Electric network for non-reversible Markov chains

Joint work with Áron Folly

Márton Balázs

University of Bristol

Random walks on graphs and potential theory

University of Warwick, 20th May 2015.

Reversible chains and resistors
- Reducing a network
- Thomson, Dirichlet principles
- Monotonicity, transience, recurrence

Irreversible chains and electric networks
- The part
- From network to chain
- From chain to network
- Effective resistance
- What works

The electric network
- Reducing the network
- Nonmonotonicity
- Dirichlet principle

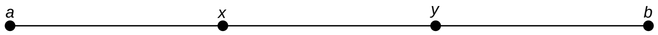
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



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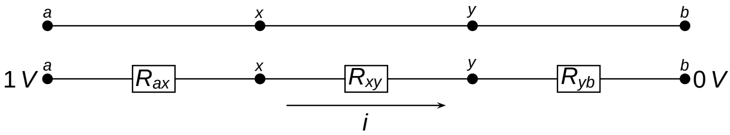

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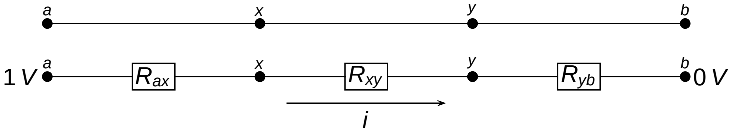

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Notice $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$, so the chain is reversible.

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Reversible chains and resistors

Let $n_x = \mathbf{E}_a$(number of visits to x before absorbed in b). Then

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$\mathbf{E}_a$(signed current $x \to y$ before absorbed in b)

$$= n_x P_{xy} - n_y P_{yx} = (u_x - u_y) C_{xy} = i_{xy}.$$ normalisation…

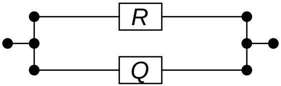
$P_{xy} = C_{xy}/C_x$ $C_x = \mu_x$

Reducing a network

Series:

$$R_{\text{eff}} = R + Q$$

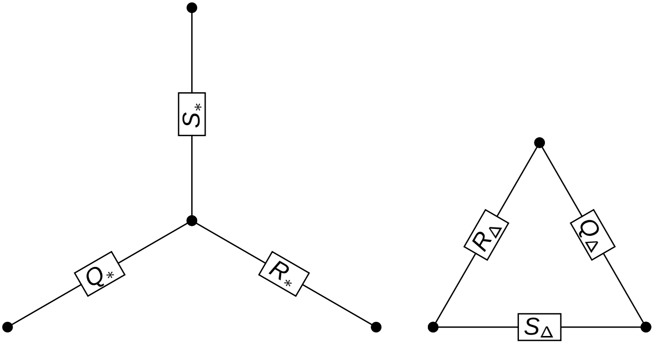
Parallel:



$$\frac{1}{R_{\text{eff}}} = \frac{1}{R} + \frac{1}{Q}$$

Reducing a network

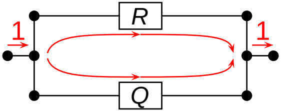
Star-Delta:



$$R_* = \frac{Q_\Delta S_\Delta}{R_\Delta + Q_\Delta + S_\Delta}, \qquad R_\Delta = \frac{R_* Q_* + R_* S_* + Q_* S_*}{R_*}.$$

Thomson, Dirichlet principles

Thomson principle:



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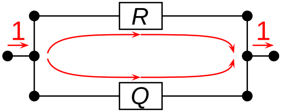

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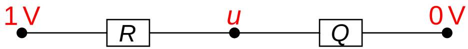

The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$.

Monotonicity, transience, recurrence

The monotonicity property:

Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

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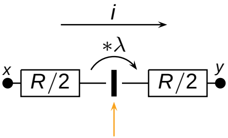
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

$\rightsquigarrow$ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

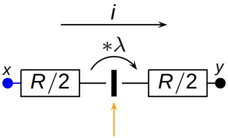
The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

The part



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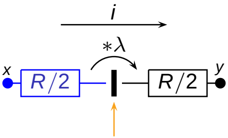

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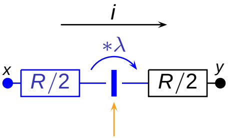

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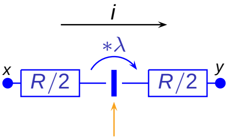

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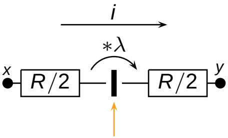

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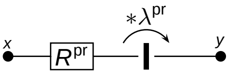
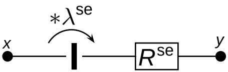


$$(u_x - i \cdot R^{pr}) \cdot \lambda^{pr} = u_y$$



$$u_x \cdot \lambda^{se} - R^{se} \cdot i = u_y$$

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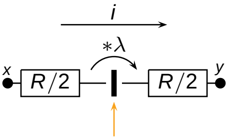

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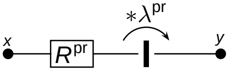

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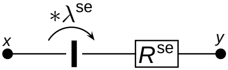

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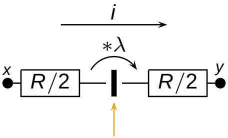

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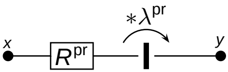

$$(u_x - i \cdot R^{pr}) \cdot \lambda^{pr} = u_y$$
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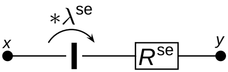
$$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$$



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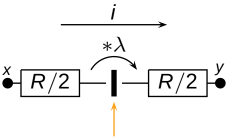

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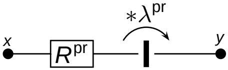

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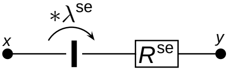

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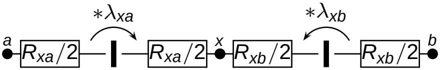
Harmonicity



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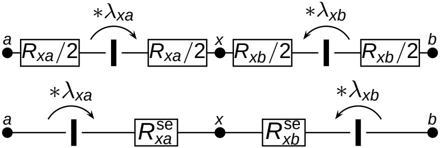

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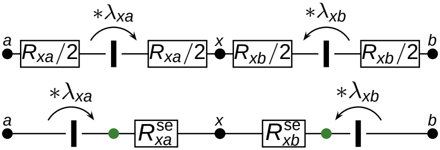

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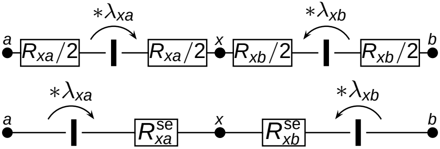

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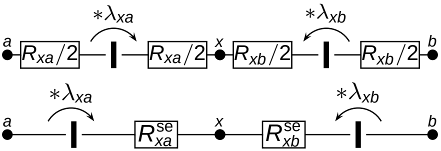

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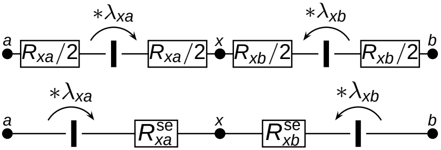

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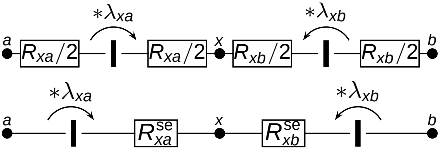

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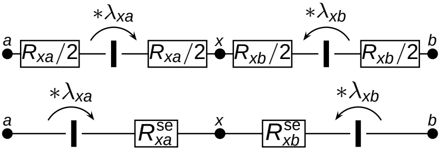

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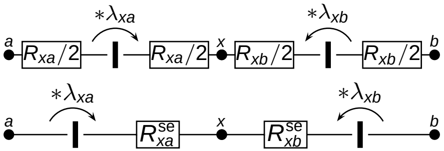

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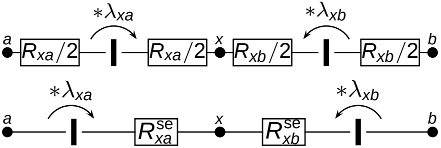
$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



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From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

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"Markovian" property

$$u_x = \sum_z P_{xz} u_z ; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

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Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

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$$\mathbf{E}_a(\text{signed current } x \to y \text{ before absorbed in } b)$$
$$= n_x P_{xy} - n_y P_{yx} = (\hat{u}_x \gamma_{xy} - \hat{u}_y \gamma_{yx}) D_{xy} = \hat{i}_{xy}.$$ normalisation...

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Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$.
Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

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$\rightsquigarrow$ In particular, i_a is proportional to $u_a - u_b$. We have effective resistance.

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Modulo normalisation. . .

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in the reversed network!

Theorem (Chandra, Raghavan, Ruzzo, Smolensky and Tiwari '96 for reversible)

Commute time $= R_{\mathit{eff}} \cdot$ *all conductances.*

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For all sets A, B, capacity$\sim$escape probability.

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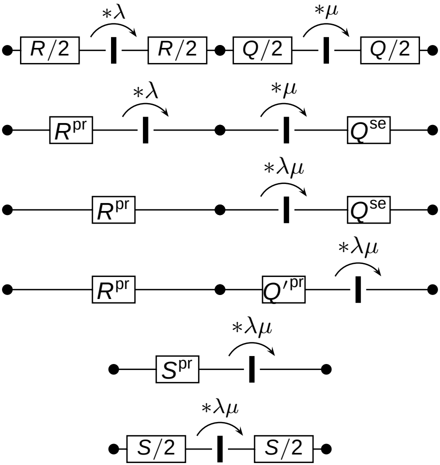
This is non-physical!

In particular, symmetrising the chain ($P_{xy}\to\frac{P_{xy}+\hat{P}_{xy}}{2}$) cannot increase escape probabilities:

- symmetrising leaves C_{xy} unchanged;
- the above sum is minimised by the symmetric voltages, not $\{u_x\}$ (Classical Dirichlet principle).

The electric network

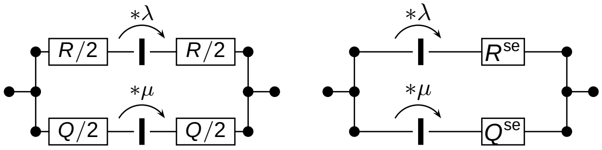
Series:



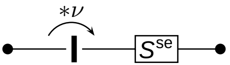
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

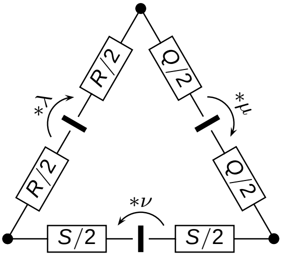
$$\nu = \frac{Q\lambda(\mu+1)+R\mu(\lambda+1)}{Q(\mu+1)+R(\lambda+1)}.$$

The electric network

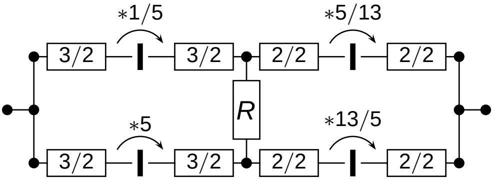
Star-Delta:

Star to Delta works,

Delta to Star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).



Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$

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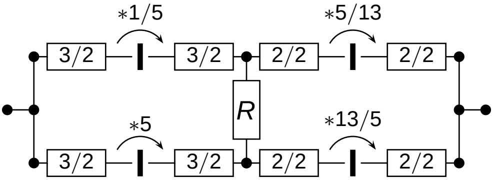

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$$(i^*_u)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

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Thank you.

A trivial statement

Theorem (Well Known Theorem)

A Markov chain is reversible if and only if for every closed cycle $x_0, x_1, x_2, \ldots, x_n = x_0$ in Ω we have

$$P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_0} = P_{x_0 x_{n-1}} \cdot P_{x_{n-1} x_{n-2}} \cdots P_{x_1 x_0}.$$

In particular, any Markov chain on a finite connected tree G is necessarily reversible.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

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$$\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1.$$

- Total multiplication factor along any loop is one.

A trivial statement

Electrical proof.

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Electrical proof.

Repeat for trees:

- There are no loops.

□

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$\square$

Second thank you.