Road layout in the KPZ class

joint with Riddhipratim Basu, Sudeshna Bhattacharjee, Karambir Das, David Harper

Márton Balázs

University of Bristol

UK Network in Stochastics, Edinburgh

17^{th} June 2024.

A naive Poisson model

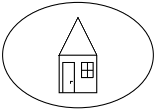
Last passage percolation

Our model

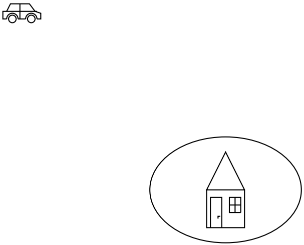
Questions

Answers

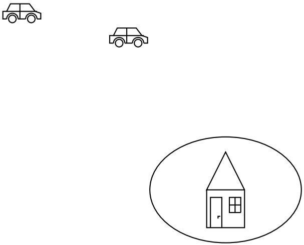
A naive Poisson model



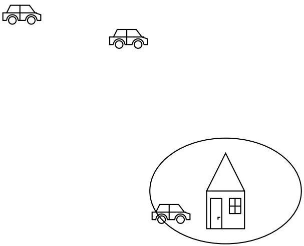
A naive Poisson model



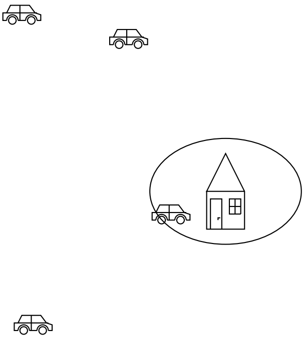
A naive Poisson model



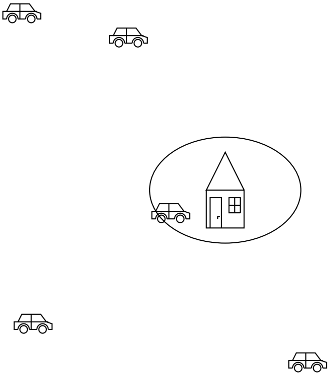
A naive Poisson model



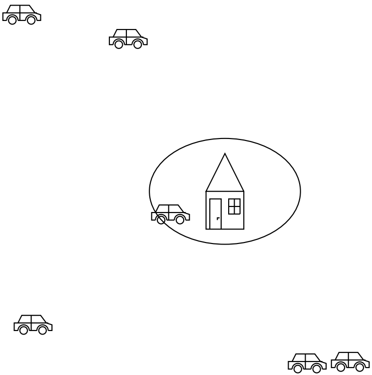
A naive Poisson model



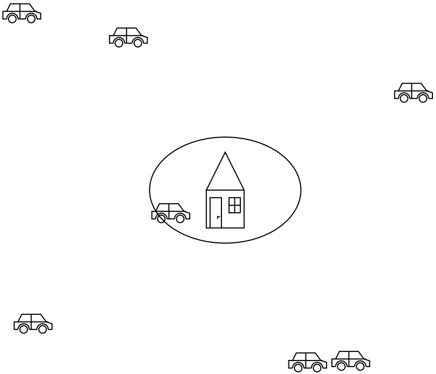
A naive Poisson model



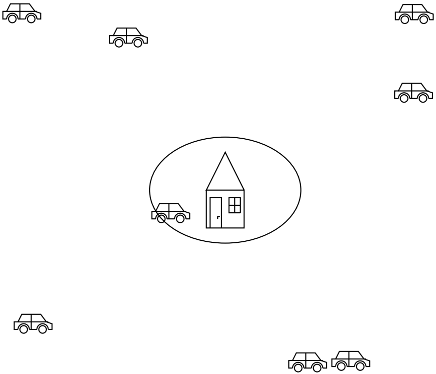
A naive Poisson model



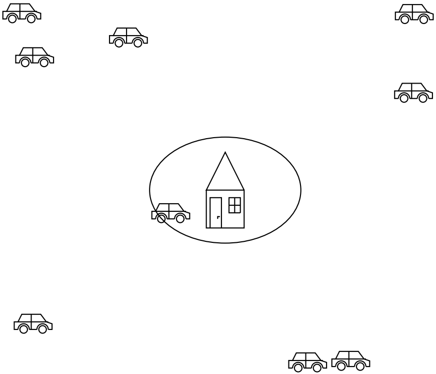
A naive Poisson model



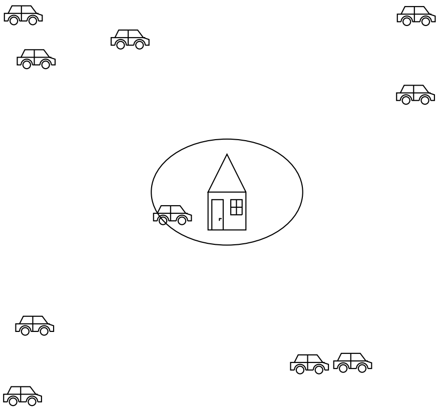
A naive Poisson model



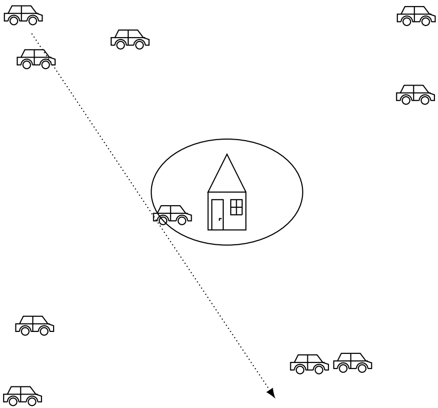
A naive Poisson model



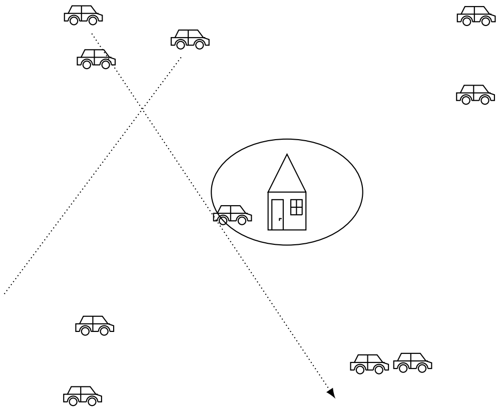
A naive Poisson model



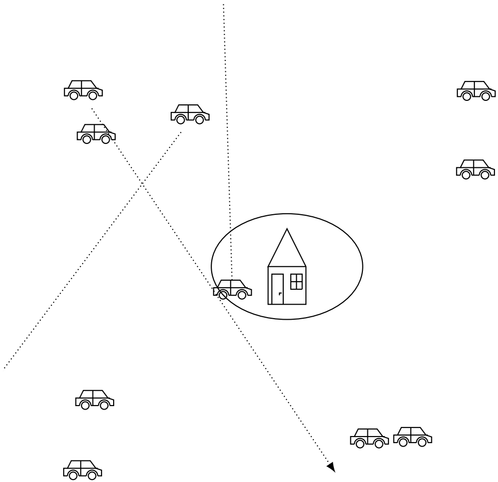
A naive Poisson model



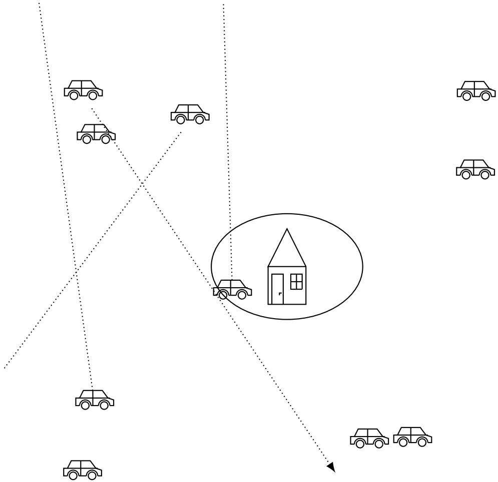
A naive Poisson model



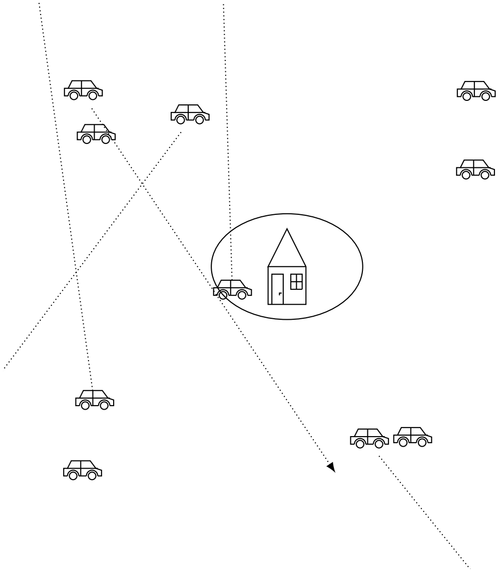
A naive Poisson model



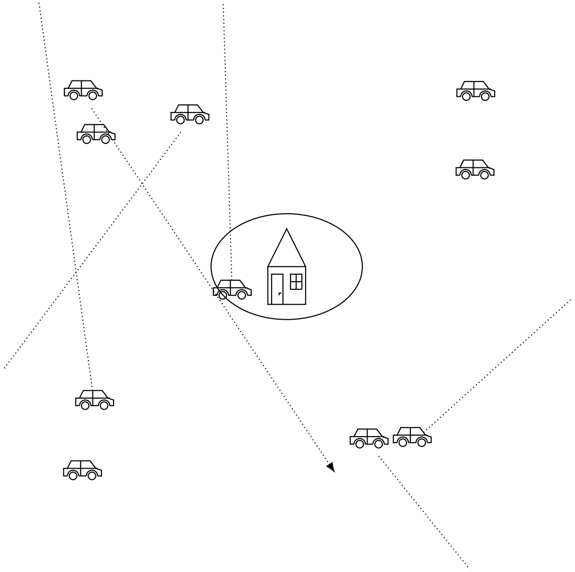
A naive Poisson model



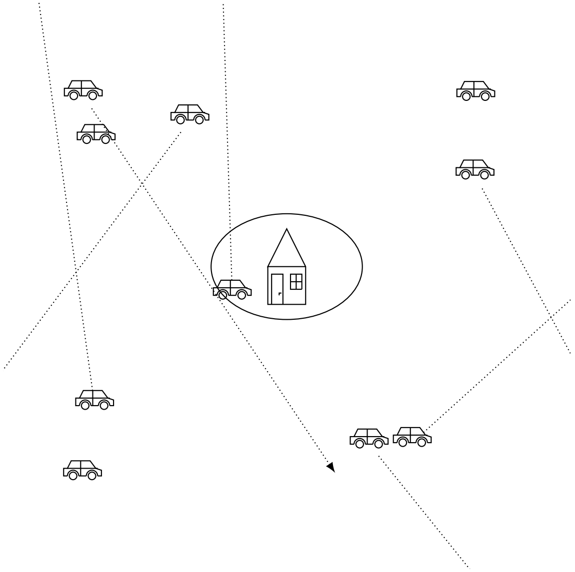
A naive Poisson model



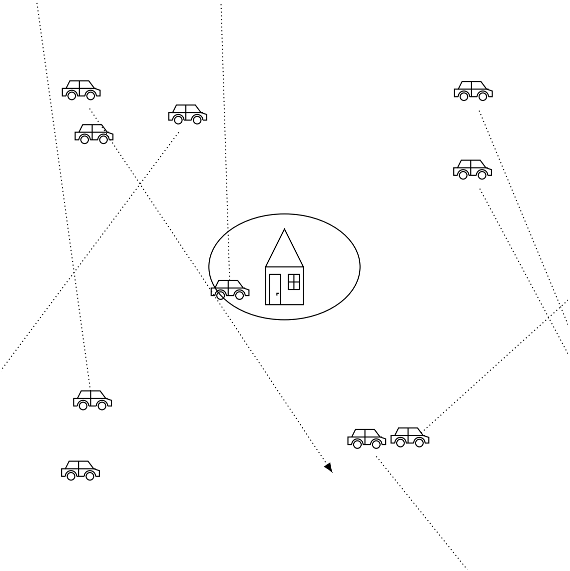
A naive Poisson model



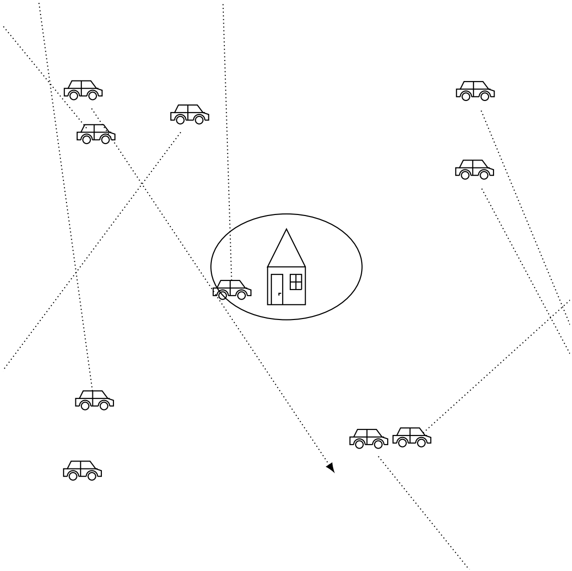
A naive Poisson model



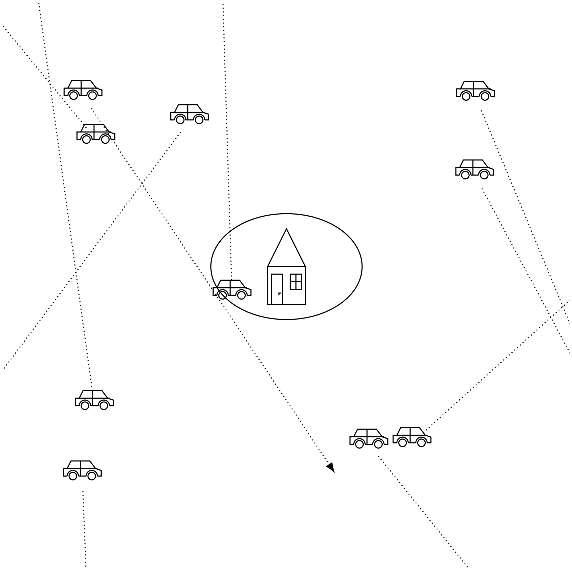
A naive Poisson model



A naive Poisson model



A naive Poisson model



A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.

A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.

A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.
- How many paths come r close to my house?

A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.
- How many paths come r close to my house?
- Mark the start points with the directions and travel lengths: still a Poisson process.

A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.
- How many paths come r close to my house?
- Mark the start points with the directions and travel lengths: still a Poisson process.
- Find the marked subspace of paths intersecting the radius r disk around my house.

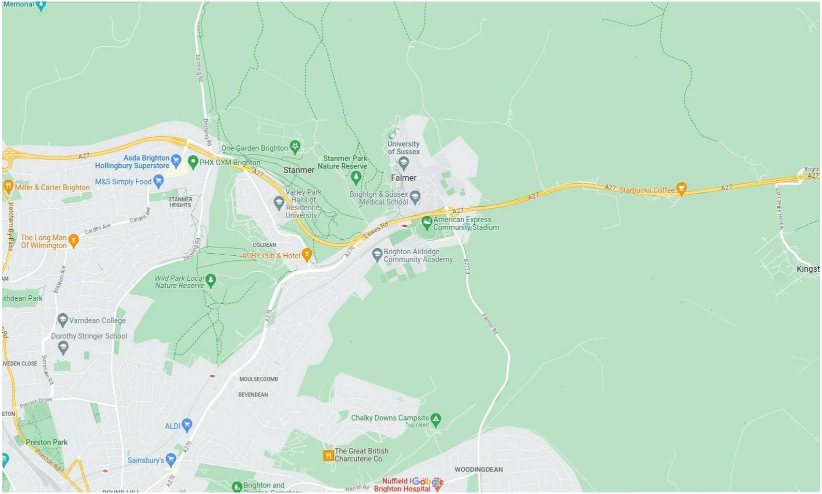
A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.
- How many paths come r close to my house?
- Mark the start points with the directions and travel lengths: still a Poisson process.
- Find the marked subspace of paths intersecting the radius r disk around my house.
- The number of paths doing this is still Poisson with mean at least $2Dr \cdot \mathrm{e}^{-r/D}$, or $\frac{2D}{\pi r} \cdot \mathrm{e}^{-r/D}$ per unit area.

A naive Poisson model

- Cars start from points of a homogeneous intensity 1 Poisson process.
- They each pick an independent uniform random direction and go straight that way for an Exponential($\frac{1}{D}$) distance.
- How many paths come r close to my house?
- Mark the start points with the directions and travel lengths: still a Poisson process.
- Find the marked subspace of paths intersecting the radius r disk around my house.
- The number of paths doing this is still Poisson with mean at least $2Dr \cdot \mathrm{e}^{-r/D}$, or $\frac{2D}{\pi r} \cdot \mathrm{e}^{-r/D}$ per unit area.
- Unfortunately $D \gg r \ldots$

A naive Poisson model



A naive Poisson model

- Clearly not a good model.

A naive Poisson model

- Clearly not a good model.
- People who first beat a path; horse drawn carriages; road builders try to minimise obstacles. Gradients, built-up objects, etc.

A naive Poisson model

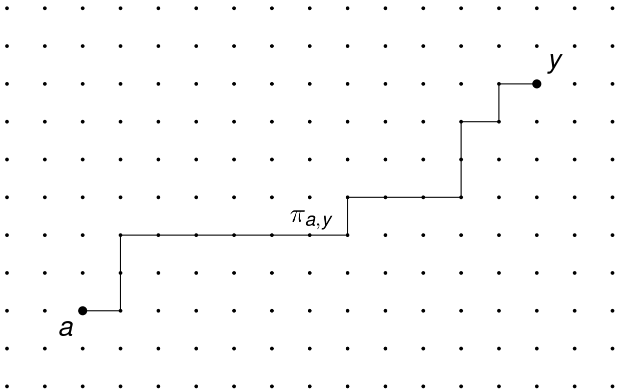
- Clearly not a good model.
- People who first beat a path; horse drawn carriages; road builders try to minimise obstacles. Gradients, built-up objects, etc.
- $\rightsquigarrow$ first passage percolation (FPP). Roads coalesce.

A naive Poisson model

- Clearly not a good model.
- People who first beat a path; horse drawn carriages; road builders try to minimise obstacles. Gradients, built-up objects, etc.
- $\rightsquigarrow$ first passage percolation (FPP). Roads coalesce.
- More tools in Exponential last passage percolation (LPP). Should behave similarly to FPP.

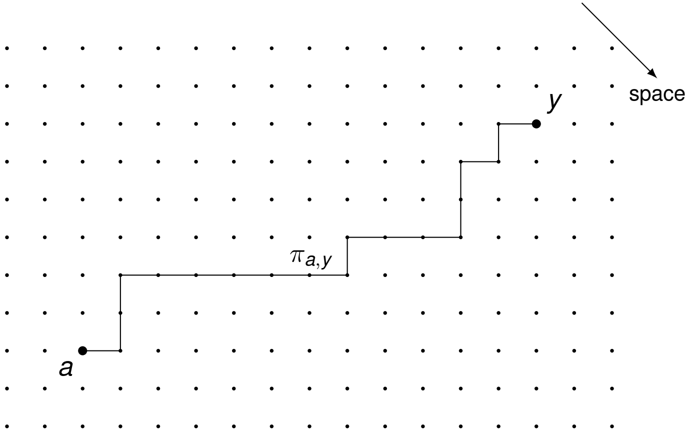
Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.



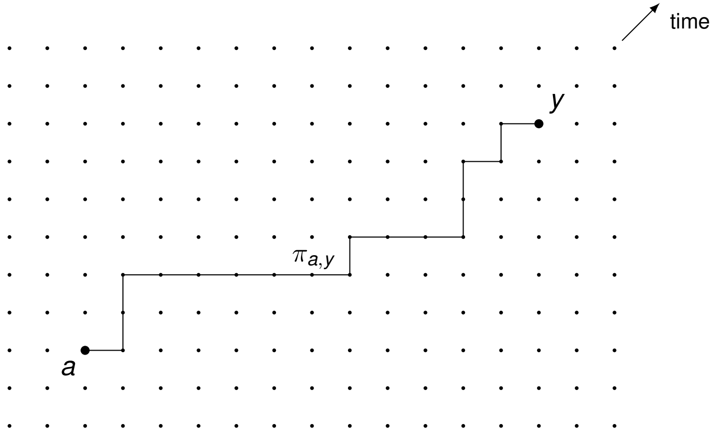
Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.

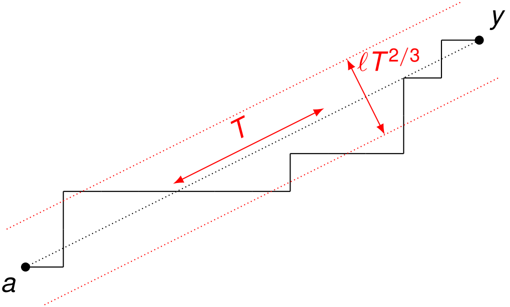


Last passage percolation

- Place ω_z i.i.d. Exp(1) for $z \in \mathbb{Z}^2$.
- The *geodesic* $\pi_{a,y}$ from a to y is the a.s. unique heaviest up-right path from a to y. Its weight is $G_{a,y}$.



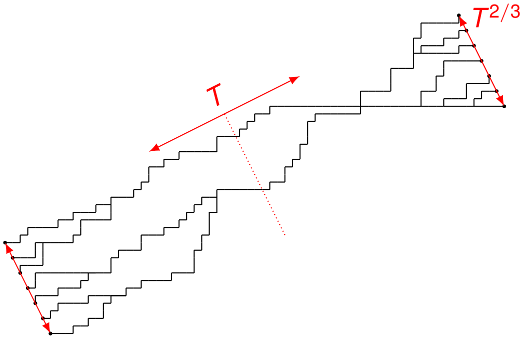
Last passage percolation: properties



$\mathbb{P}\{$geodesic exits width $\ell T^{2/3}\} \leq$ const $\cdot\, \mathrm{e}^{-C\ell^3}$ [Basu, Sarkar, Sly '19; Busani, Ferrari '22]

(KPZ transversal fluctuations).

Last passage percolation: properties



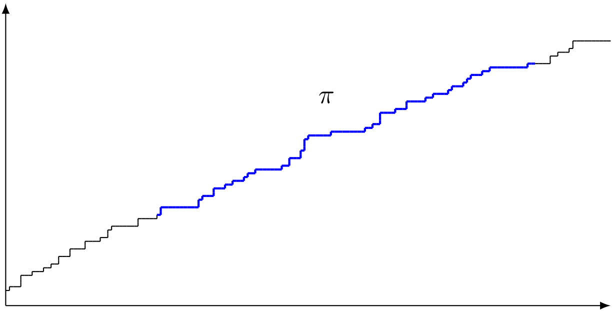
$\mathbb{P}\{$more than ℓ geodesics at mid-line$\} \le$ const $\cdot\, \mathrm{e}^{-C\ell^{1/128}}$

[Basu, Hoffman, Sly '22]

(Midpoint problem).

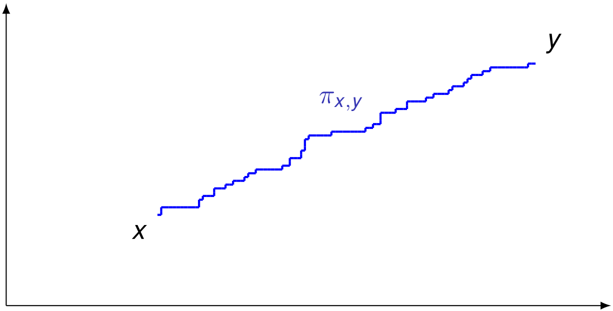
Last passage percolation

A *semi-infinite geodesic* is one that starts from a point and any of its segments is itself a geodesic between the two endpoints.



Last passage percolation

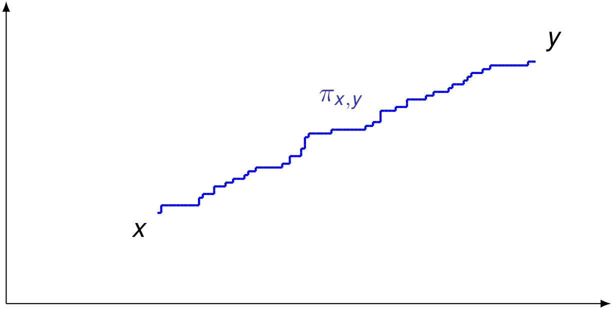
A *semi-infinite geodesic* is one that starts from a point and any of its segments is itself a geodesic between the two endpoints.



Last passage percolation

A *semi-infinite geodesic* is one that starts from a point and any of its segments is itself a geodesic between the two endpoints.

For any fixed direction this a.s. exists and is unique. [Newman (et al) '96]; [Wüthricht '02]; [Ferrari, Pimentel '05]; [Coupier '11]; [Janjigian, Rassoul-Agha, Seppäläinen '23]



Our model

- Throw i.i.d. Exp(1) weights on $\mathbb{Z}^2$.

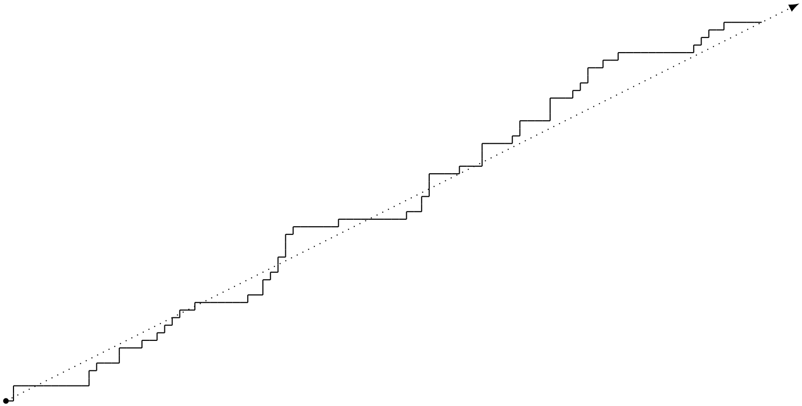
Our model

- Throw i.i.d. Exp(1) weights on $\mathbb{Z}^2$.
- Give each point on $\mathbb{Z}^2$ Uniform$(\varepsilon, \frac{\pi}{2} - \varepsilon)$ independent angles. Cars start from everywhere, in random directions.

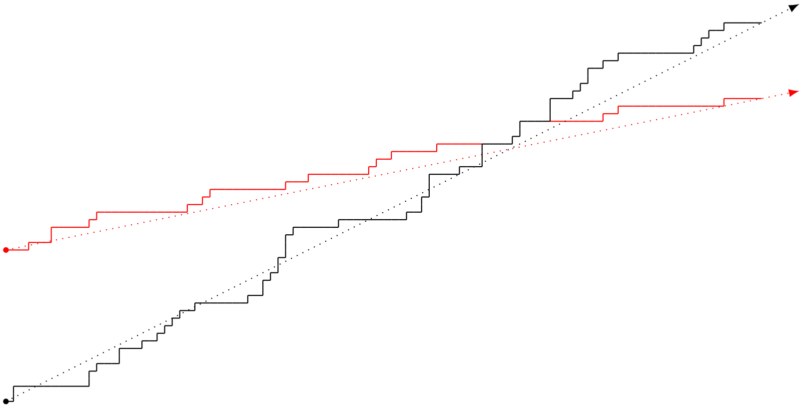
Our model

- Throw i.i.d. Exp(1) weights on $\mathbb{Z}^2$.
- Give each point on $\mathbb{Z}^2$ Uniform(ε, $\frac{\pi}{2} - \varepsilon$) independent angles. Cars start from everywhere, in random directions.
- Draw the a.s. semi-infinite geodesic for each point to its chosen direction. Many of these will coalesce when the angles are close. That's our road map with traffic data on it. A road segment is *busy* when many geodesics use that edge.

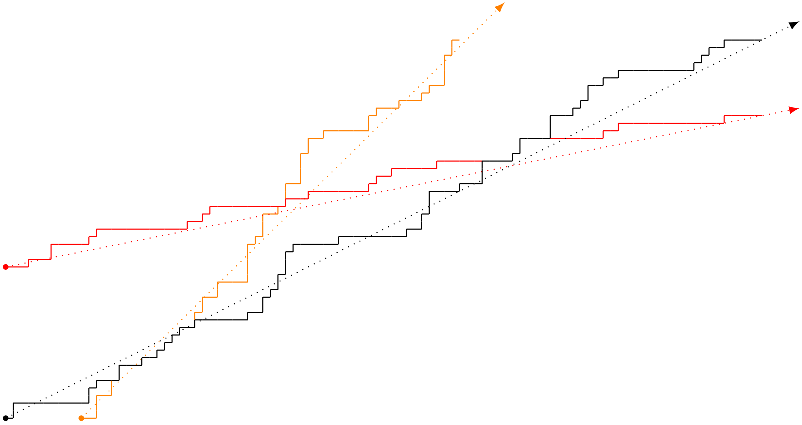
Our model



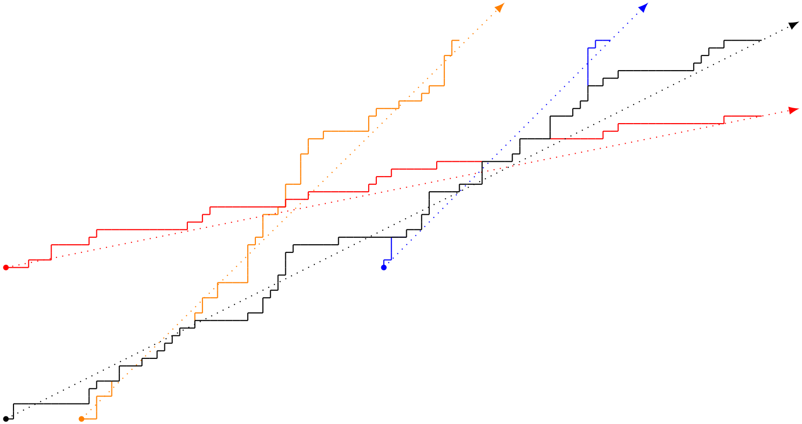
Our model



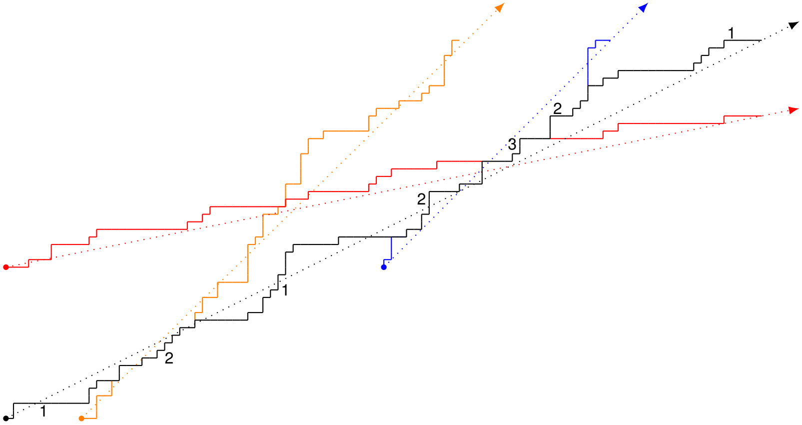
Our model



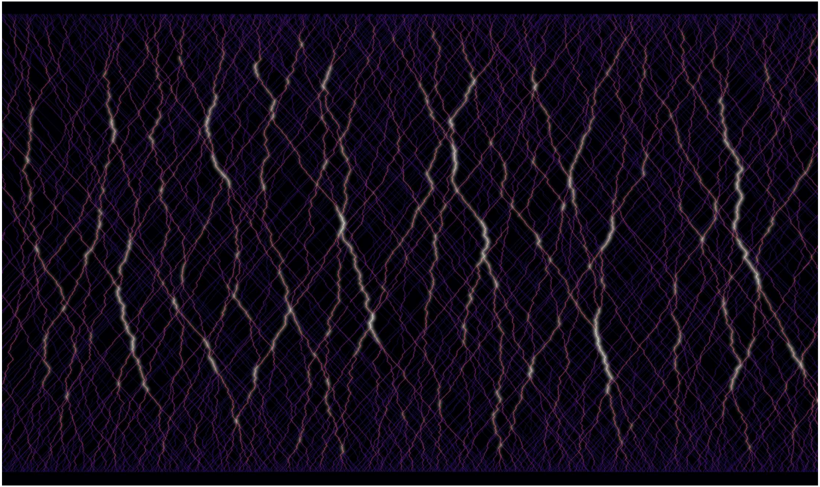
Our model



Our model



Our model



Simulation by David Harper

Questions:

- How many cars go through the origin (my house, that is)?

Questions:

- How many cars go through the origin (my house, that is)?
- From how far do cars come and visit the origin (distant guests to my house)?

Questions:

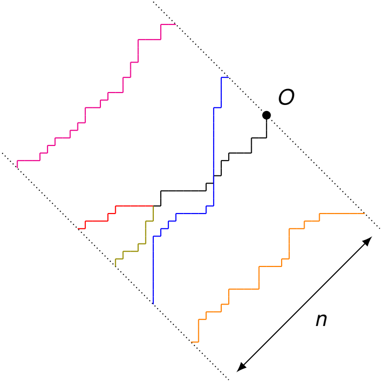
- How many cars go through the origin (my house, that is)?
- From how far do cars come and visit the origin (distant guests to my house)?
- How far is the nearest busy road? I.e., within distance k in the space-direction, what is the probability to find an edge with k^α geodesics on it (and what is the interesting α)?

Questions:

- How many cars go through the origin (my house, that is)?
- From how far do cars come and visit the origin (distant guests to my house)?
- How far is the nearest busy road? I.e., within distance k in the space-direction, what is the probability to find an edge with k^α geodesics on it (and what is the interesting α)?
- Is this actually a good model of real road networks out there?

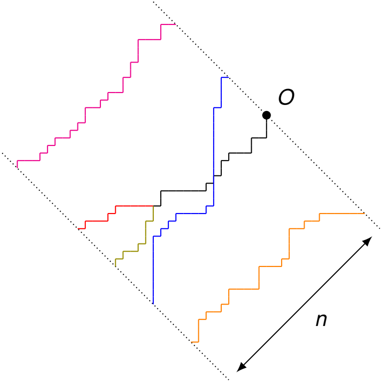
How many cars go through the origin?

The number N_n that start from distance n and go through O is mean 1 (mass transport principle).



How many cars go through the origin?

The number N_n that start from distance n and go through O is mean 1 (mass transport principle).



From all layers: $N = \sum_{n=1}^{\infty} N_n$ is of infinite mean.

Answers

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

Theorem

$$\mathbb{P}\{\textit{road with} \geq k^4 \textit{ cars within distance } k\} \sim \mathcal{O}\left(\tfrac{1}{2}\right).$$

Answers

Theorem

$$cn^{-1/3} \le \mathbb{P}\{\textit{a car from distance} \ge n \textit{ visits } O\} \le Cn^{-1/3}.$$

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

Theorem

$$\mathbb{P}\{\textit{road with} \ge k^4 \textit{ cars within distance } k\} \sim \mathcal{O}\big(\tfrac{1}{2}\big).$$

We don’t believe the log.

Busy road close by?

To be more precise:

Theorem

$$\mathbb{P}\left\{\textit{no road with} \geq k^4 \textit{ cars within distance } \frac{\delta k}{\log k}\right\} \geq 1 - C\delta.$$

Busy road close by?

To be more precise:

Theorem

$$\mathbb{P}\left\{\textit{no road with} \geq k^4 \textit{ cars within distance } \frac{\delta k}{\log k}\right\} \geq 1 - C\delta.$$

We don't believe the log.

Busy road close by?

To be more precise:

Theorem

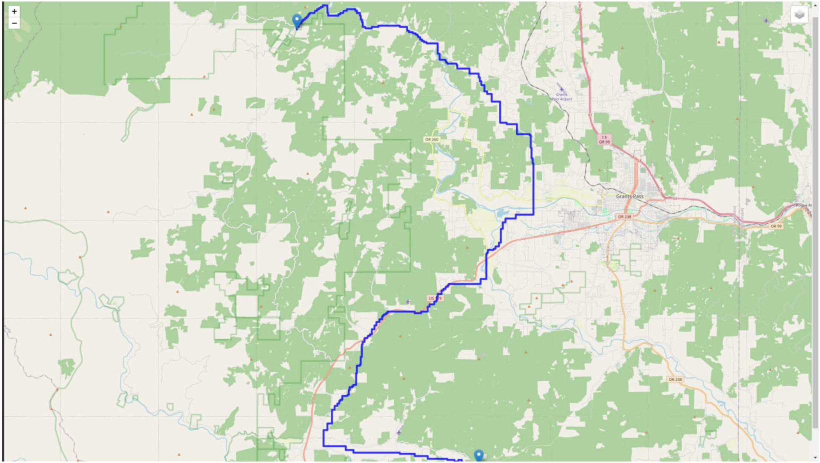
$$\mathbb{P}\left\{\textit{no road with} \geq k^4 \textit{ cars within distance } \frac{\delta k}{\log k}\right\} \geq 1 - C\delta.$$

We don't believe the log.

Theorem

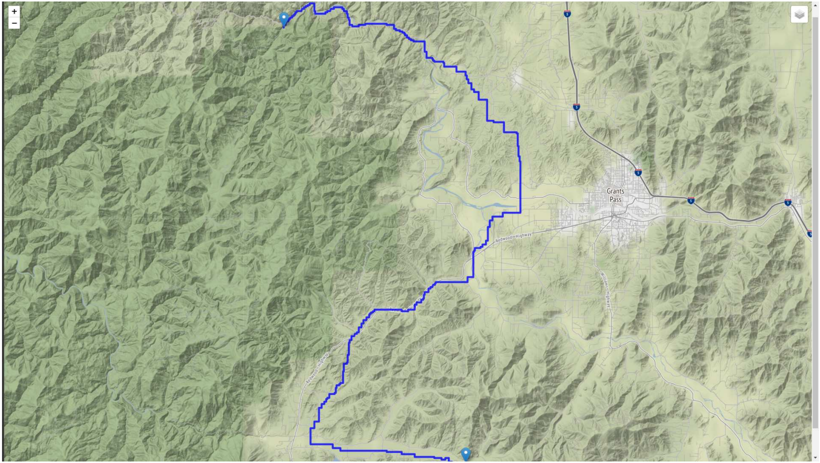
$$\mathbb{P}\{\textit{yes, road with} \geq \textit{const} \cdot k^4 \textit{ cars within distance } k\} \geq c.$$

Is this all any good?



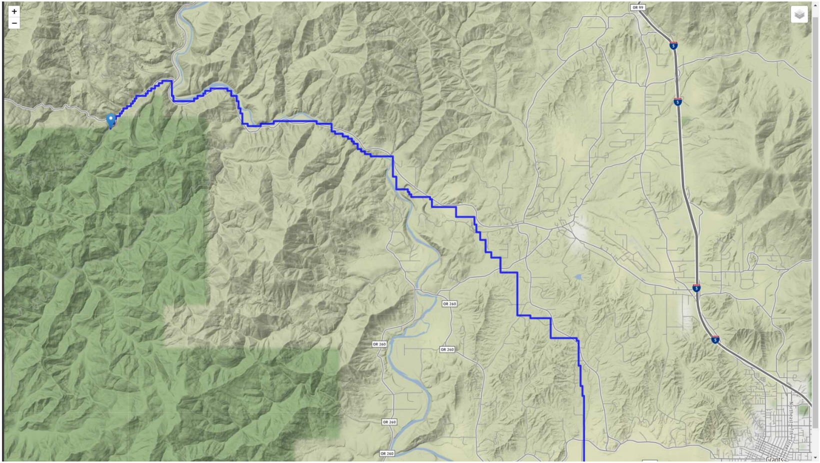
Simulation by David Harper

Is this all any good?



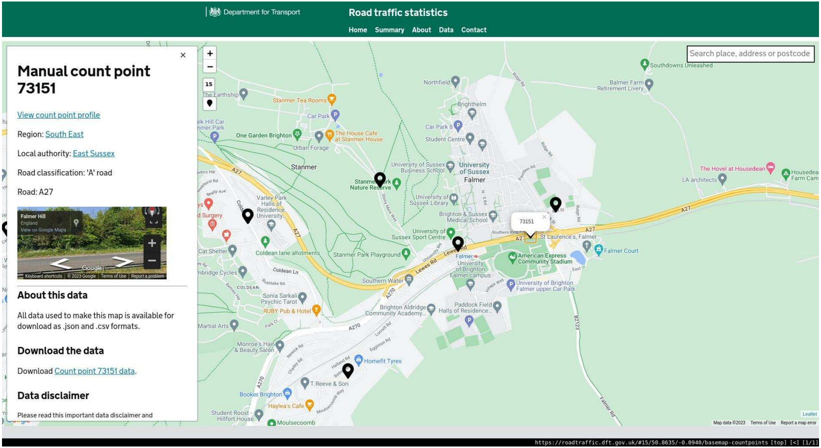
Simulation by David Harper

Is this all any good?



Simulation by David Harper

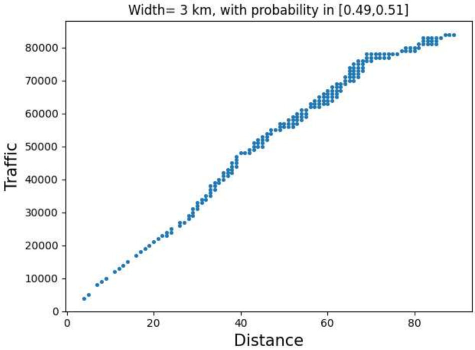
Is this all any good?



Is this all any good?

$\mathbb{P}\{$road with $\geq \ell$ cars within distance $k\}\ldots$?

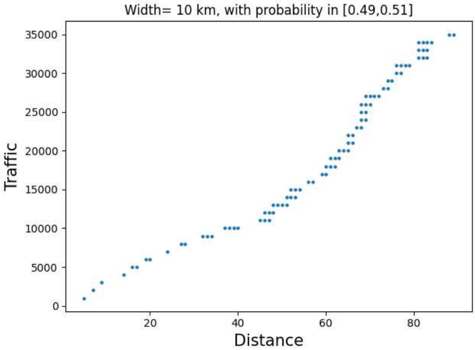
Is this all any good? On the South:



Between 49% and 51% of startpoints have at least this much traffic within the distance shown.

Thm: $\mathbb{P}\{\text{road with} \geq k^4 \text{ cars within distance } k\} \sim \mathcal{O}\left(\frac{1}{2}\right).$

Is this all any good? On the North and the West:



Between 49% and 51% of startpoints have at least this much traffic within the distance shown.

Thm: $\mathbb{P}\{\text{road with} \geq k^4 \text{ cars within distance } k\} \sim \mathcal{O}\left(\frac{1}{2}\right)$.

Is this all any good?

- Average length of a UK car journey: 10...15 km.

Is this all any good?

- Average length of a UK car journey: 10...15 km.
- Big (i.e., well measured) roads are 20...30 km apart.

Is this all any good?

- Average length of a UK car journey: 10...15 km.
- Big (i.e., well measured) roads are 20...30 km apart.
- $20^{3/2} \ldots 30^{3/2} \simeq 90 \ldots 160$.

Is this all any good?

- Average length of a UK car journey: 10…15 km.
- Big (i.e., well measured) roads are 20…30 km apart.
- $20^{3/2} \ldots 30^{3/2} \simeq 90 \ldots 160$.
- Most car trips just don't span far enough to reach a big road.

Is this all any good?

- Average length of a UK car journey: 10. . . 15 km.
- Big (i.e., well measured) roads are 20. . . 30 km apart.
- $20^{3/2} \dots 30^{3/2} \simeq 90 \dots 160$.
- Most car trips just don't span far enough to reach a big road.

Thank you.

From how far?

Theorem

$$cn^{-1/3} \le \mathbb{P}\{\textit{a car from distance} \ge n \textit{ visits } O\} \le Cn^{-1/3}.$$

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.

From how far?

Theorem

$$cn^{-1/3} \le \mathbb{P}\{\textit{a car from distance} \ge n \textit{ visits } O\} \le Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.

From how far?

Theorem

$$cn^{-1/3} \le \mathbb{P}\{\textit{a car from distance} \ge n \textit{ visits } O\} \le Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- Then either cars wandered in wrong directions (transversal fluctuations) or not enough coalescence happened from distance n (midpoint problem).

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- Then either cars wandered in wrong directions (transversal fluctuations) or not enough coalescence happened from distance n (midpoint problem).
- $\rightsquigarrow$ upper bound.

From how far?

Theorem

$$cn^{-1/3} \le \mathbb{P}\{\textit{a car from distance} \ge n \textit{ visits } O\} \le Cn^{-1/3}.$$

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations).

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations).
- Complications with inclusion-exclusion; independence of different enough directions...

From how far?

Theorem

$$cn^{-1/3} \leq \mathbb{P}\{\textit{a car from distance} \geq n \textit{ visits } O\} \leq Cn^{-1/3}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations).
- Complications with inclusion-exclusion; independence of different enough directions...
- $\rightsquigarrow$ lower bound.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- We already know they cannot come from too far.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- We already know they cannot come from too far.
- Then either too many cars picked the direction of the origin (independent choices), or many wandered in from a wrong direction (transversal fluctuations).

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- We already know they cannot come from too far.
- Then either too many cars picked the direction of the origin (independent choices), or many wandered in from a wrong direction (transversal fluctuations).
- $\rightsquigarrow$ upper bound.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too large then the event is expected to happen often on the space-line.
- We already know they cannot come from too far.
- Then either too many cars picked the direction of the origin (independent choices), or many wandered in from a wrong direction (transversal fluctuations).
- $\rightsquigarrow$ upper bound.
- We don't believe the log.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations) or they haven't coalesced enough (midpoint problem).

How many cars go through the origin?

Theorem

$$\frac{c}{k} \leq \mathbb{P}\{N \geq k^4\} \leq \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations) or they haven't coalesced enough (midpoint problem).
- Complications with inclusion-exclusion; independence of different enough directions...

How many cars go through the origin?

Theorem

$$\frac{c}{k} \le \mathbb{P}\{N \ge k^4\} \le \frac{C \log k}{k}.$$

- Break up the angle space into intervals of length $n^{-1/3}$: only need to deal with geodesic trees of a fixed direction.
- Averaging argument: if $\mathbb{P}\{\cdot\}$ is too small then the event is not expected to happen much in a segment of the space-line.
- Then either no cars picked this direction (independent choices) or they wandered in wrong directions (transversal fluctuations) or they haven't coalesced enough (midpoint problem).
- Complications with inclusion-exclusion; independence of different enough directions...
- $\rightsquigarrow$ lower bound.