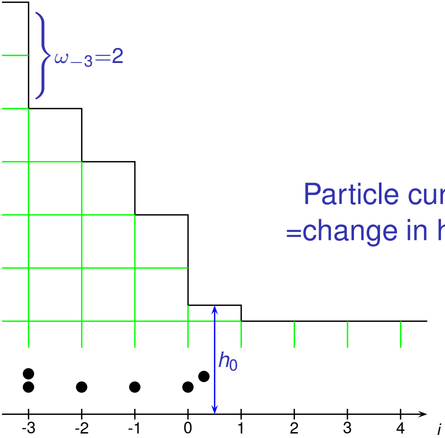

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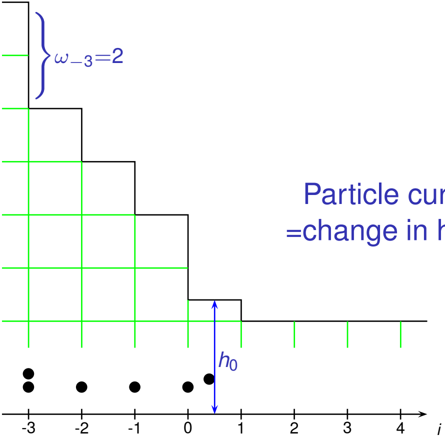

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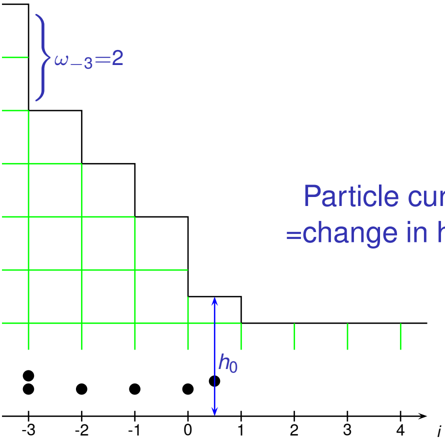

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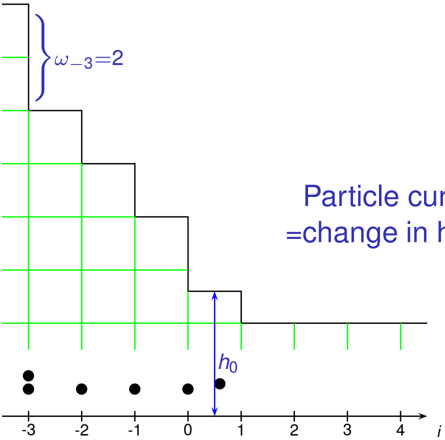

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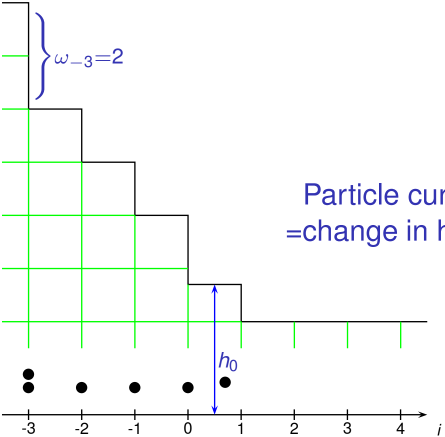

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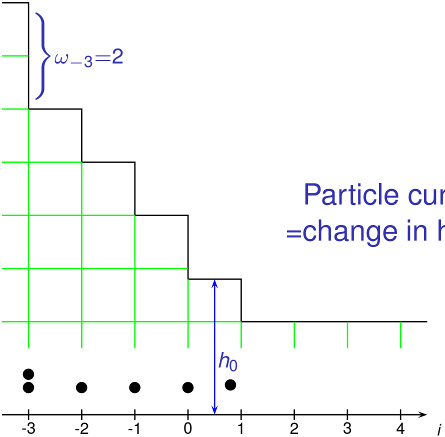

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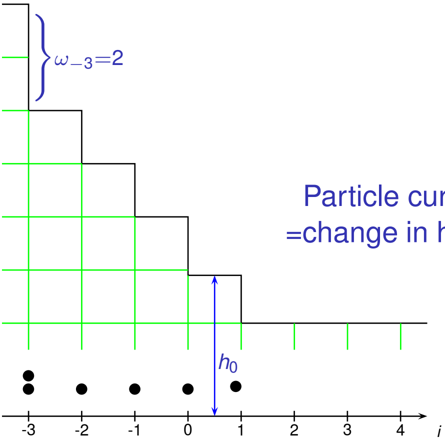

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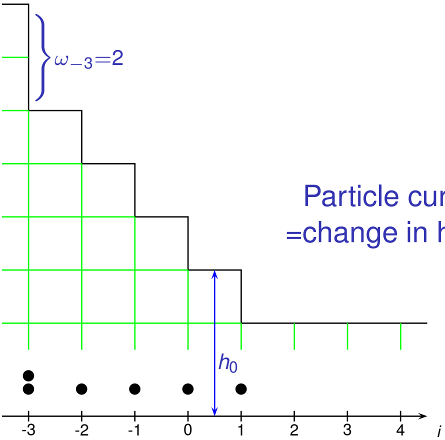

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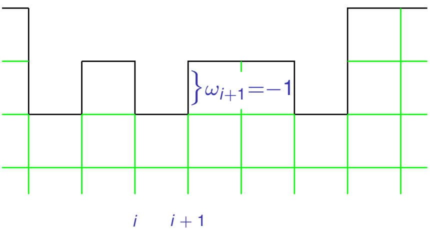
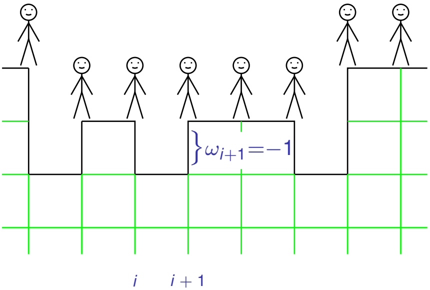
The asymmetric bricklayers process



$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

The asymmetric bricklayers process



$$\begin{pmatrix} \omega_i \\ \omega_{i+1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

a brick is added with rate $p \cdot [r(\omega_i) + r(-\omega_{i+1})]$
a brick is removed with rate $q \cdot [r(-\omega_i) + r(\omega_{i+1})]$.

$(r(\omega) \cdot r(1-\omega) = 1; \quad r$ non-decreasing; $\quad q = 1 - p < p)$.

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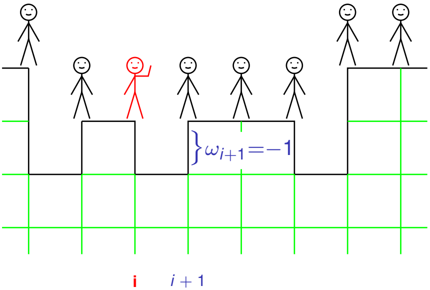

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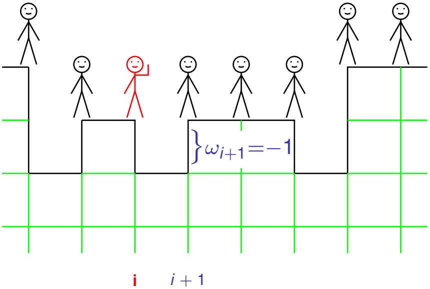

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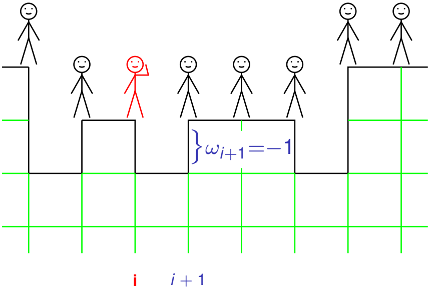

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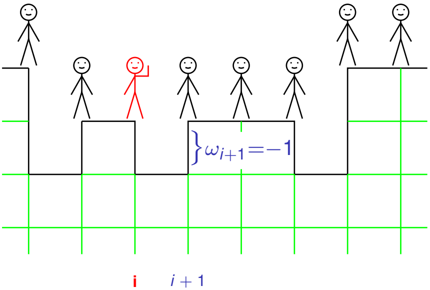

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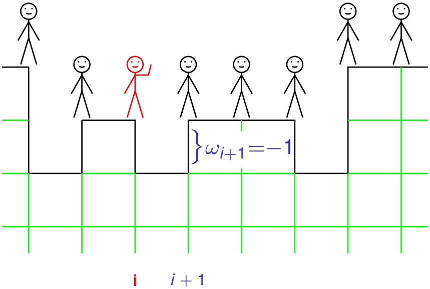

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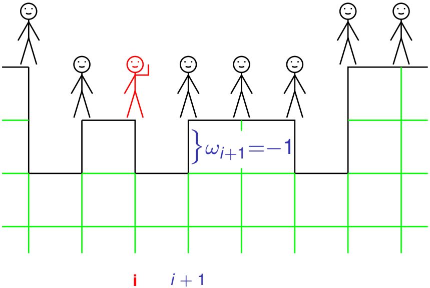

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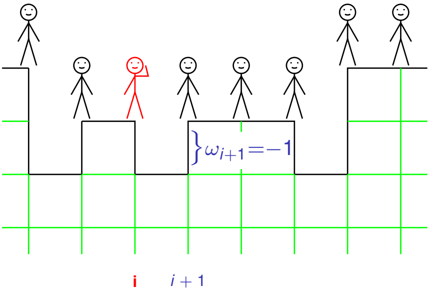

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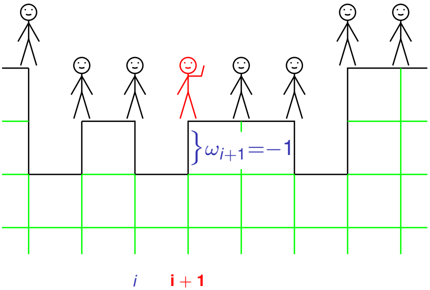

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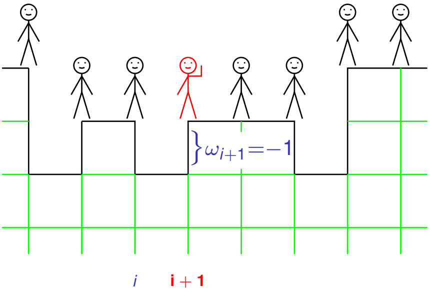
The asymmetric bricklayers process



$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

a brick is added with rate $p\cdot[r(\omega_i)+\mathbf{r}(-\omega_{\mathbf{i+1}})]$
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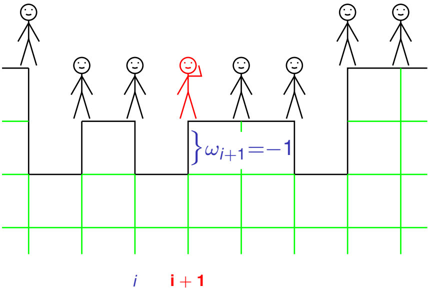

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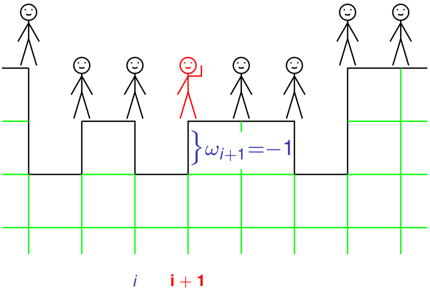

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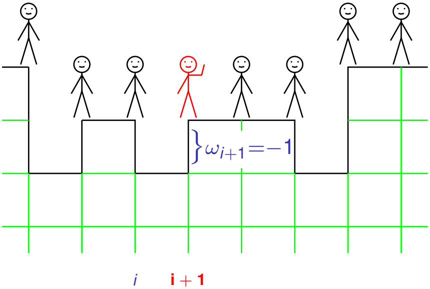

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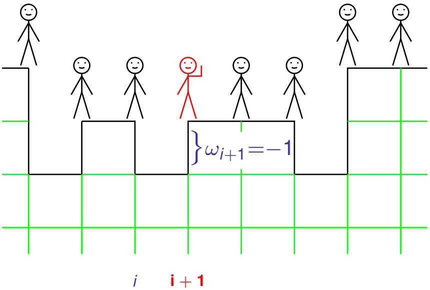

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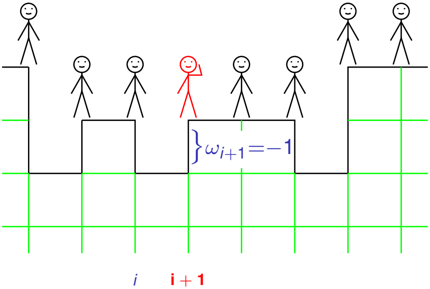

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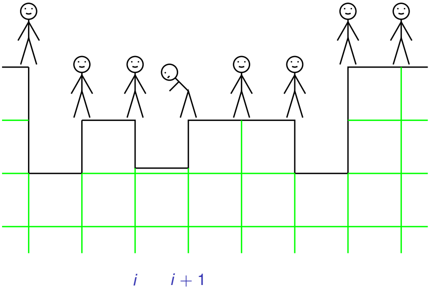

$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \to \begin{pmatrix}\omega_i - 1\\ \omega_{i+1}+1\end{pmatrix}$$
$$\| \qquad\qquad \|$$
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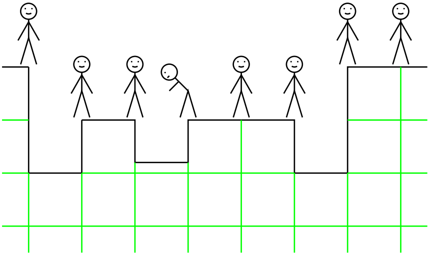

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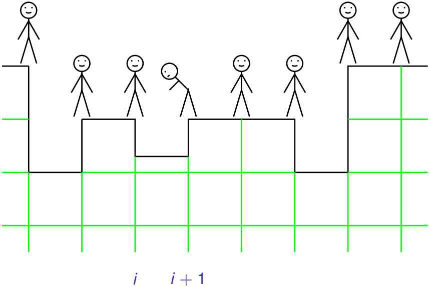

$$\begin{array}{ccc}
\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} & \to & \begin{pmatrix}\omega_i - 1\\ \omega_{i+1}+1\end{pmatrix}\\
\| & & \|\\
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\end{array}$$

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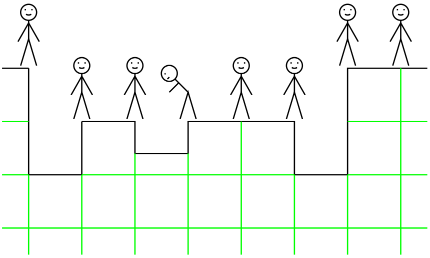

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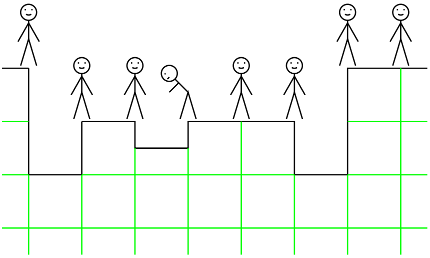

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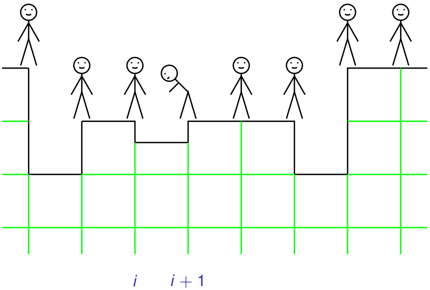

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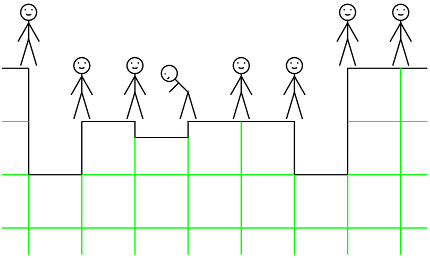

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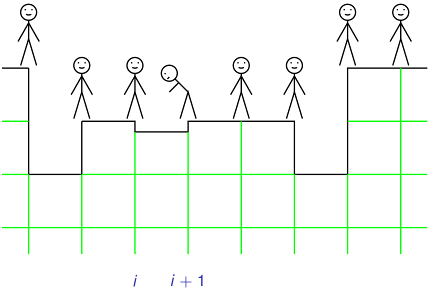

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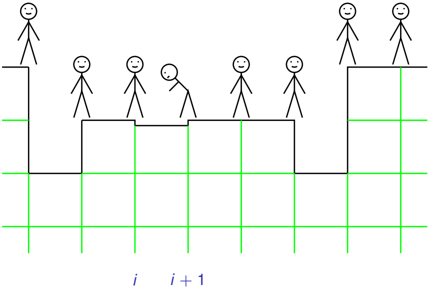

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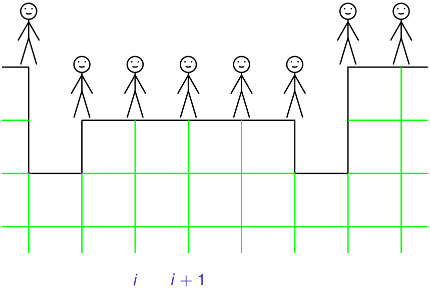

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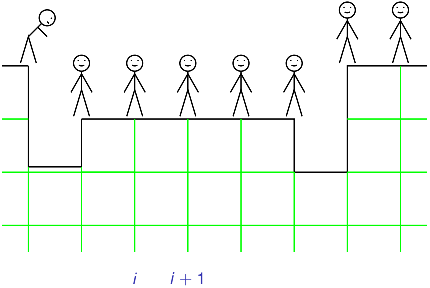

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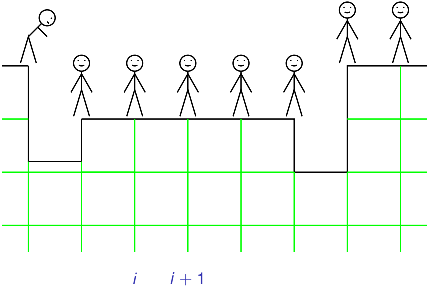

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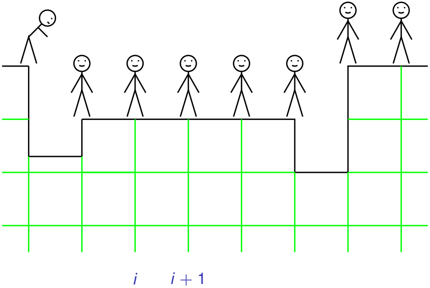

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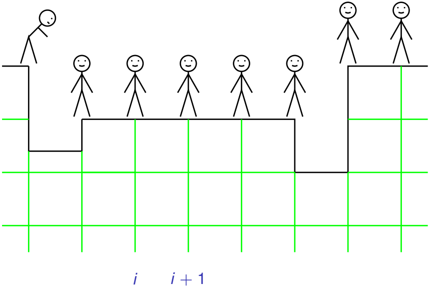

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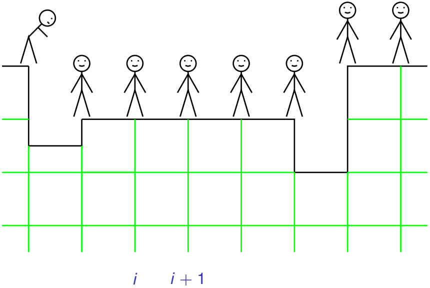

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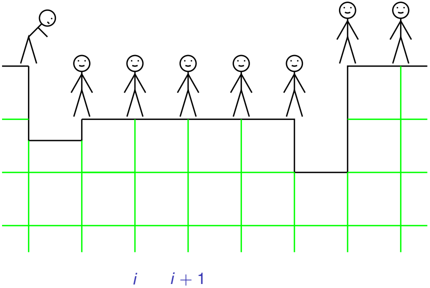

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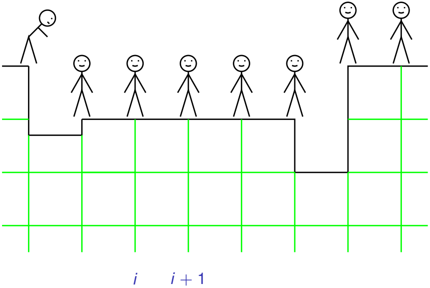

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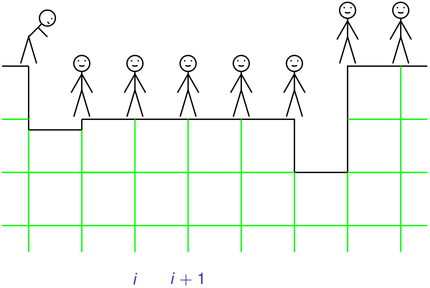

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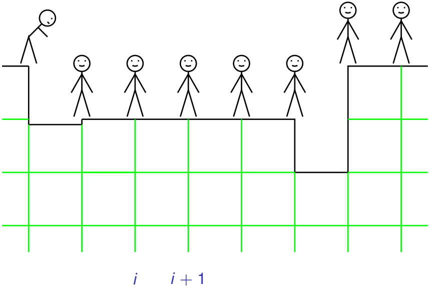

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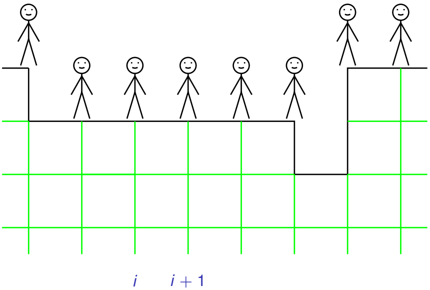

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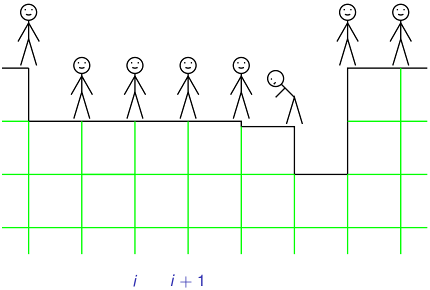

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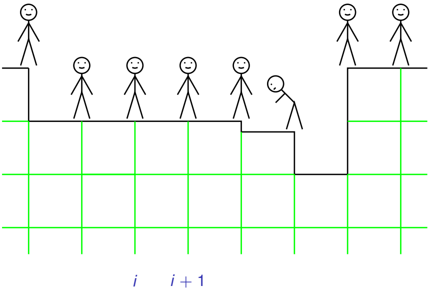

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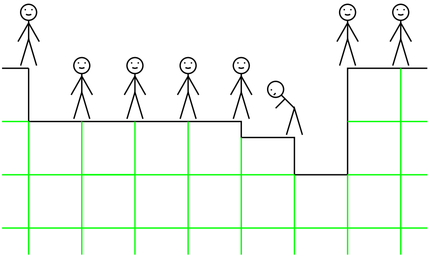

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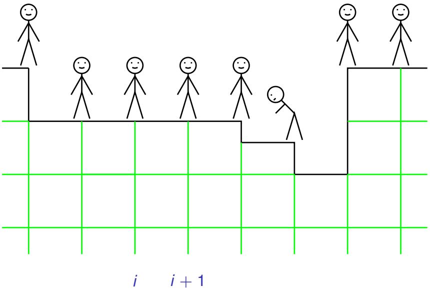

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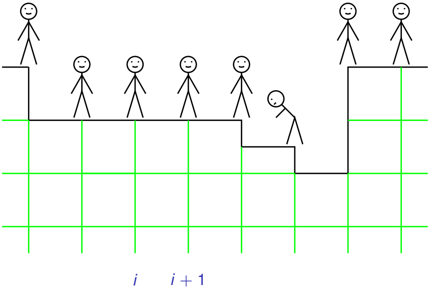

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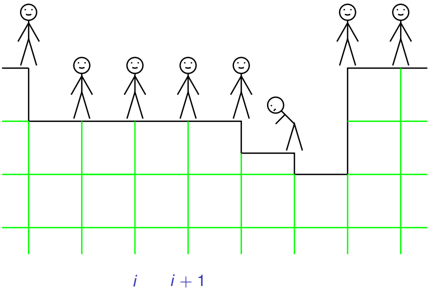

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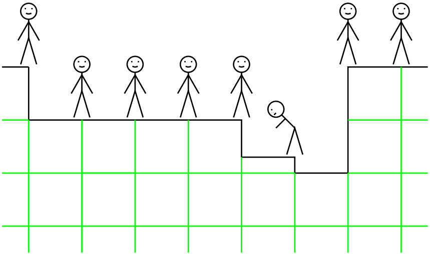

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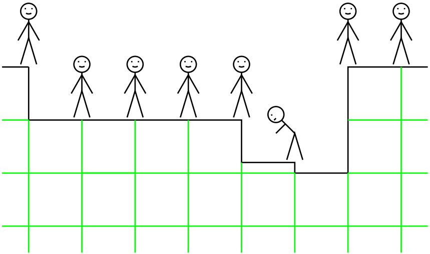

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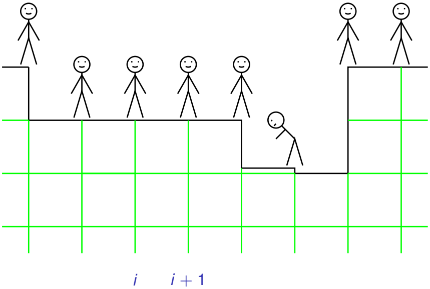

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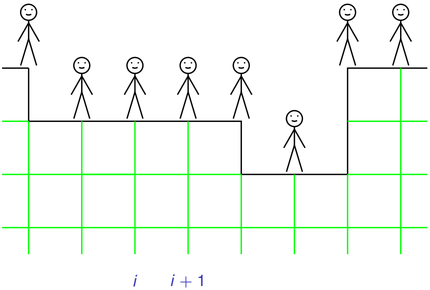

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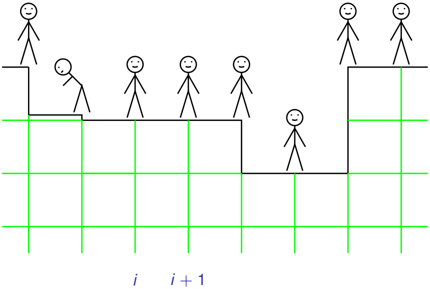

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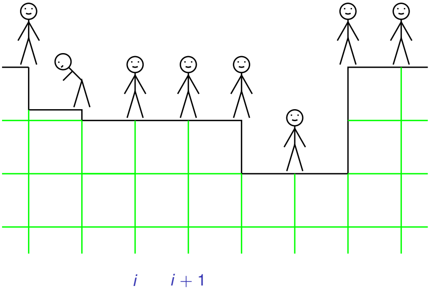

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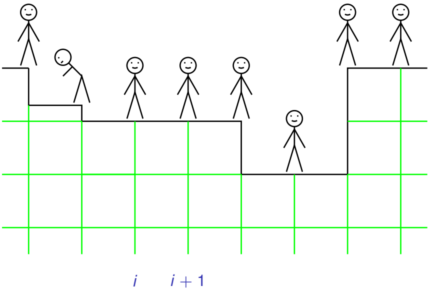

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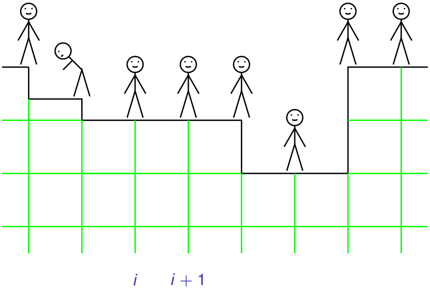

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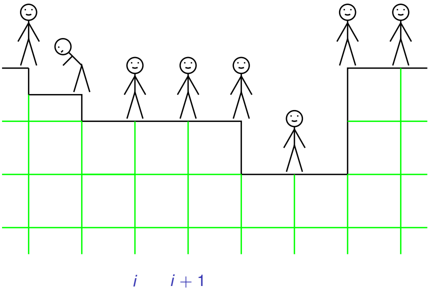

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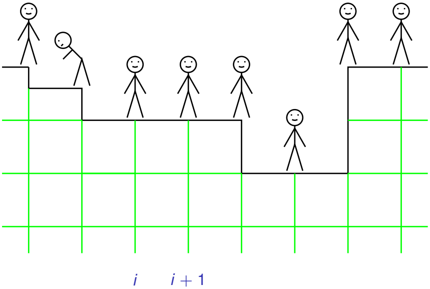

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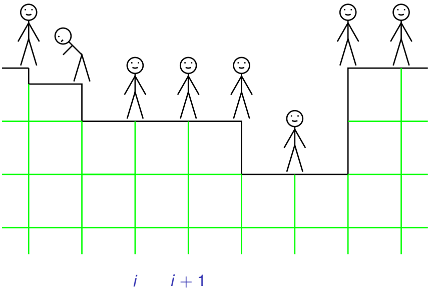

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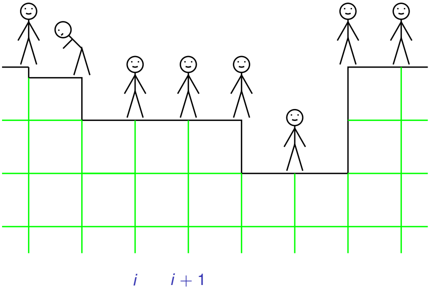

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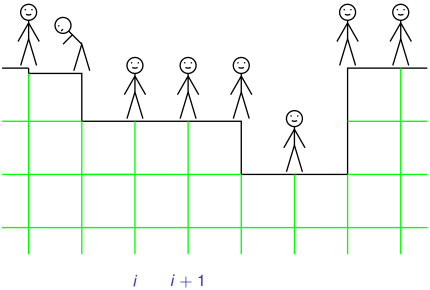

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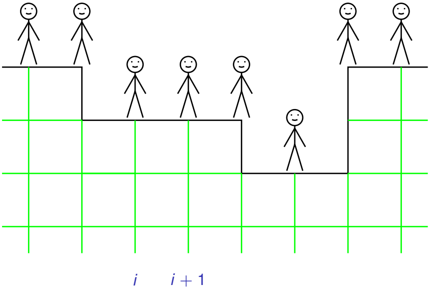

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- they satisfy some algebraic conditions to get a product stationary distribution for the process.

Stationarity

Under these conditions, for every *feasible* density ϱ,

$$(\omega_i)_{i\in\mathbb{Z}} \sim \text{i.i.d. } \mu^\varrho =: \underline{\mu}^\varrho$$

is stationary for some explicit μ^ϱ probability distribution on $\mathbb{Z}$.

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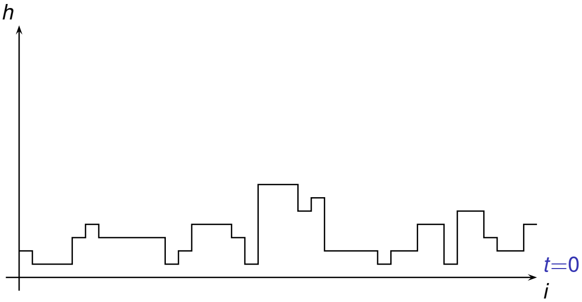
Theorem

For each feasible ϱ, the process $\big(\underline{\omega}(t)\big)_{t\in\mathbb{R}}$ in stationary distribution $\underline{\mu}^\varrho$ is ergodic (for time shifts).

Integrated particle current



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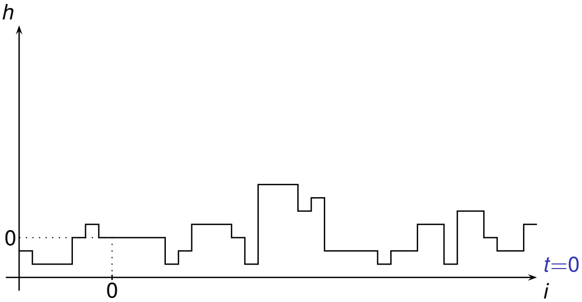

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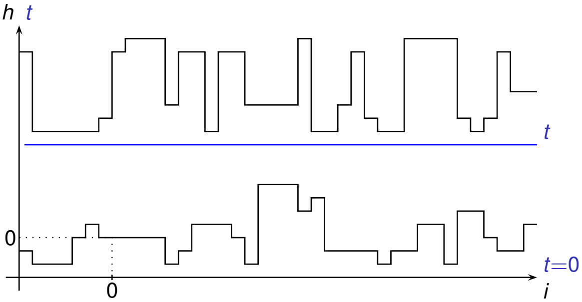

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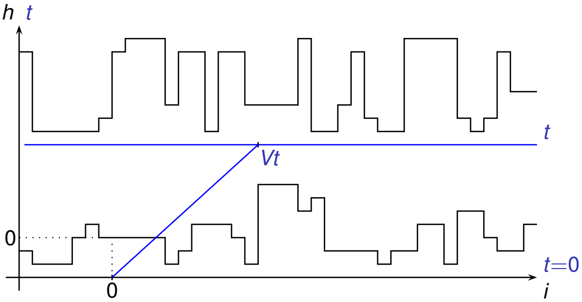

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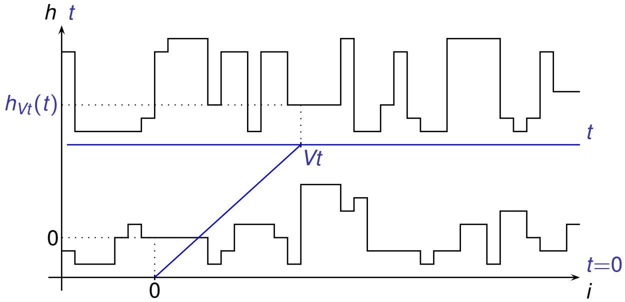

$h_{Vt}(t)$ = height as seen by a moving observer of velocity V.

= net number of particles passing the window $s \mapsto Vs$.

(Remember: particle current=change in height.)

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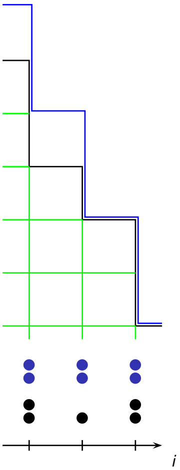
Tool: the second class particle

States η and ω only differ at one site.



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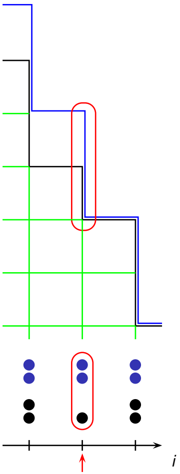

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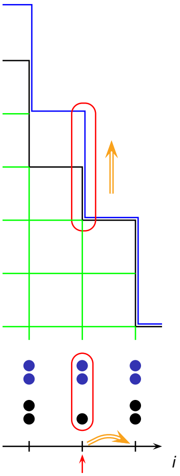

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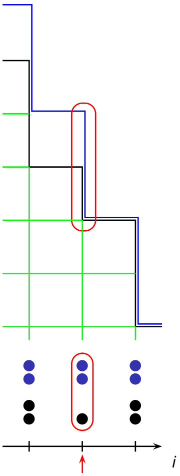

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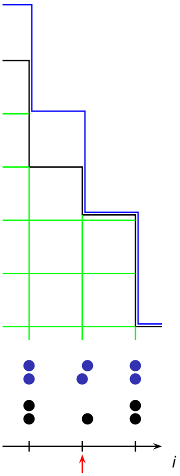

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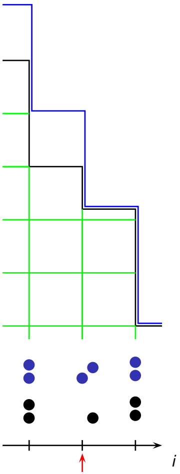

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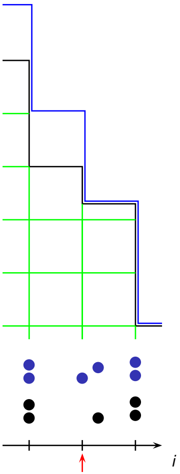

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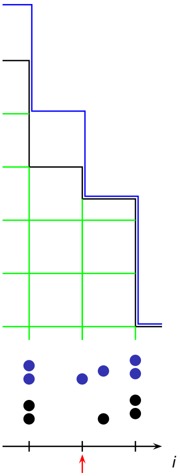

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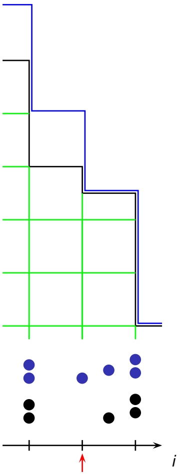

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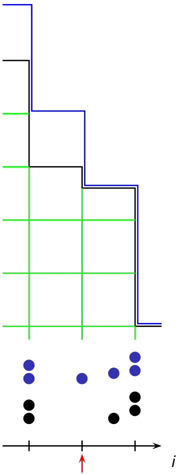

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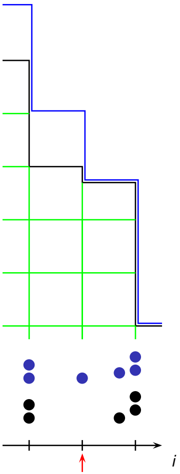

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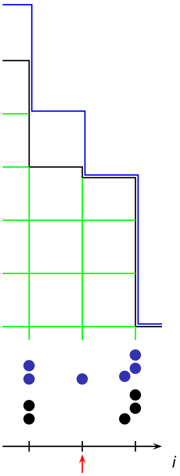

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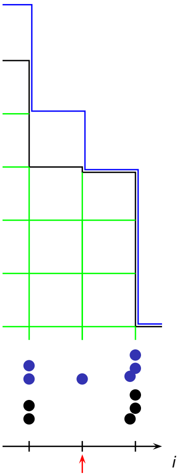

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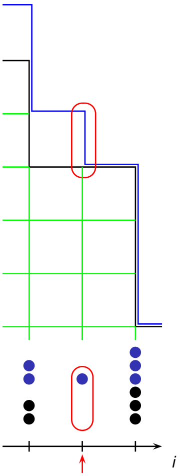

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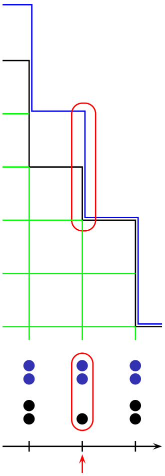

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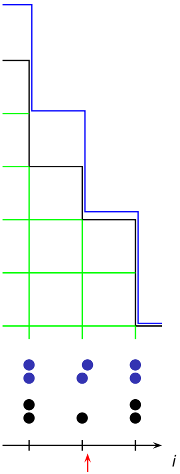

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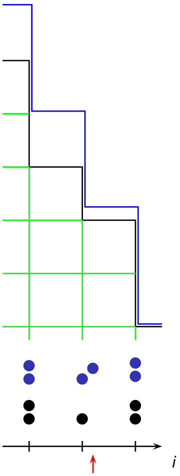

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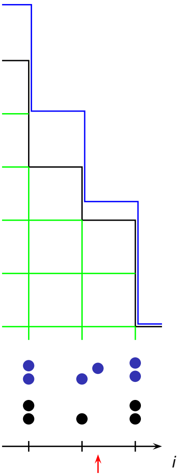

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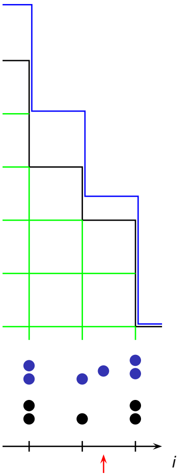

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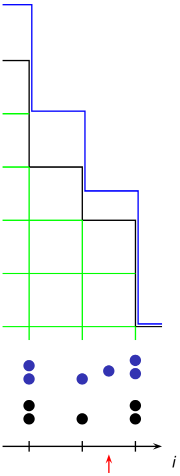

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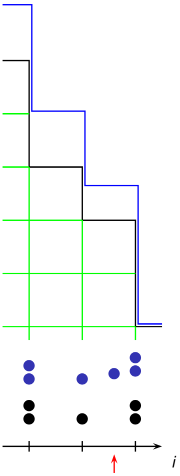

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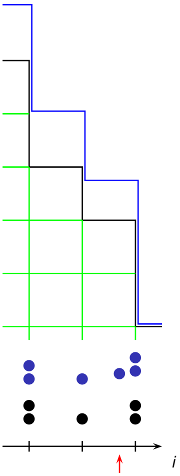

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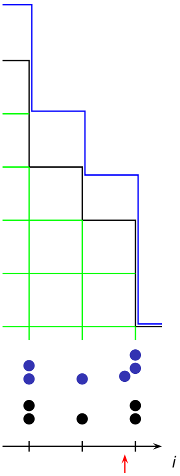

Growth on the right:
rate$\leq$rate
with rate-rate:

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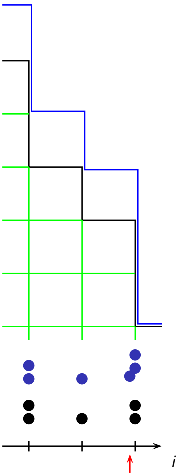

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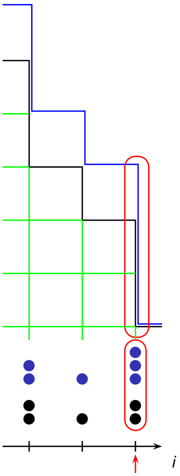

Growth on the right:
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Growth on the left:
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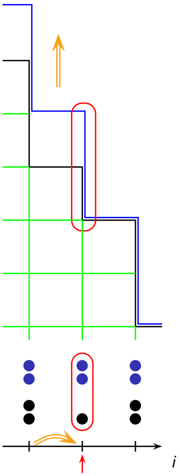

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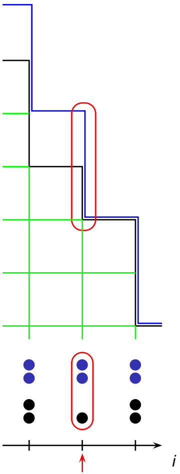

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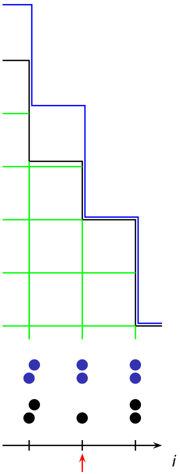

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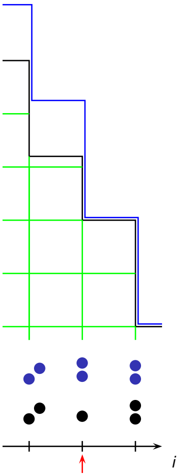

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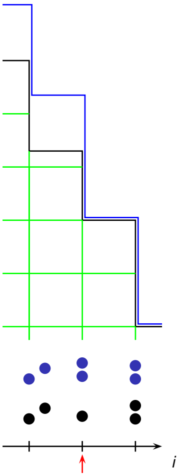

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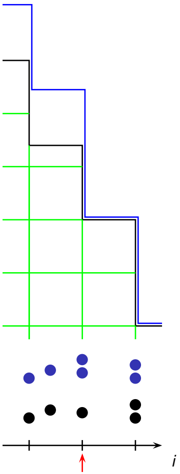

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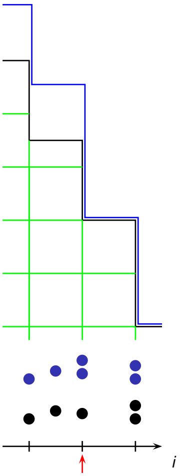

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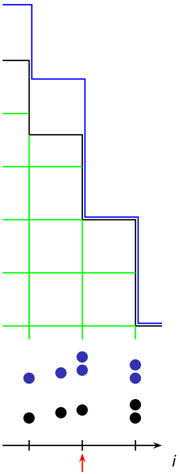

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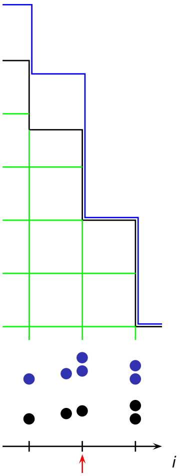

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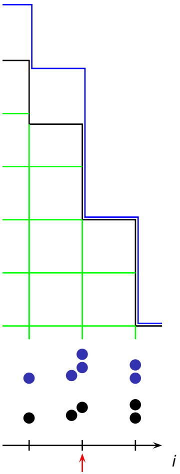

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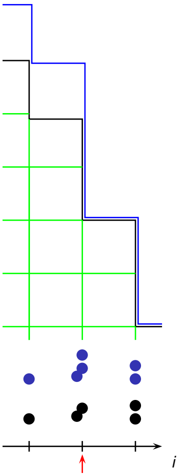

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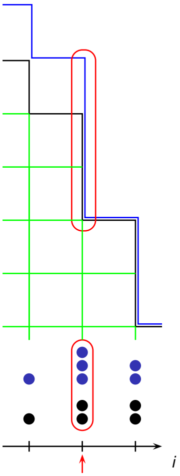

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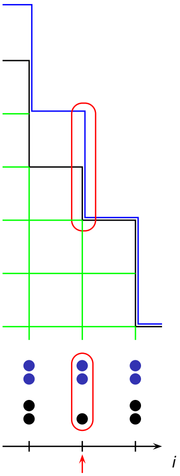

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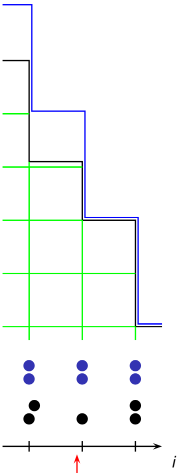

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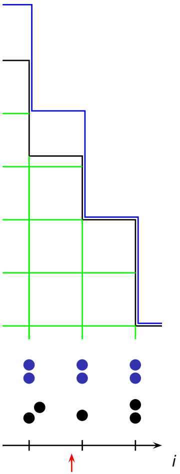

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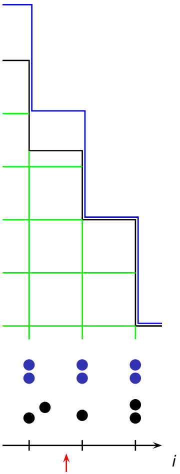

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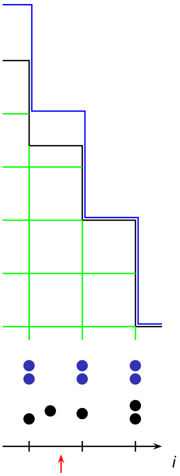

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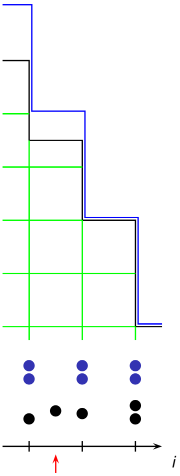

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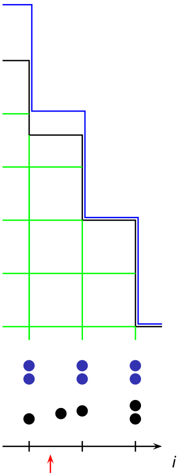

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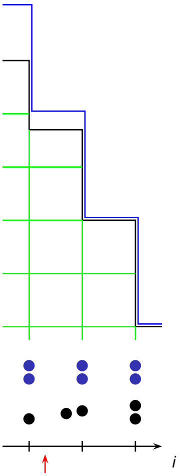

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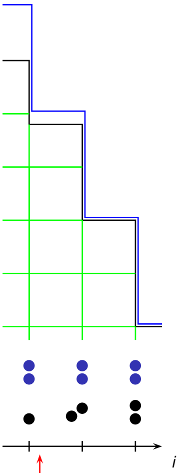

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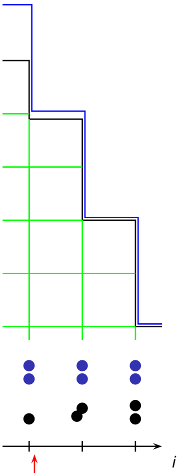

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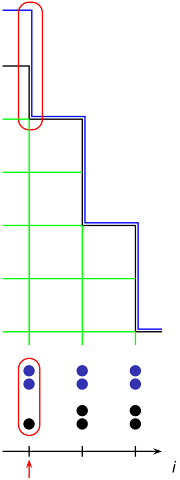

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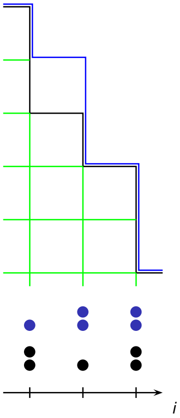
States η and ω only differ at two neighbouring sites:



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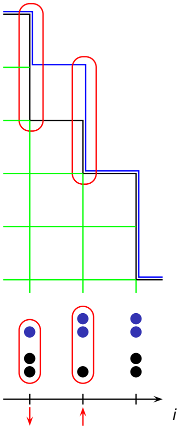

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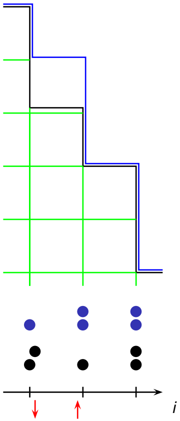

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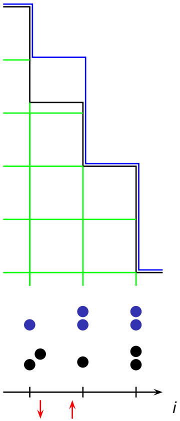

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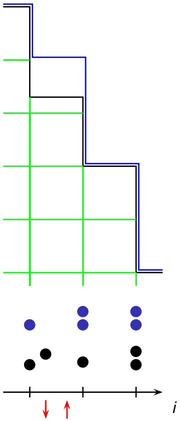

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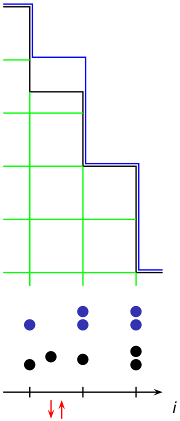

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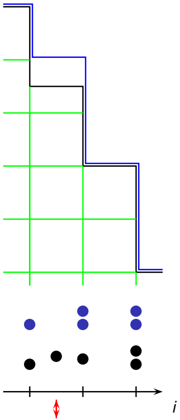

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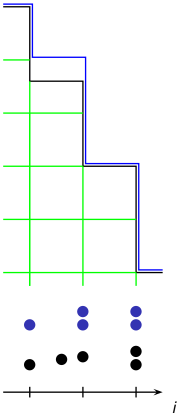

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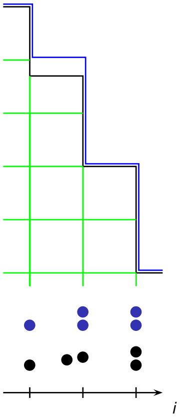

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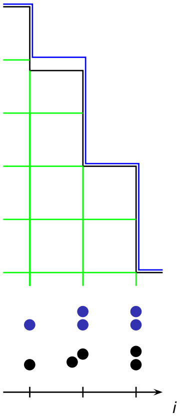

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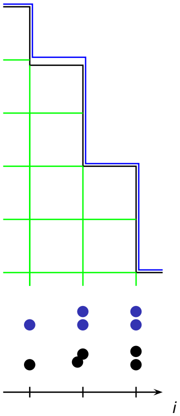

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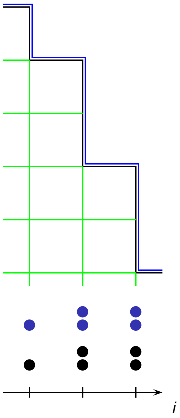

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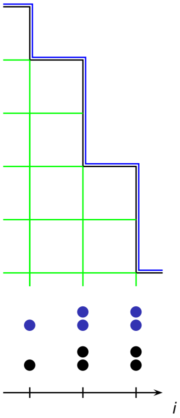
States η and ω only differ at two neighbouring sites:
Difference vanishes by any time with positive probability.



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- Hewitt-Savage 0-1 law: a function that's invariant to any permutation of i.i.d. random variables is a.s. constant. QED.

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- Let ψ be an invariant function (in time) of $\underline{\omega}(t)$, w.t.s. that it's a constant (on states) $\underline{\mu}^{\varrho}$-a.s.
- For that pair $\underline{\omega}(0)$, $\underline{\eta}(0) = \underline{\omega}(0) + \uparrow + \downarrow$:

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