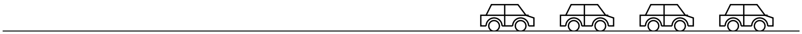
A simple model for traffic jams

Márton Balázs

School of Mathematics

Open Day, University of Bristol, 26th October, 2024

Arriving to a traffic jam



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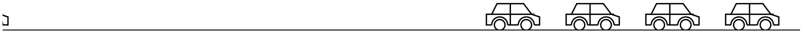

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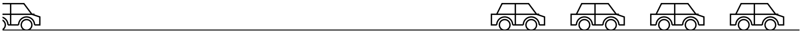

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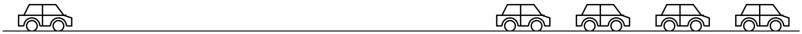

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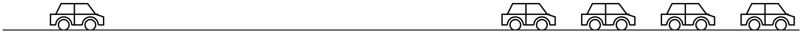

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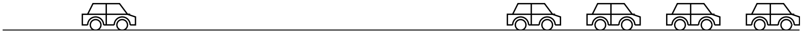

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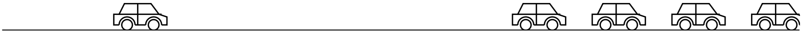

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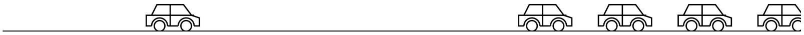

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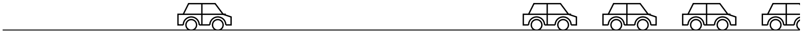

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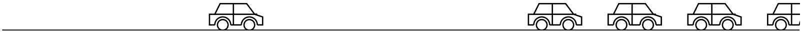

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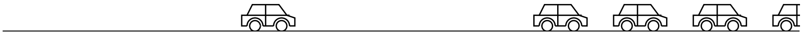

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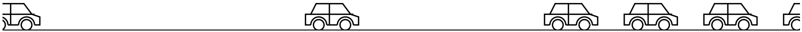

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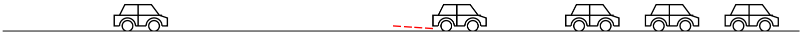

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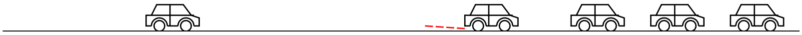

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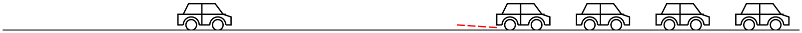

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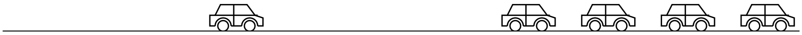

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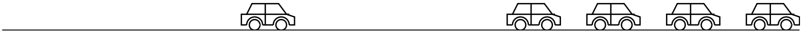

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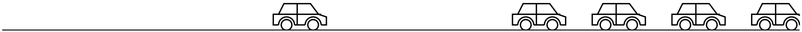

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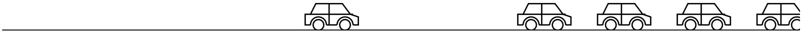

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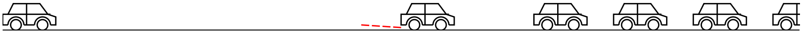

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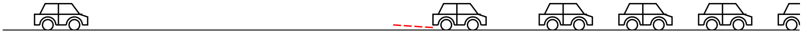

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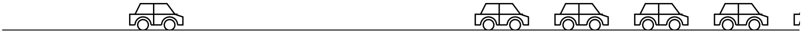

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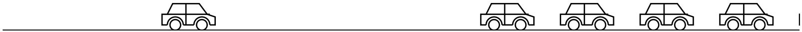

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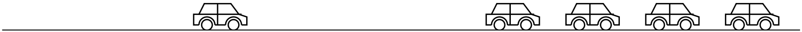

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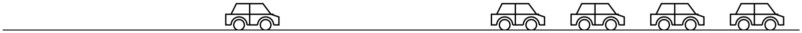

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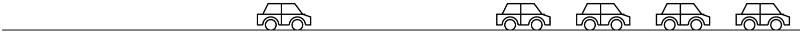

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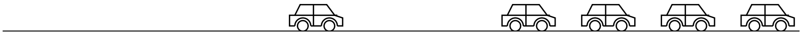

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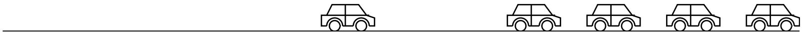

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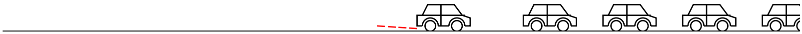

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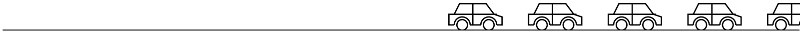

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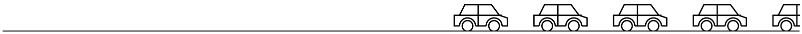

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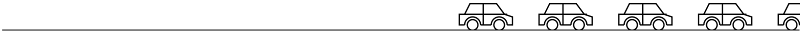

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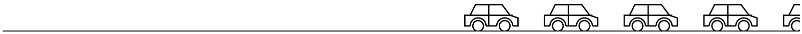

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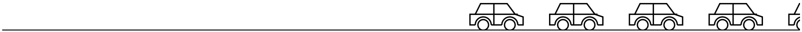

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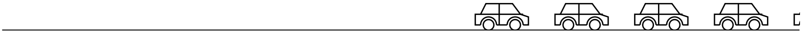

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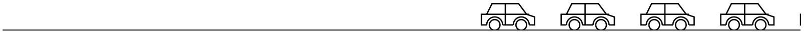

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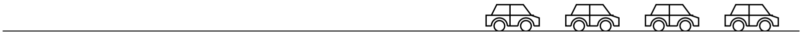

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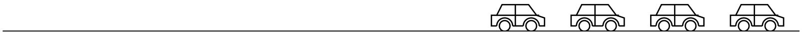

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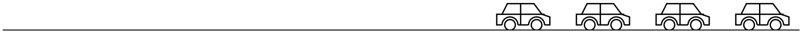

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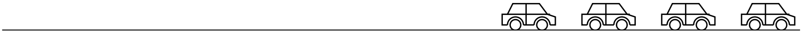

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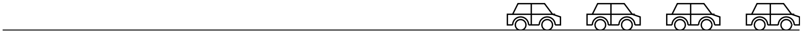

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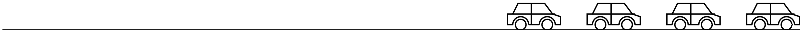

We notice the slow cars $\rightsquigarrow$ strong braking immediately.

Arriving to a traffic jam is always sharp.

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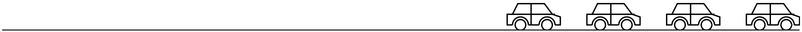
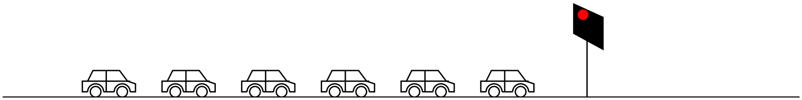


We notice the slow cars $\rightsquigarrow$ strong braking immediately.

Arriving to a traffic jam is always sharp.

This is one aspect that makes motorways dangerous places.

Leaving a traffic jam



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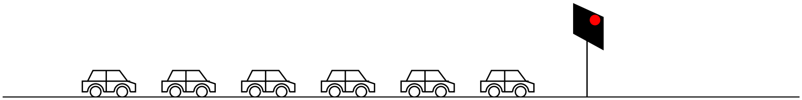

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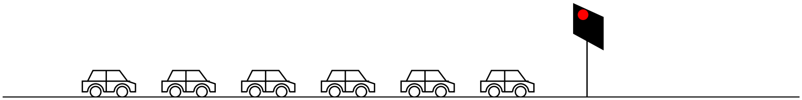

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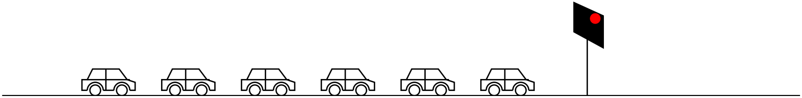

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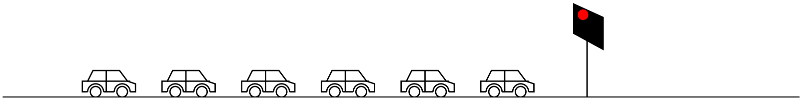

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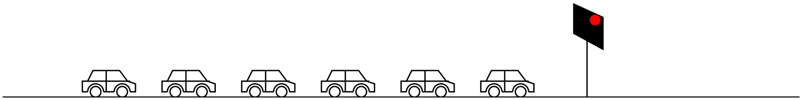

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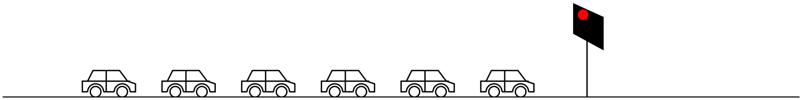

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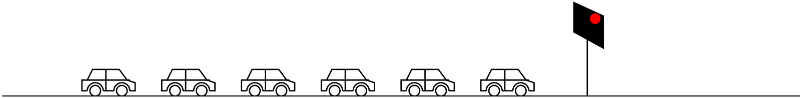

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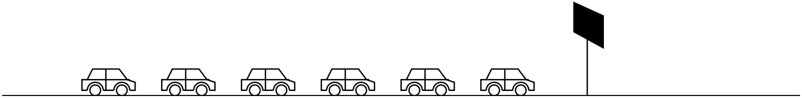

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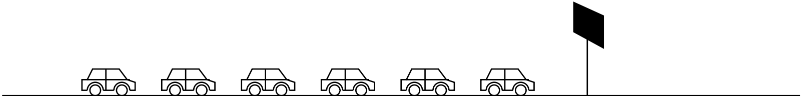

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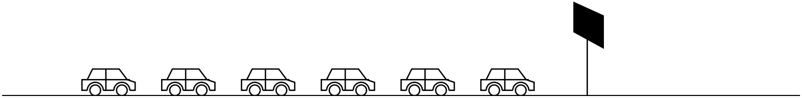

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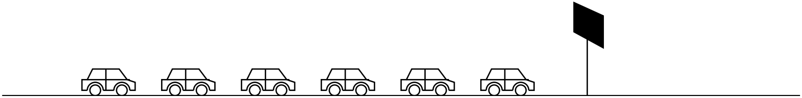

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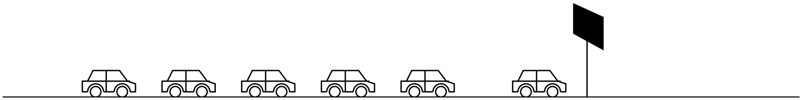

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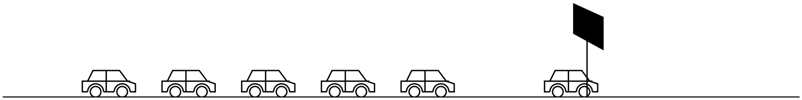

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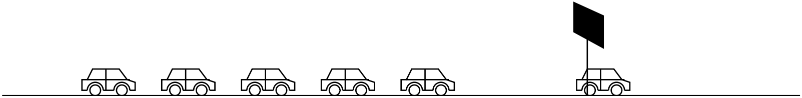

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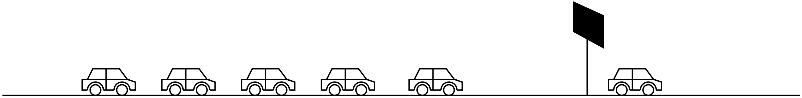

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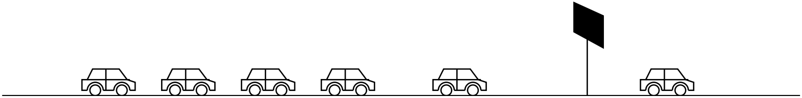

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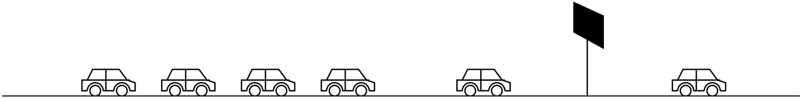

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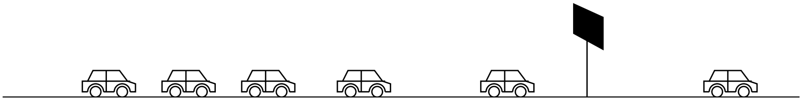

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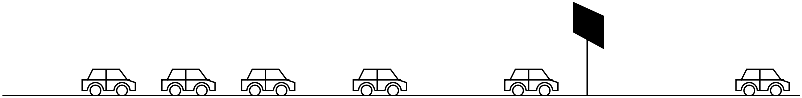

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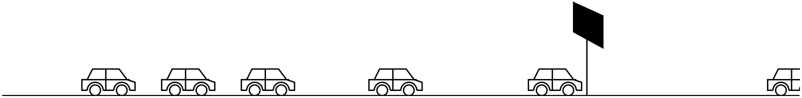

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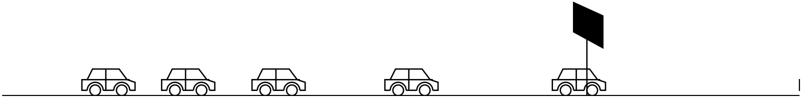

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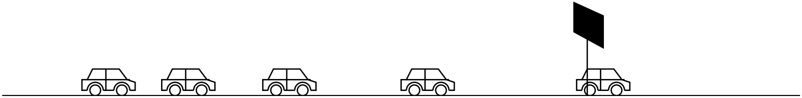

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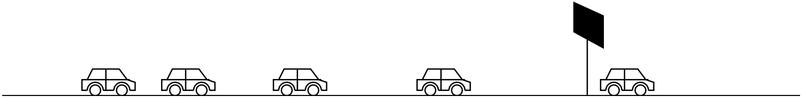

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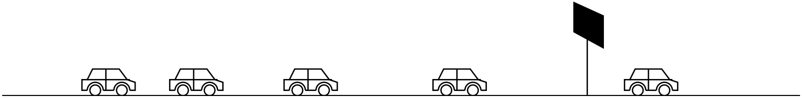

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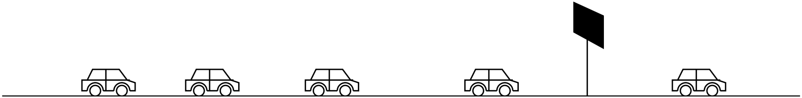

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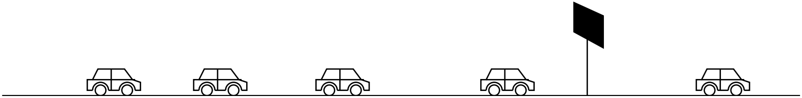

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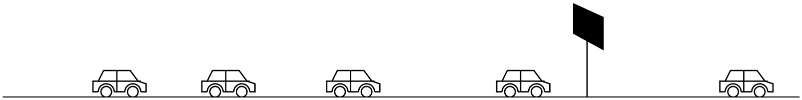

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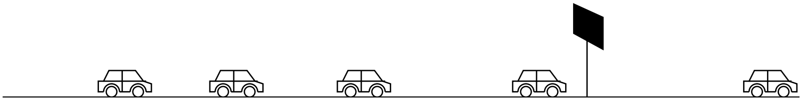

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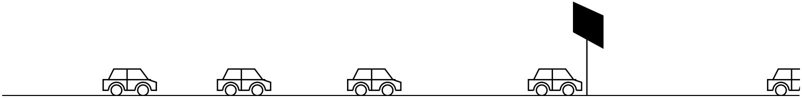

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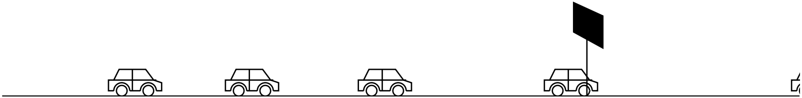

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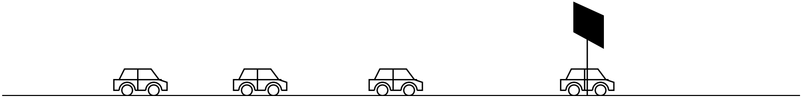

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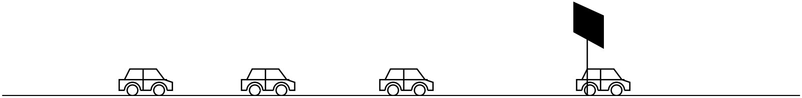

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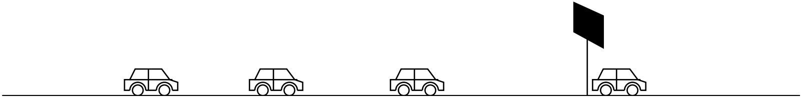

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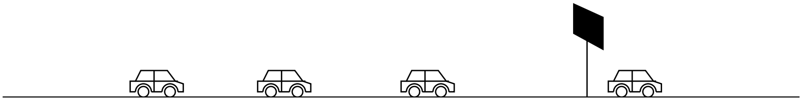

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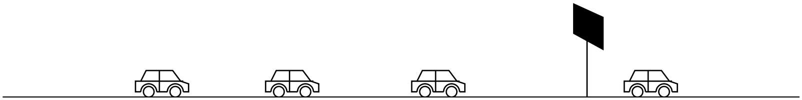

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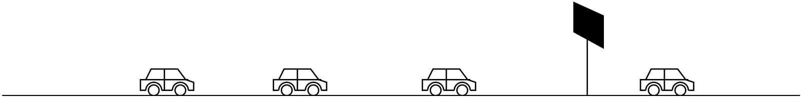

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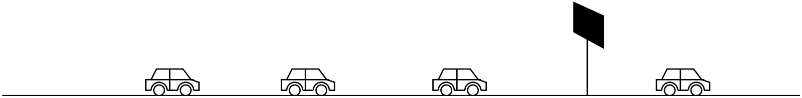

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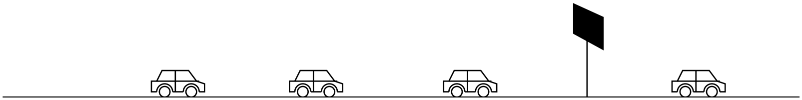

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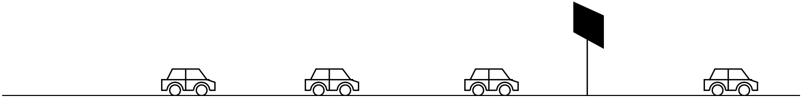

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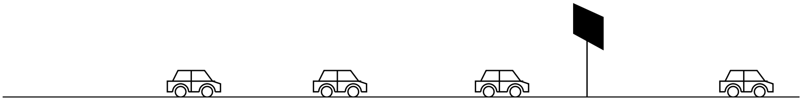

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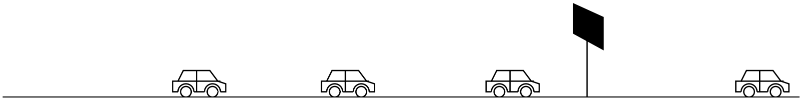

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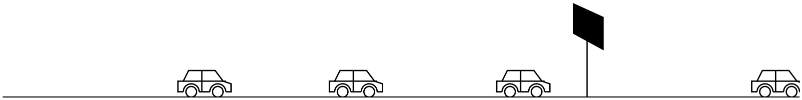

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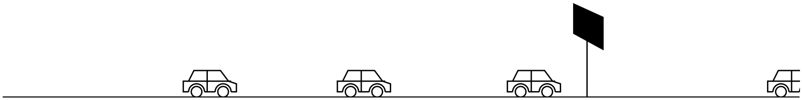

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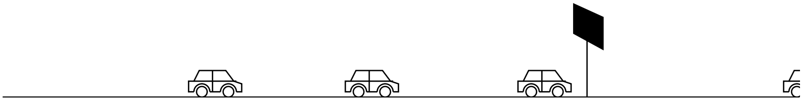

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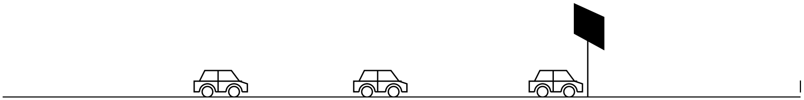

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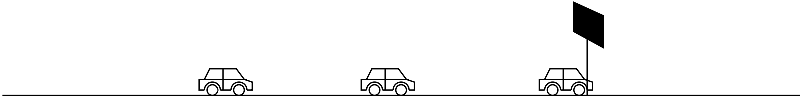

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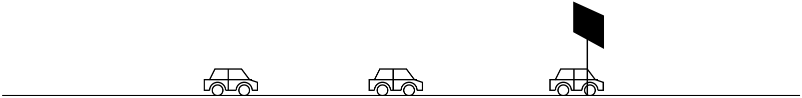

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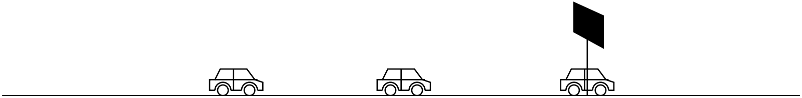

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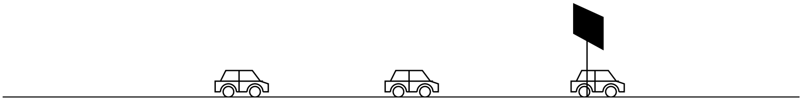

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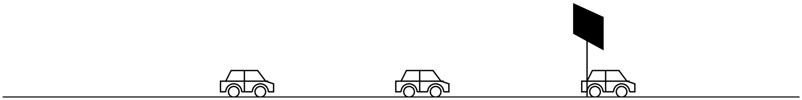

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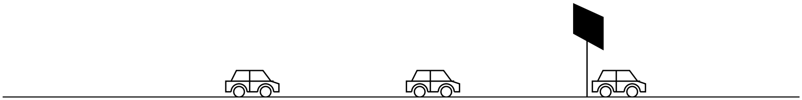

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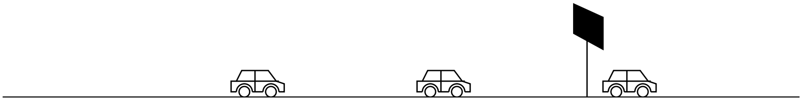

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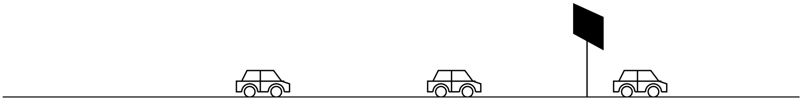

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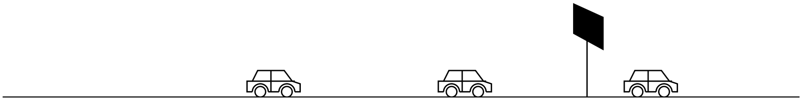

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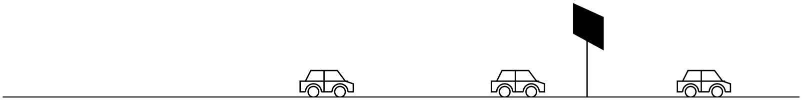

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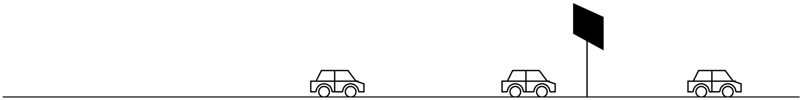

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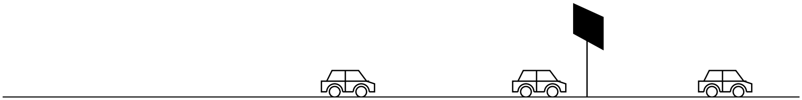

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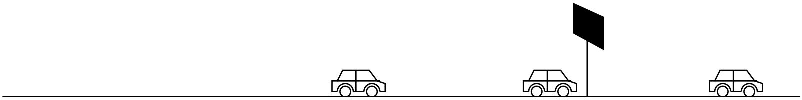

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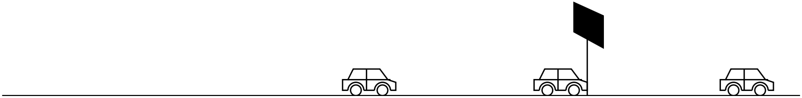

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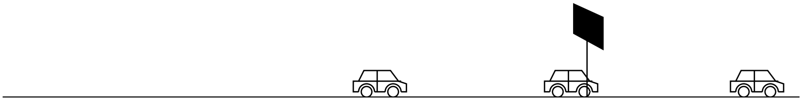

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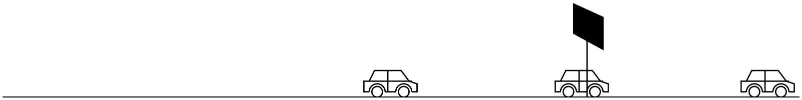

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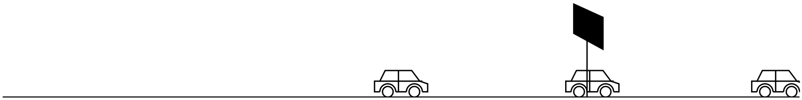

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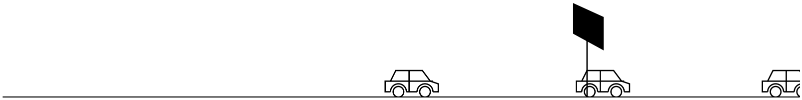

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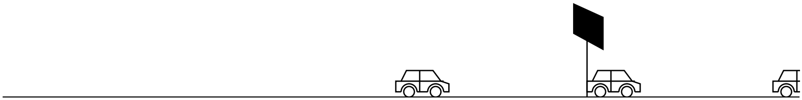

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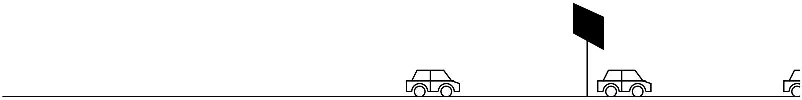

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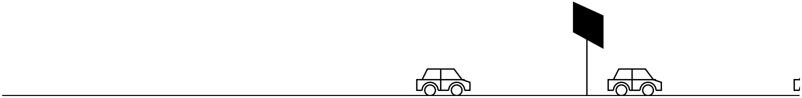

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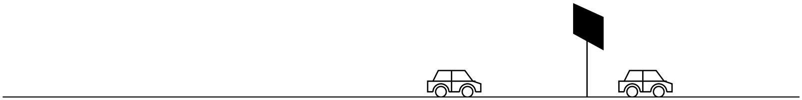

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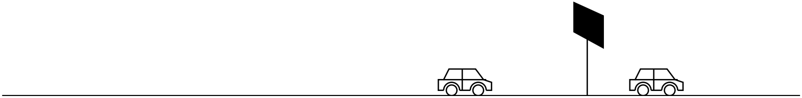

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Continuous, long acceleration for those starting from the rear

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Leaving a traffic jam is always soft, “blurry”.

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Why is there such a difference between the two ends of a traffic jam?

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Continuous, long acceleration for those starting from the rear

Leaving a traffic jam is always soft, “blurry”.

Why is there such a difference between the two ends of a traffic jam?

Totally asymmetric simple exclusion process: an explanation

Being ageless

We first seek a random time that does not remember its past.
Let $\tau > 0$ be a random time such that

$$\mathbf{P}\{\tau > t\} = \mathrm{e}^{-t} \qquad \text{for all } t > 0.$$

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Suppose that this time has not passed yet at t. What is the probability that it will not pass before another s seconds? That is, out of those cases when $\tau > t$ occurs, in what percentage will $\tau > t + s$ also occur?

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The answer is:

$$\frac{\mathbf{P}\{\tau > t + s\}}{\mathbf{P}\{\tau > t\}} = \frac{\mathrm{e}^{-(t+s)}}{\mathrm{e}^{-t}} = \mathrm{e}^{-s}.$$

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The same as $\mathbf{P}\{\tau > s\}$, regardless of t!

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The same as $\mathbf{P}\{\tau > s\}$, regardless of t!
We have found the secret of being ageless.

Being ageless

⏰ $\leftarrow$ This will be the ageless alarm clock that rings at time τ

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$\rightsquigarrow$ What is the probability that an  rings within a small time t?

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$$\mathbf{P}\{\tau \le t\} = 1 - \mathbf{P}\{\tau > t\} = 1 - \mathrm{e}^{-t} \simeq 1 - (1 - t) + \text{error} = t + \text{error}.$$

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$\rightsquigarrow$ What is the probability that an ⏰ rings within a small time t?

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$\rightsquigarrow$ What is the probability that *two* independent ⏰ ⏰ both ring within a small time t?

Being ageless

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$\rightsquigarrow$ What is the probability that *two* independent ⏰ ⏰ both ring within a small time t?

$$\mathbf{P}\{\tau \le t\} \cdot \mathbf{P}\{\tau \le t\} \simeq t^2 + \text{error} = \text{error}.$$

Being ageless

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$\rightsquigarrow$ What is the probability that an ⏰ rings within a small time t?

$$\mathbf{P}\{\tau \le t\} = 1-\mathbf{P}\{\tau > t\} = 1-\mathrm{e}^{-t} \simeq 1-(1-t)+\text{error} = t+\text{error}.$$

$\rightsquigarrow$ What is the probability that *two* independent ⏰ ⏰ both ring within a small time t?

$$\mathbf{P}\{\tau \le t\} \cdot \mathbf{P}\{\tau \le t\} \simeq t^2 + \text{error} = \text{error}.$$

$\rightarrow$ More ⏰ 's, even smaller probability.

Being ageless

⇝ What is the probability that *none* of k independent  's ring within a small time t?

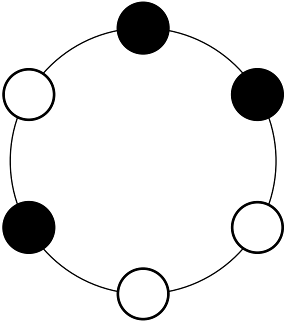
Being ageless

$\rightsquigarrow$ What is the probability that *none* of k independent  's ring within a small time t?

$$
\begin{aligned}
\mathbf{P}\{\text{none of them ring}\} &= \mathbf{P}\{\tau > t\}^k \\
&= \mathrm{e}^{-kt} \\
&\simeq (1 - kt) + \text{error.}
\end{aligned}
$$

The Totally Asymmetric Simple Exclusion Process

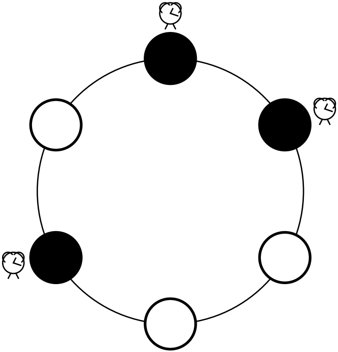
TASEP



m balls in N possible slots.

The Totally Asymmetric Simple Exclusion Process

TASEP



m balls in N possible slots.

Each listening to its own  . When that rings, the ball tries to jump to the right.

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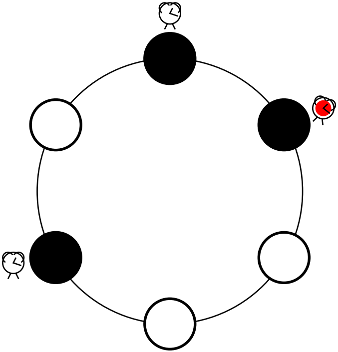

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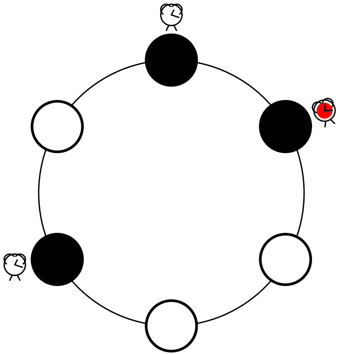

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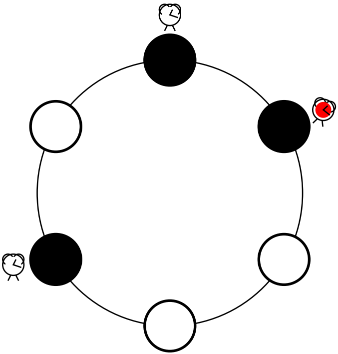

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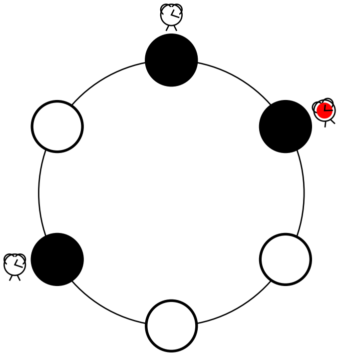

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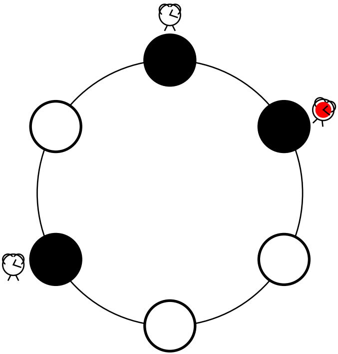

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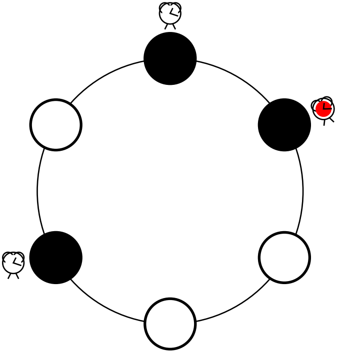

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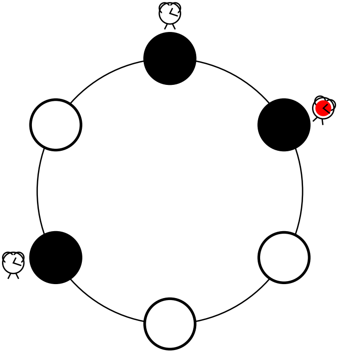

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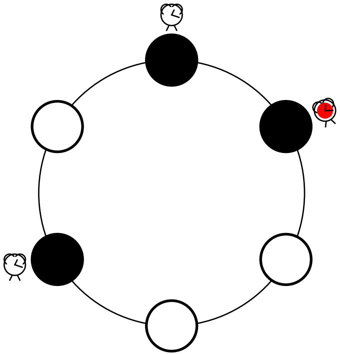

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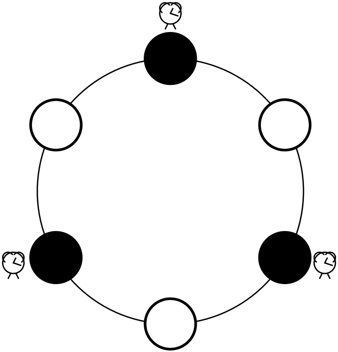

m balls in N possible slots.

Each listening to its own  . When that rings, the ball tries to jump to the right.

The Totally Asymmetric Simple Exclusion Process

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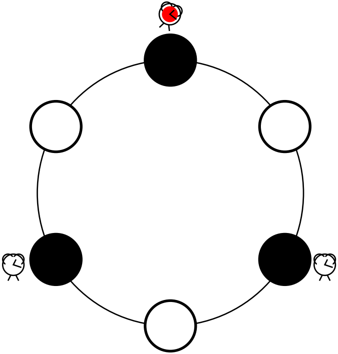

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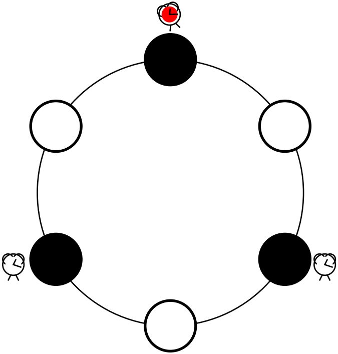

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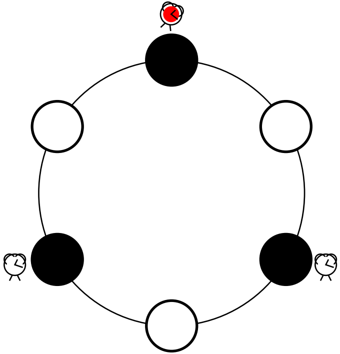

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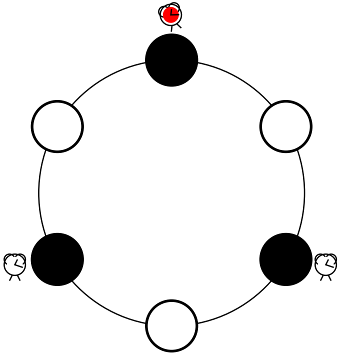

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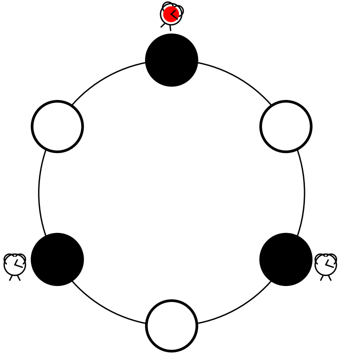

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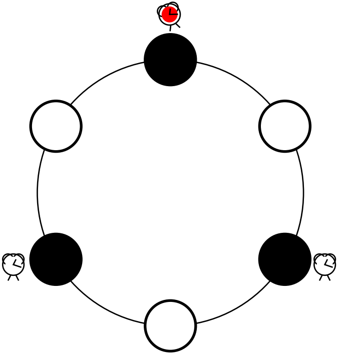

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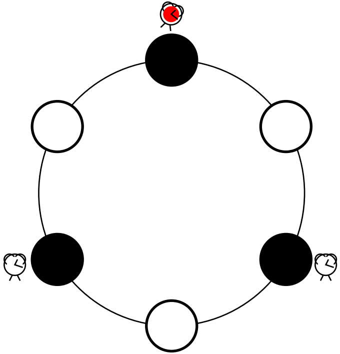

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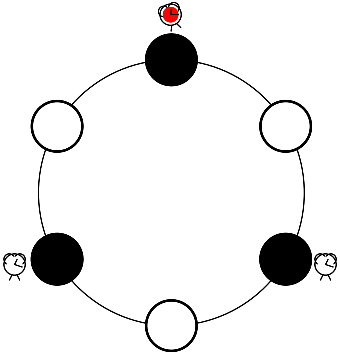

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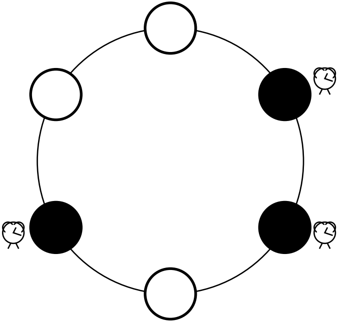

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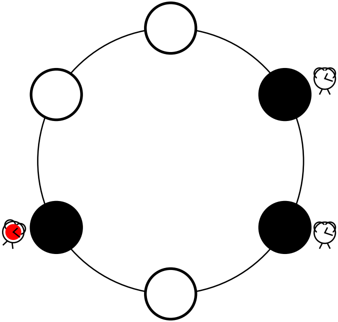

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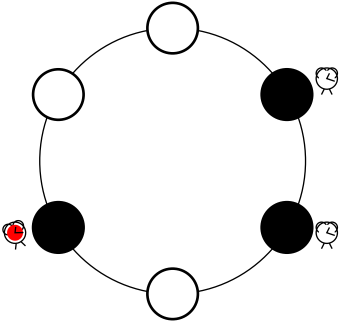

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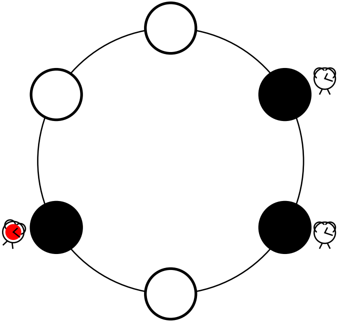

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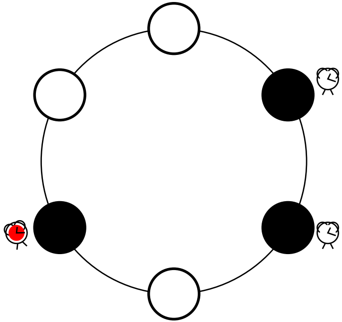

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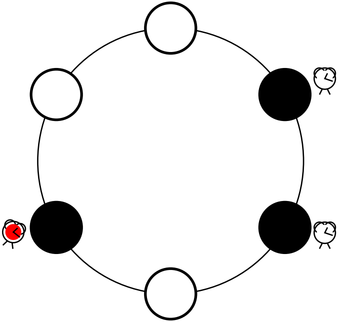

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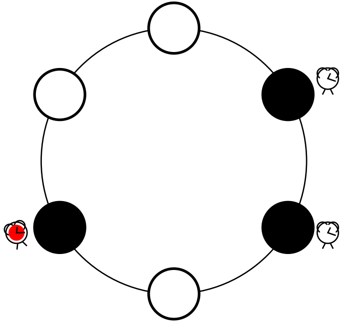

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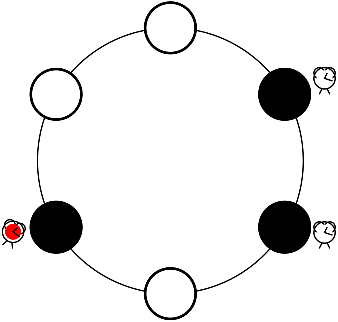

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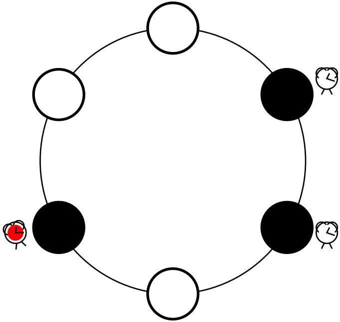

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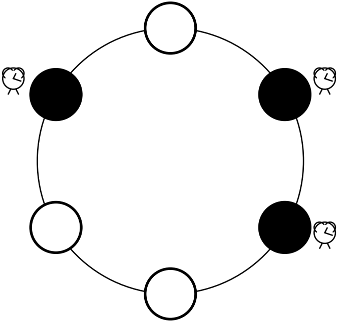

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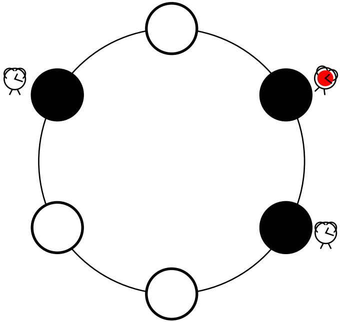

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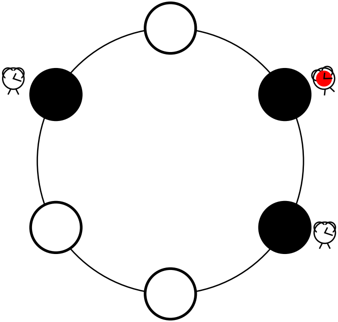

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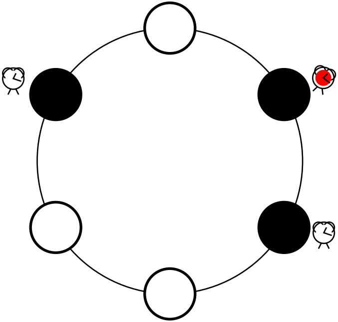

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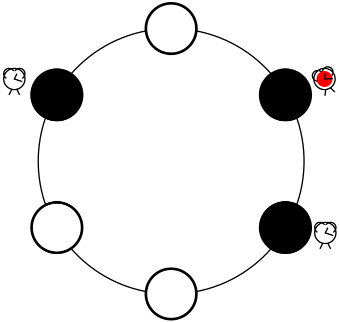

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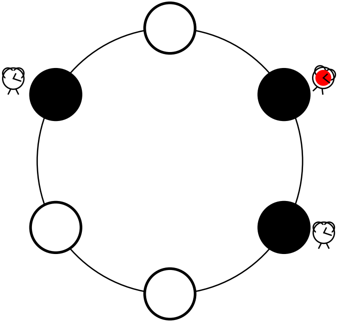

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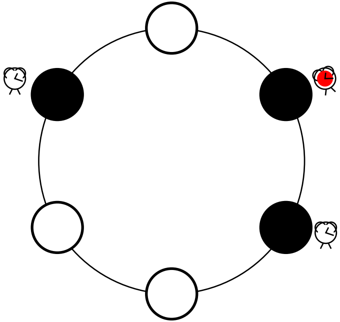

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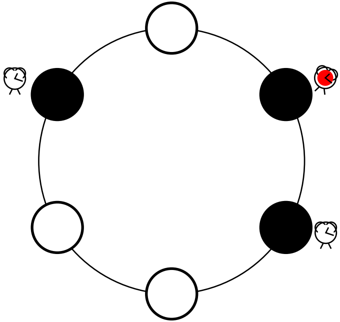

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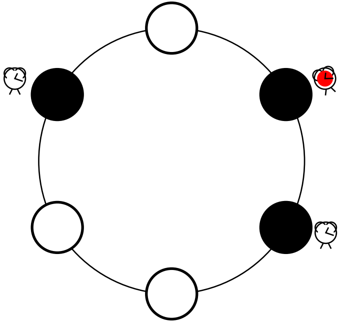

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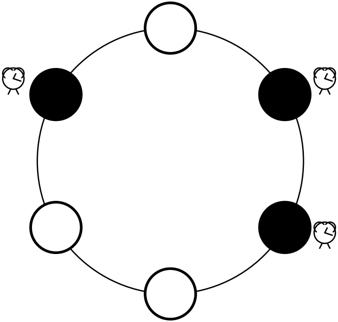

m balls in N possible slots.

Each listening to its own ⏰. When that rings, the ball tries to jump to the right. But sometimes it's blocked.

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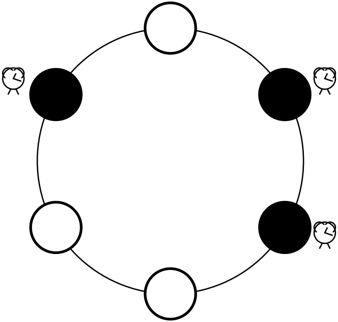

m balls in N possible slots.

Each listening to its own . When that rings, the ball tries to jump to the right. But sometimes it's blocked.
Ageless, independent 's $\Rightarrow$ if we know the present, no need to know the past. *Markov property*, makes things handy.

Stationary distribution

Random process $\rightsquigarrow$ need to talk about *distributions*.

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What is the stationary distribution the one that's unchanged in time?

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Theorem

With N and m fixed, the distribution that gives equal chance to each (m-ball) configuration, is stationary.

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In this case every configuration occurs with probability $1/\binom{N}{m}$.

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With N and m fixed, the distribution that gives equal chance to each (m-ball) configuration, is stationary.

1st remark.

In this case every configuration occurs with probability $1/\binom{N}{m}$.

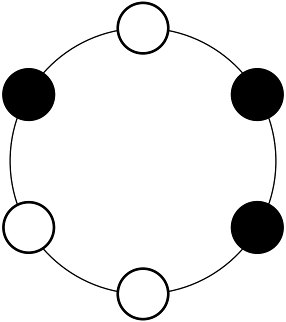
2nd remark.

With fixed N, m, there is no other stationary distribution.

Stationary distribution

Almost proof

A state ω



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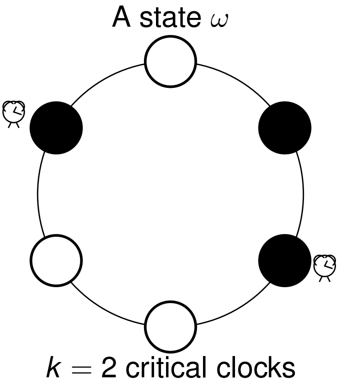

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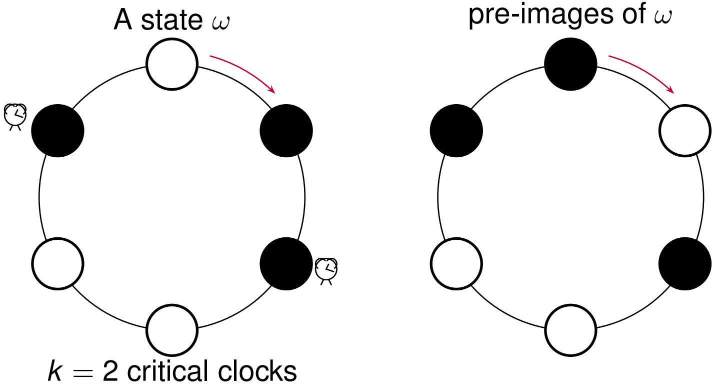

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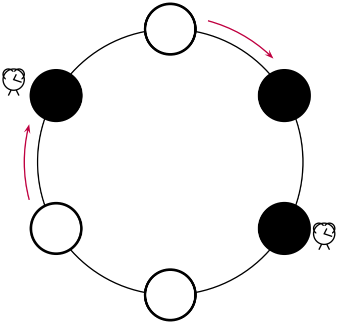

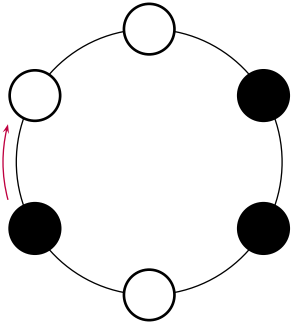

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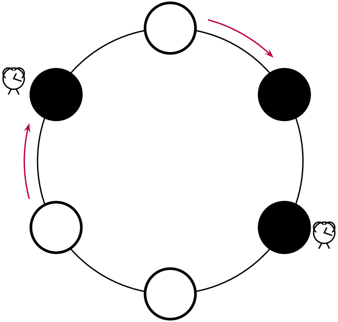

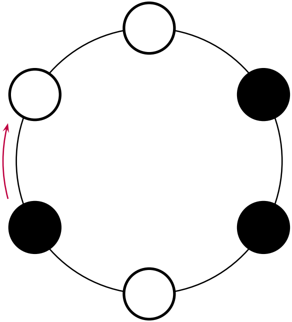

The number of critical clocks for ω = the number of pre-images of $\omega = k$

Stationary distribution

Almost proof

Suppose that each configuration has the same probability p at time s. What is the probability of the state ω after a small time t?

Stationary distribution

Almost proof

Suppose that each configuration has the same probability p at time s. What is the probability of the state ω after a small time t?

$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

Stationary distribution

Almost proof

Suppose that each configuration has the same probability *p* at time *s*. What is the probability of the state ω after a small time *t*?

$$\begin{aligned}
&\mathbf{P}\{\omega \text{ at time } s+t\} \\
&= \mathbf{P}\{\omega \text{ at time } s \text{ and no jumps within time } t\} \\
&\quad + \mathbf{P}\{\text{was a pre-image of } \omega \text{ at time } s\text{, and jumps to } \omega\} \\
&\quad + \text{error (at least two jumps occur within the small time } t\text{)}
\end{aligned}$$

Stationary distribution

Almost proof

Suppose that each configuration has the same probability p at time s. What is the probability of the state ω after a small time t?

$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

$$= \mathbf{P}\{\omega \text{ at time } s \text{ and no jumps within time } t\}$$

$$+ \mathbf{P}\{\text{was a pre-image of } \omega \text{ at time } s\text{, and jumps to } \omega\}$$

$+$ error (at least two jumps occur within the small time t)

$= \mathbf{P}\{\omega$ at time s and none of the k critical 's ring$\}$

$+ \sum_{\eta \text{ is a pre-image of } \omega} \mathbf{P}\{\eta$ at time s and the right critical  rings$\}$

$+$ error

Stationary distribution

Almost proof

$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

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$+$ error

$$= p \cdot (1 - kt) + \sum_{\eta \text{ is a pre-image of } \omega} p \cdot t + \text{error}$$

Stationary distribution

Almost proof

$$\mathbf{P}\{\omega \text{ at time } s+t\}$$

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$$= p\cdot(1-kt) + k\cdot p\cdot t + \text{error}$$

Stationary distribution

Almost proof

$$
\begin{aligned}
&\mathbf{P}\{\omega \text{ at time } s+t\} \\
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&\quad + \sum_{\eta \text{ is a pre-image of } \omega} \mathbf{P}\{\eta \text{ at time } s \text{ and the right critical ⏰ rings}\} \\
&\quad + \text{error} \\
&= p \cdot (1 - kt) + \sum_{\eta \text{ is a pre-image of } \omega} p \cdot t + \text{error} \\
&= p \cdot (1 - kt) + k \cdot p \cdot t + \text{error} = p + \text{error}.
\end{aligned}
$$

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$$\begin{aligned}
&\mathbf{P}\{\omega \text{ at time } s+t\}\\
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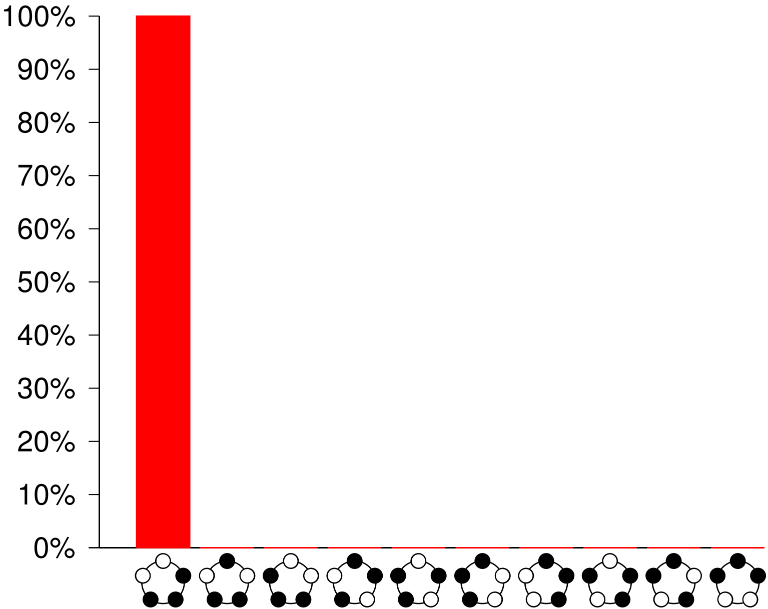
&\quad + \text{error}\\
&= p\cdot(1-kt) + \sum_{\eta \text{ is a pre-image of } \omega} p\cdot t + \text{error}\\
&= p\cdot(1-kt) + k\cdot p\cdot t + \text{error} = p + \text{error}.
\end{aligned}$$

In fact error $\simeq t^2$, stays small if summed up for more and more smaller and smaller intervals of length t. □

Stationary distribution



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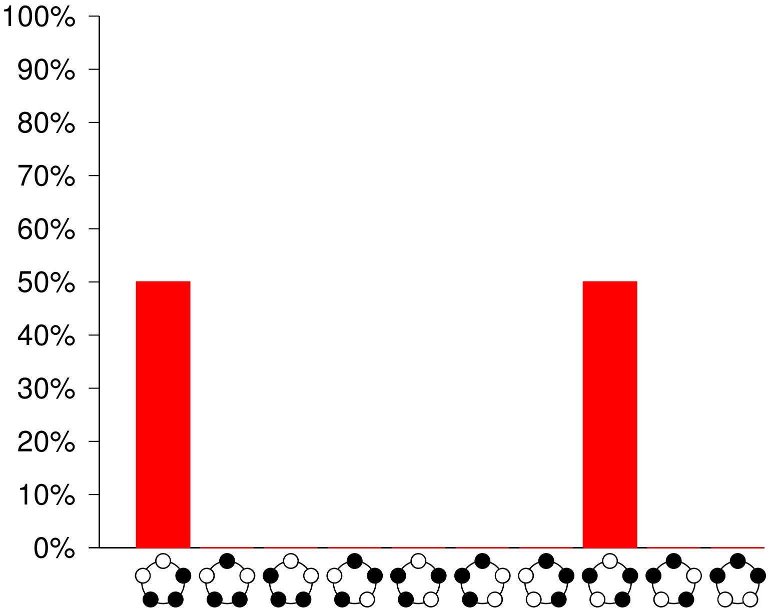

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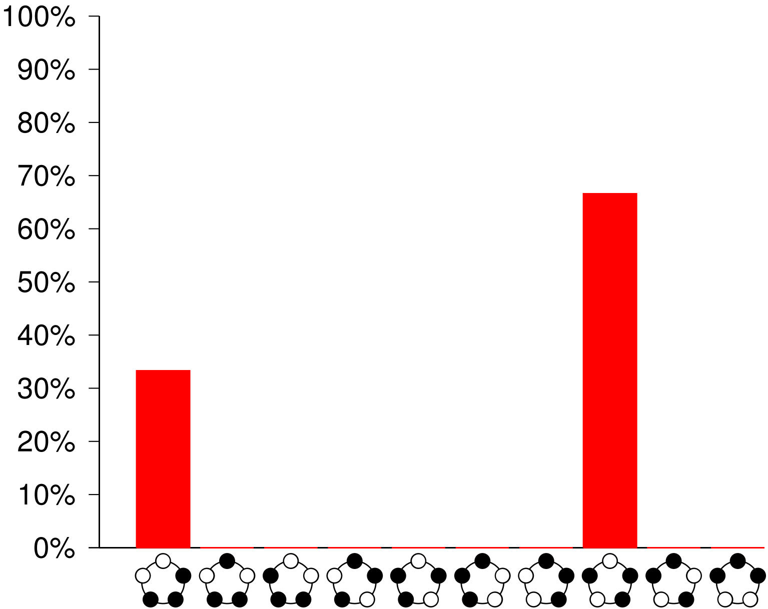

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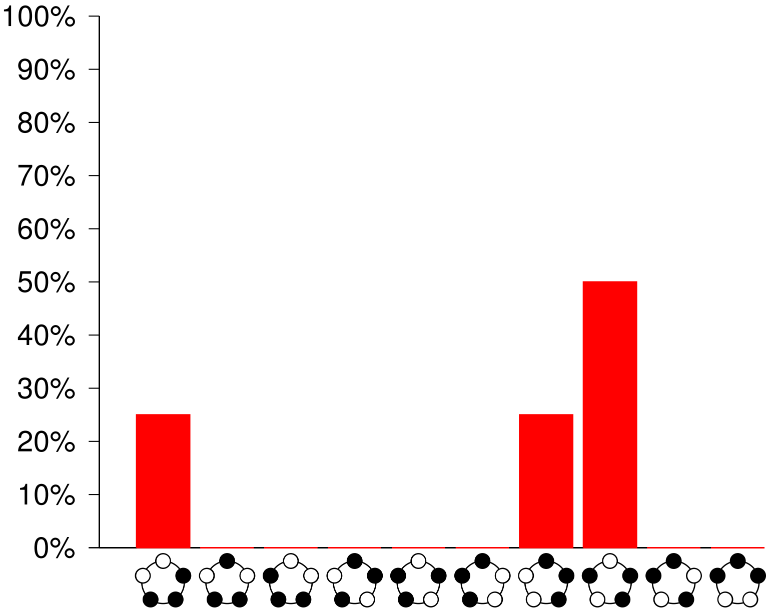

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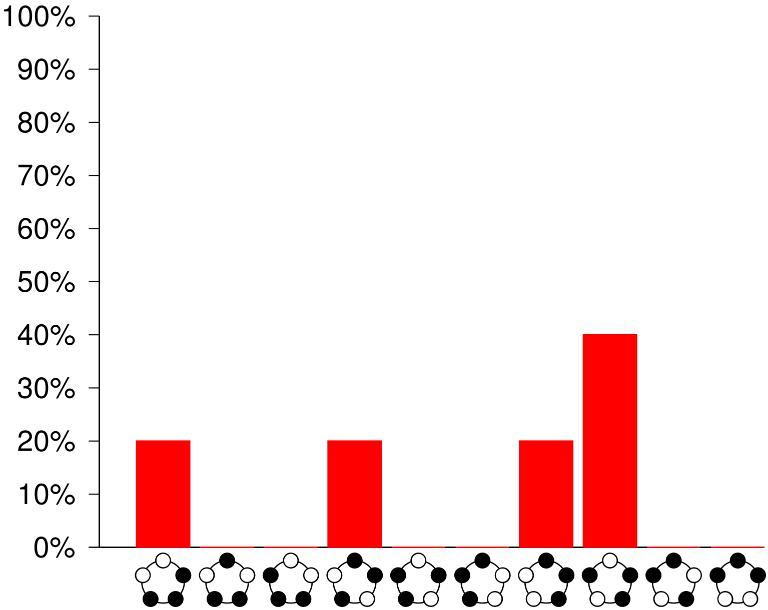

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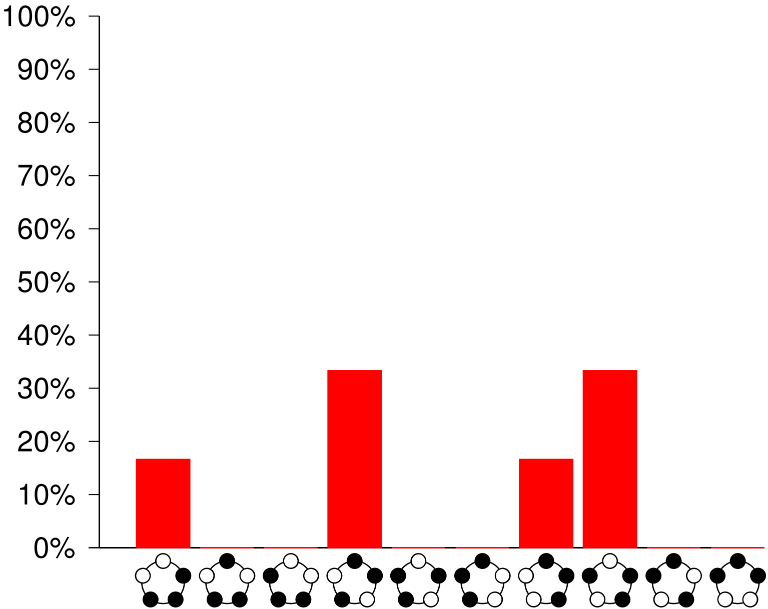

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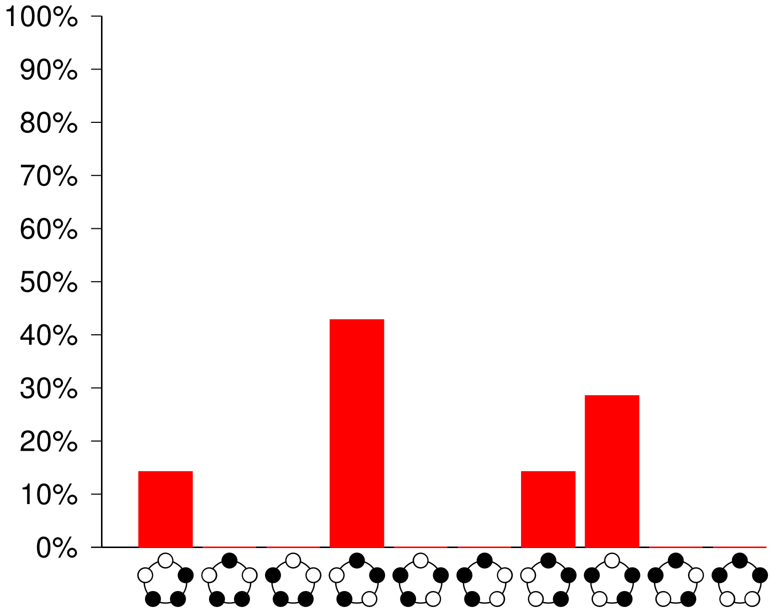

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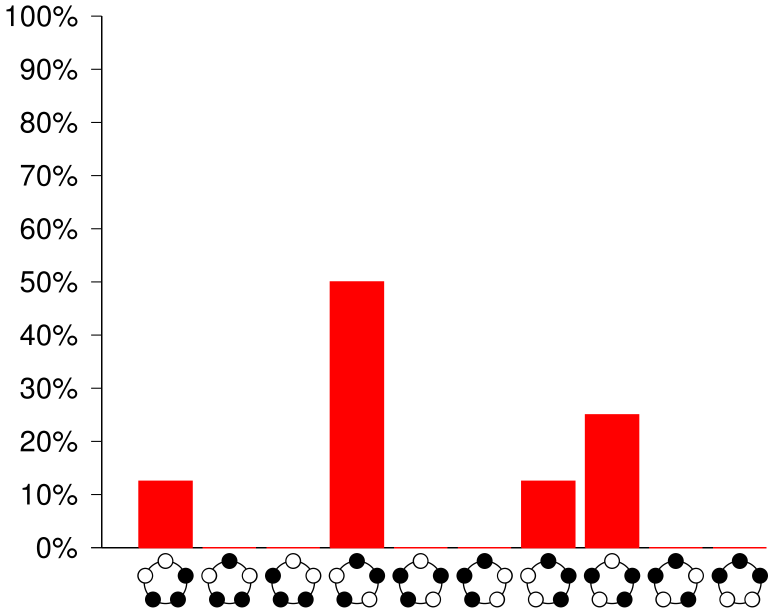

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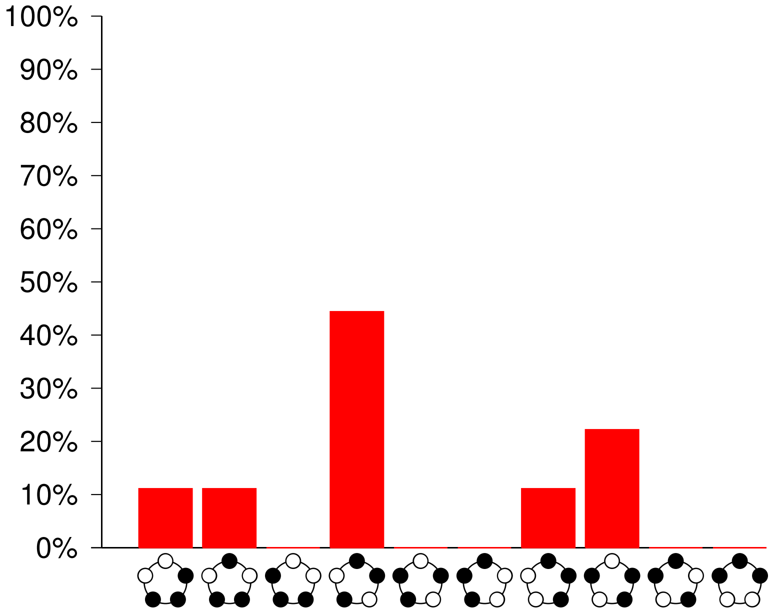

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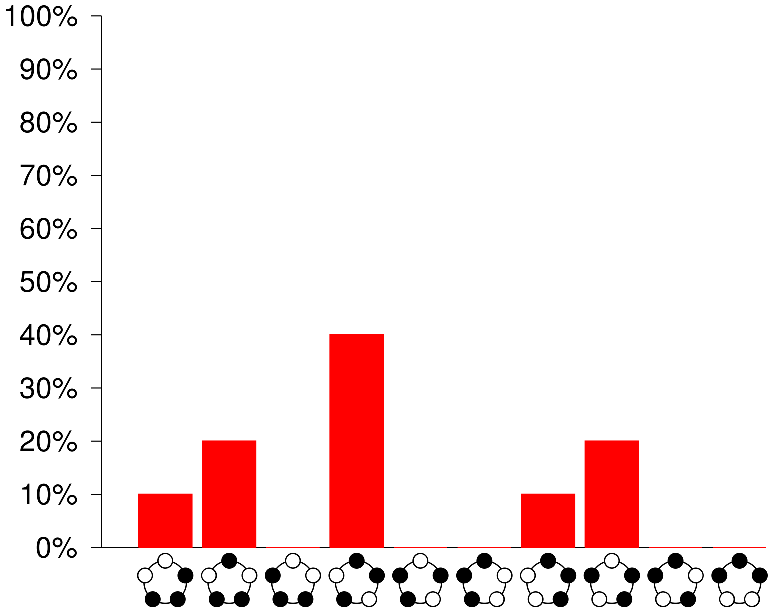

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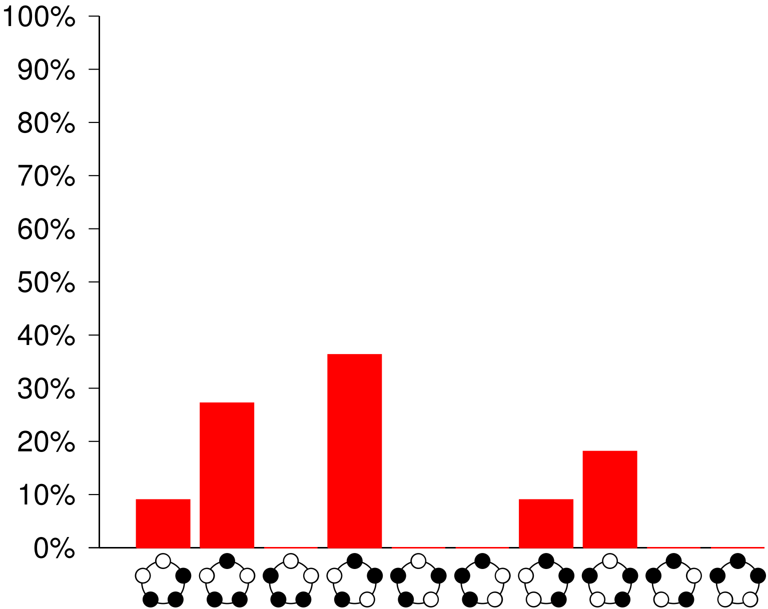
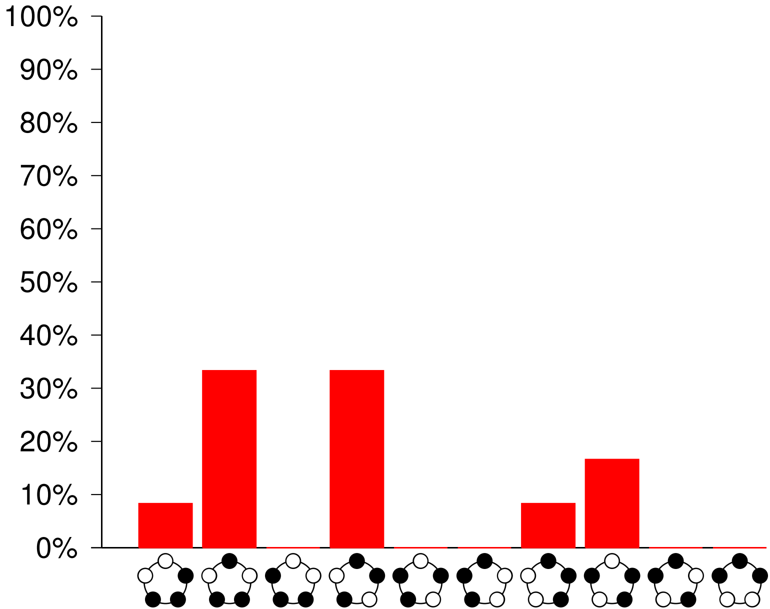


Stationary distribution



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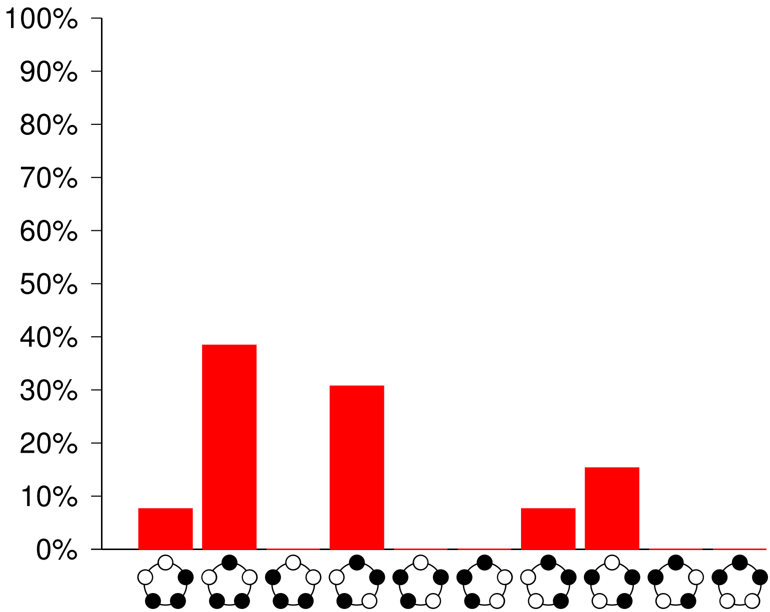

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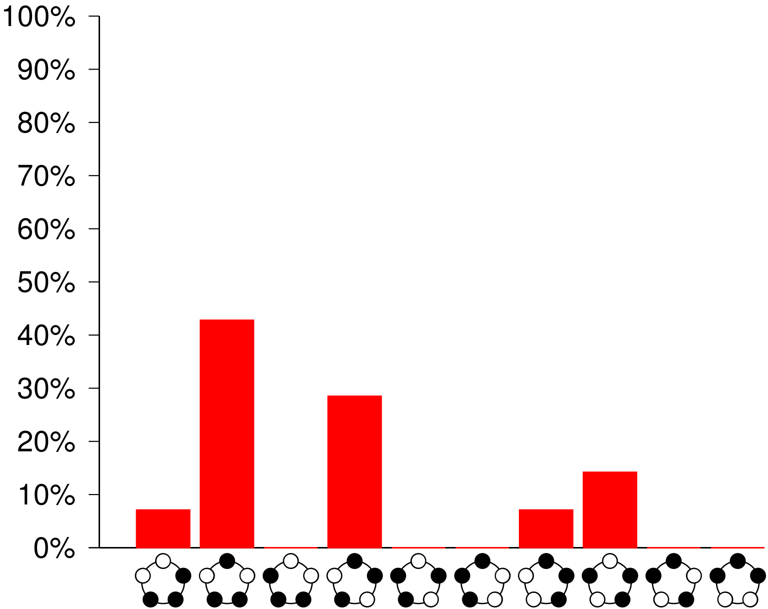

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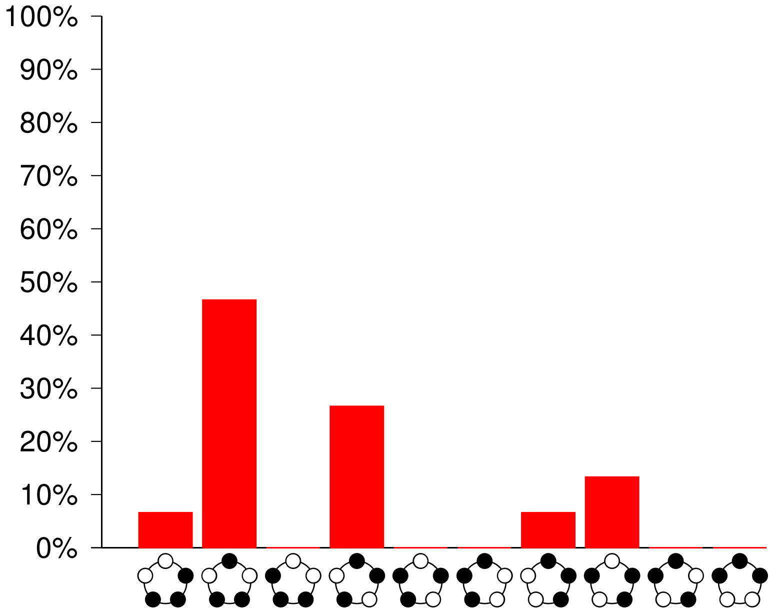

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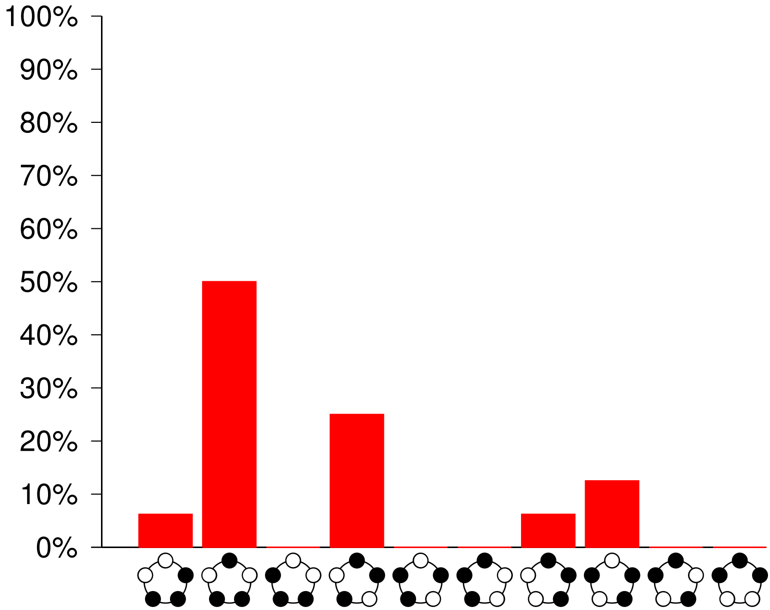

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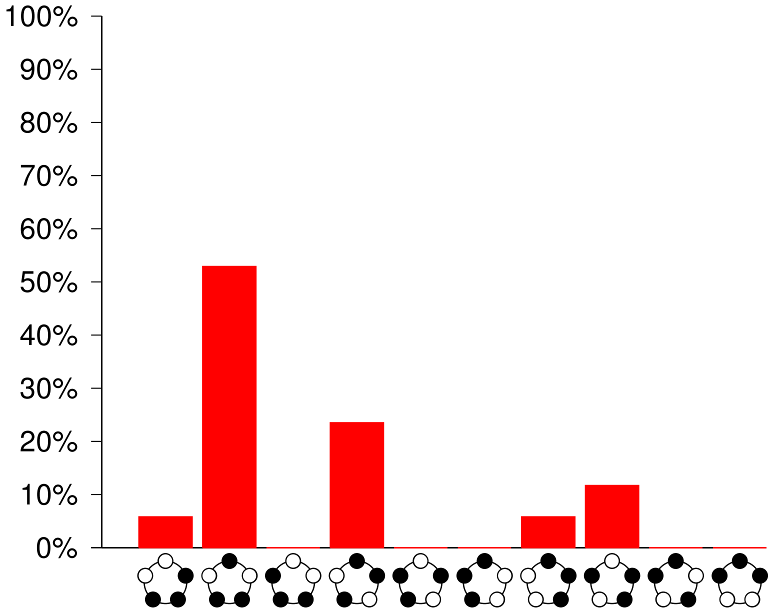

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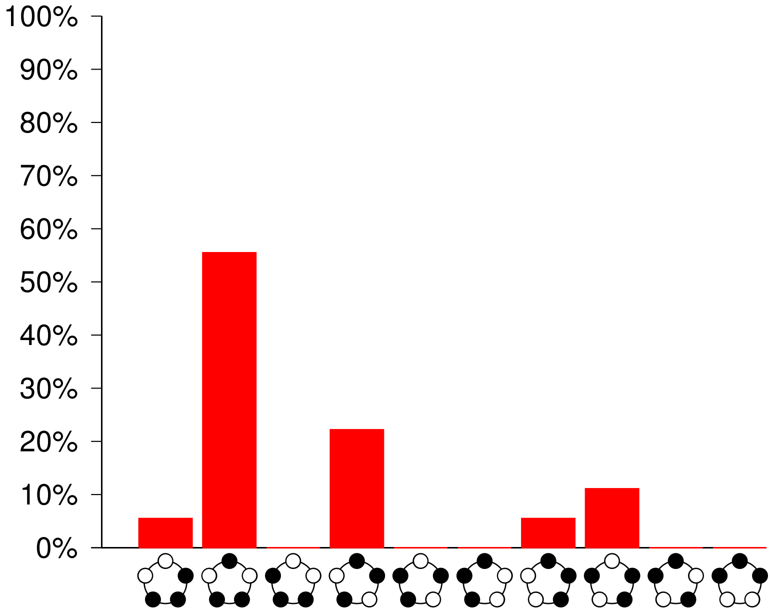

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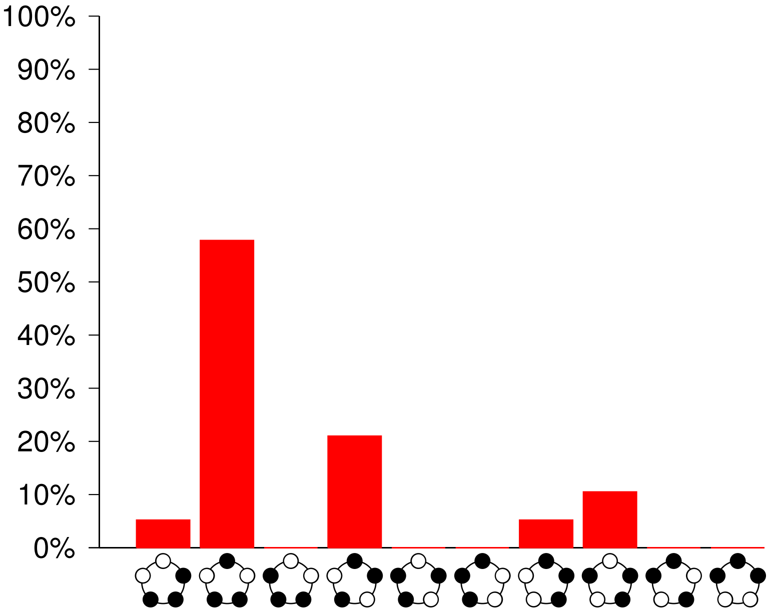

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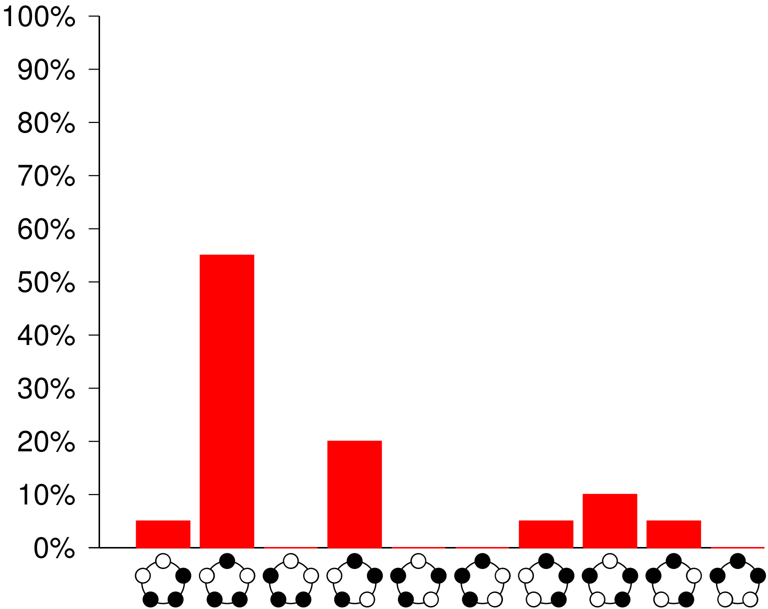

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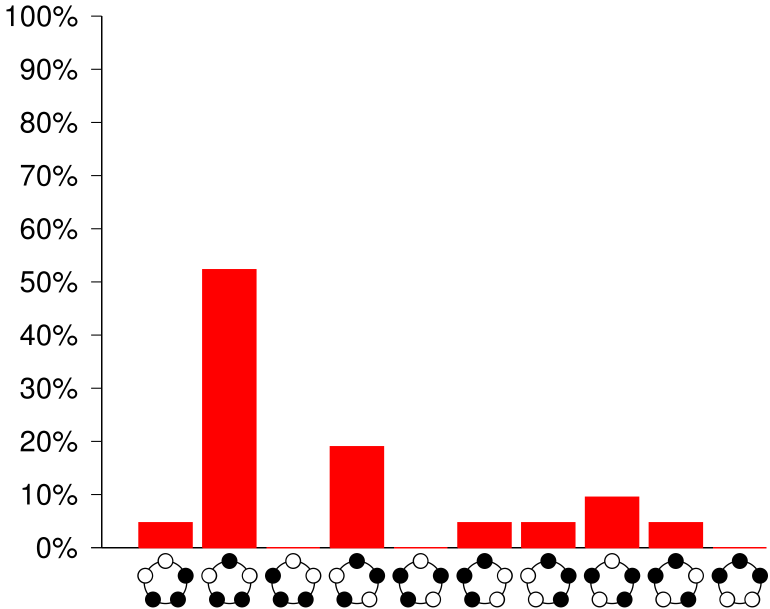

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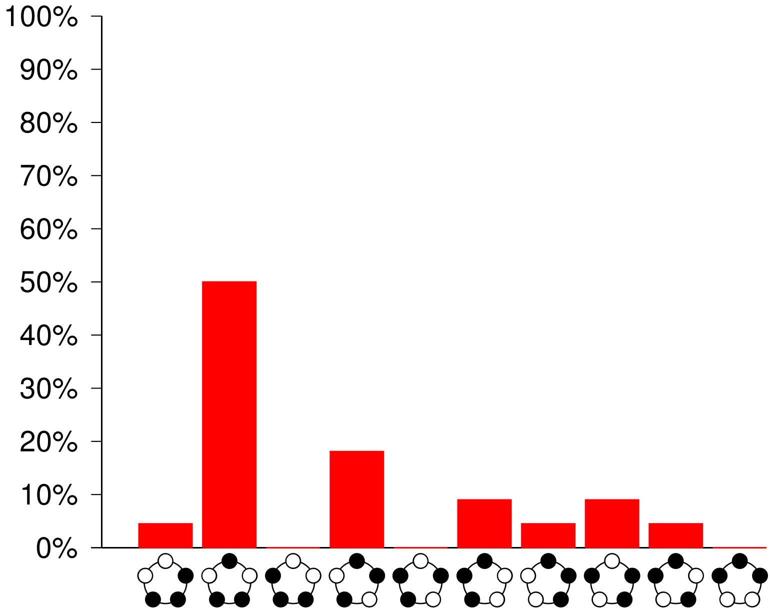

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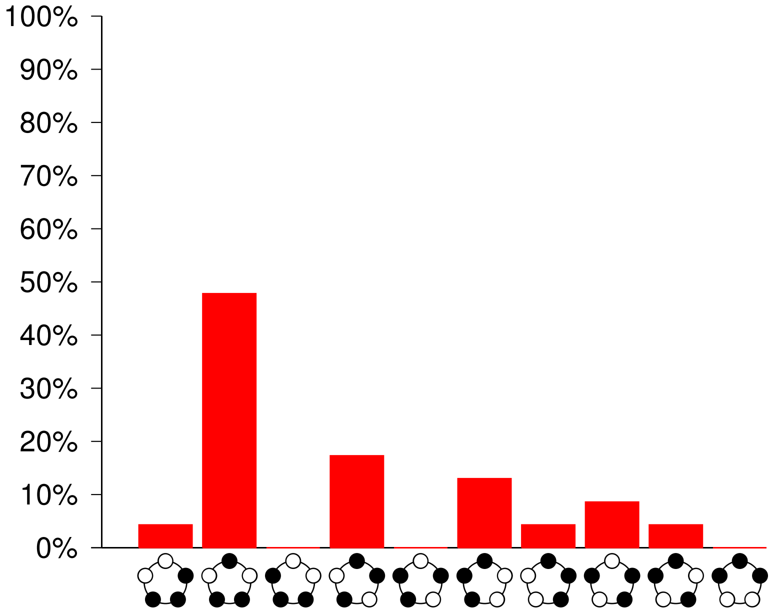

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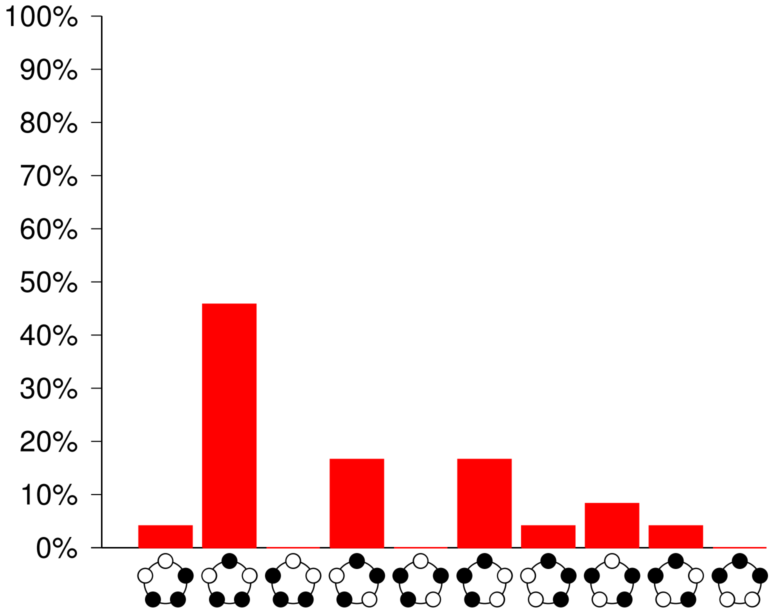

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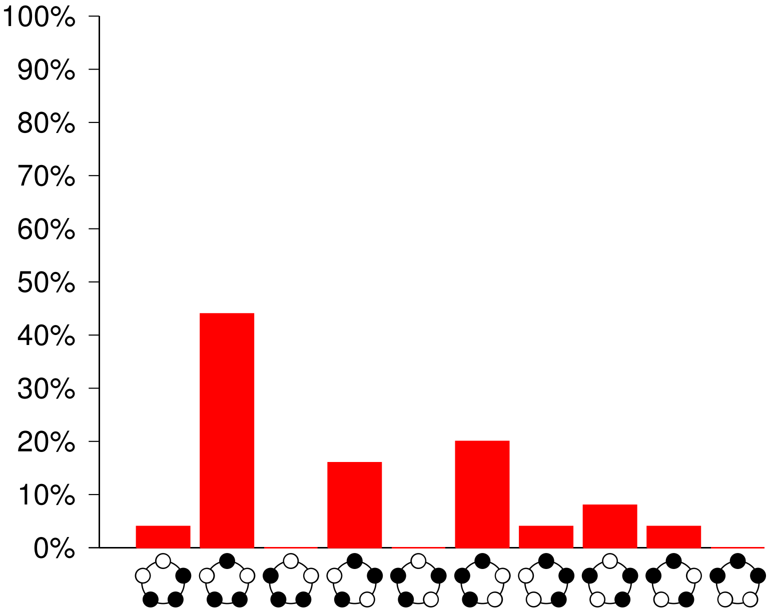

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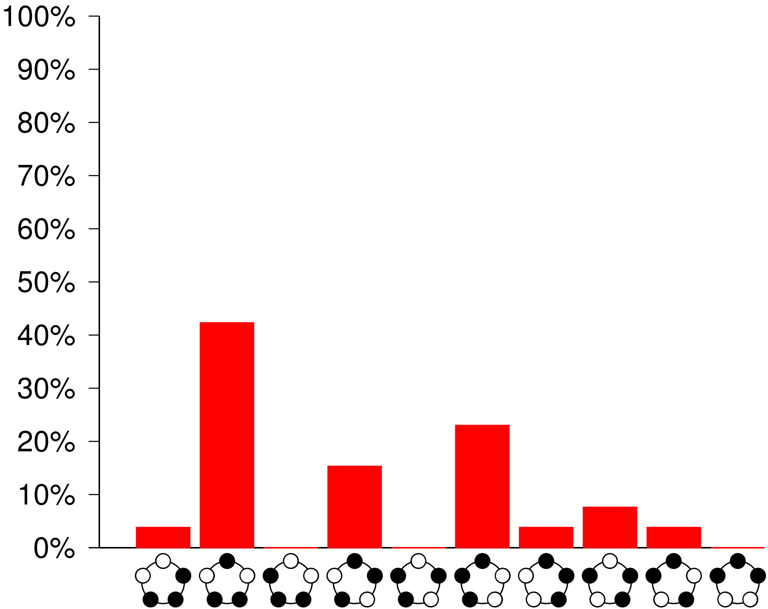

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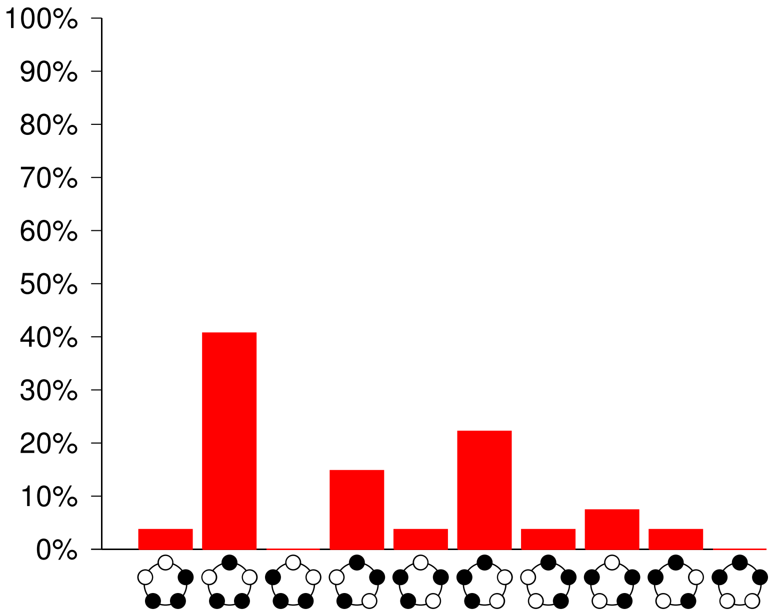

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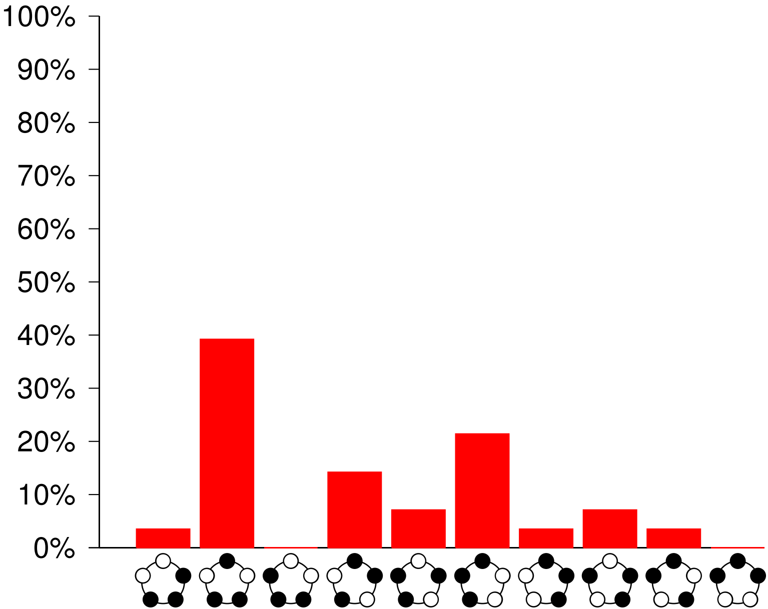

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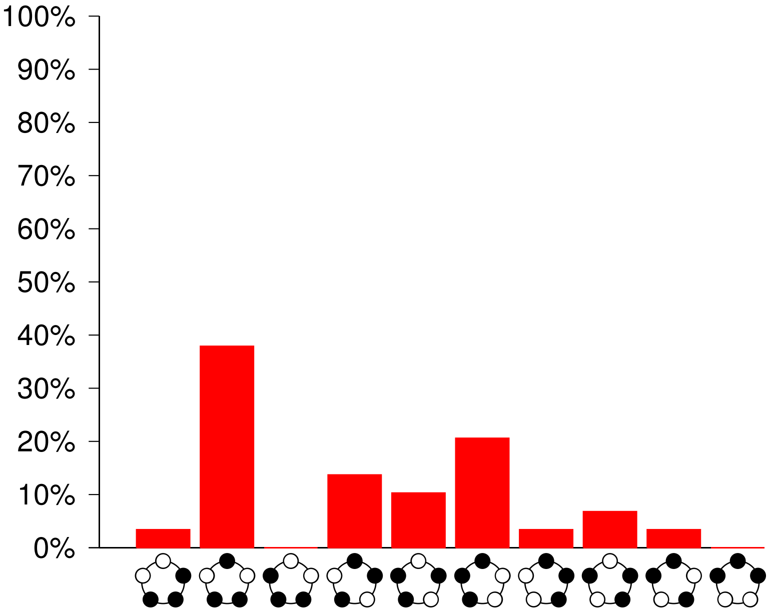

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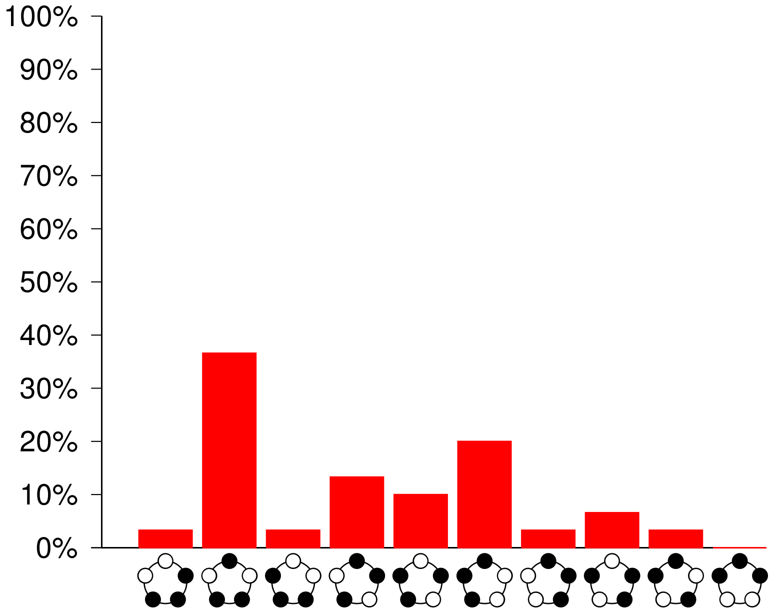

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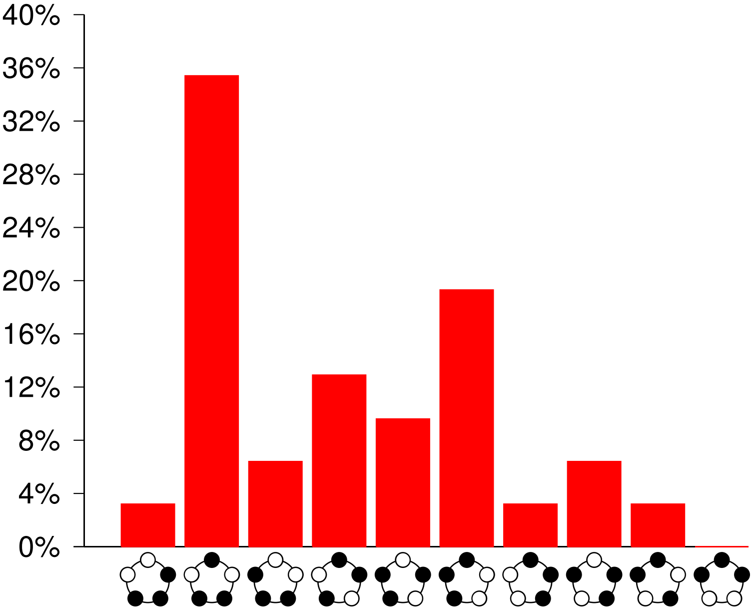

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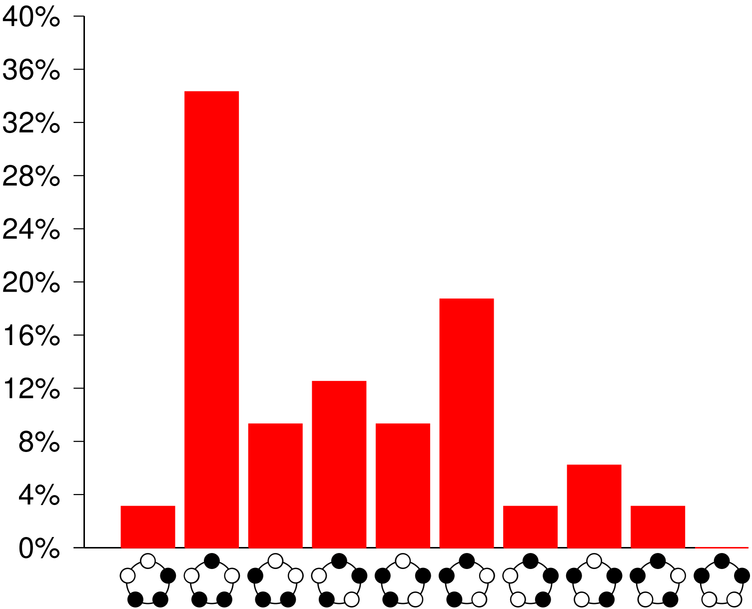

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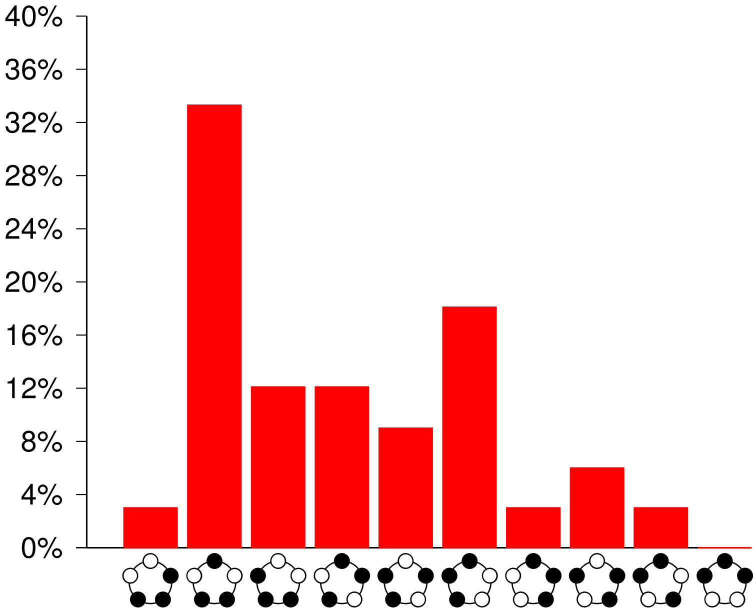

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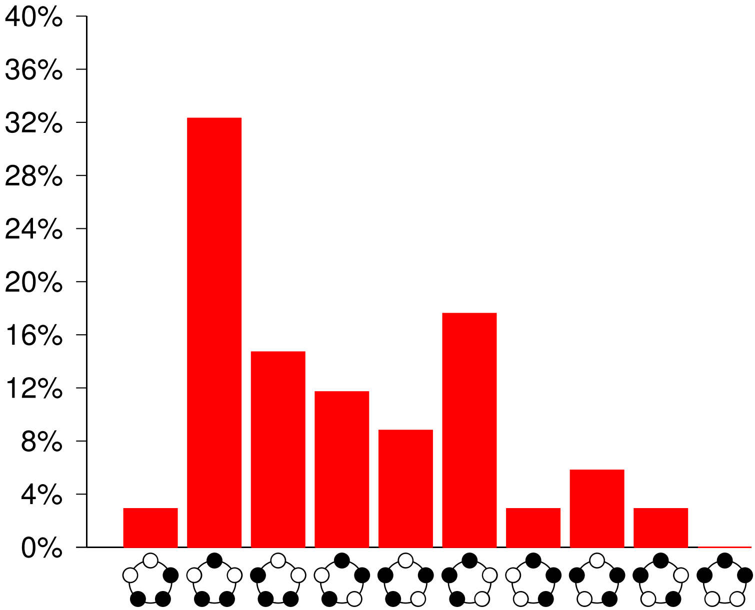

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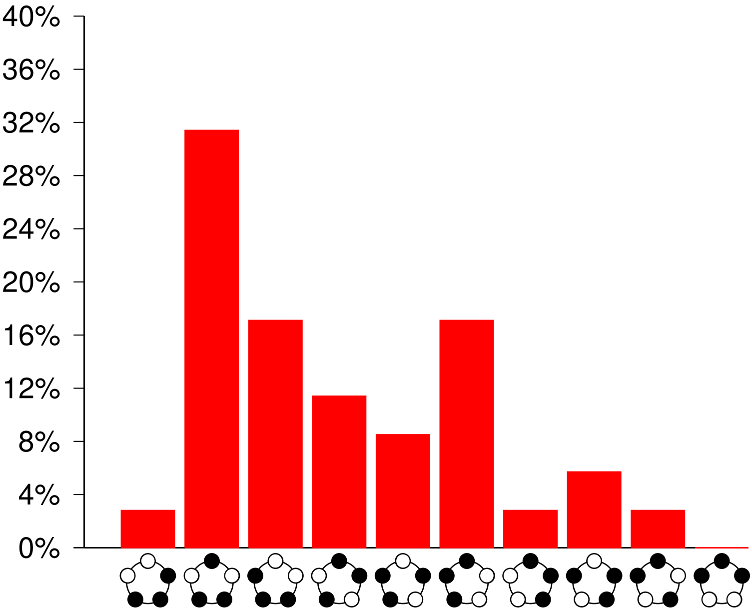

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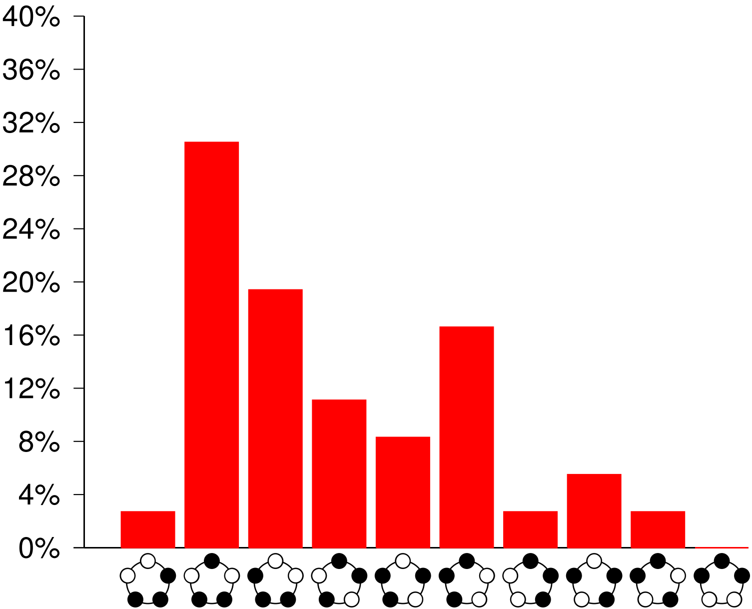

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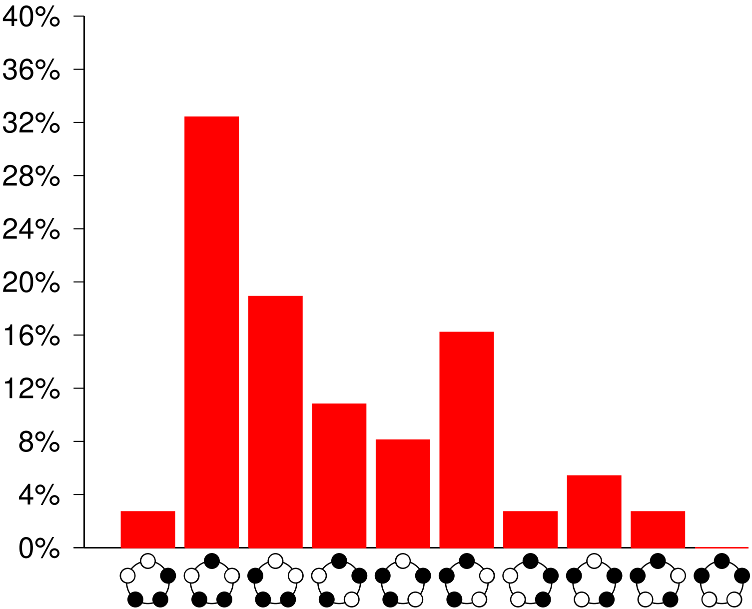

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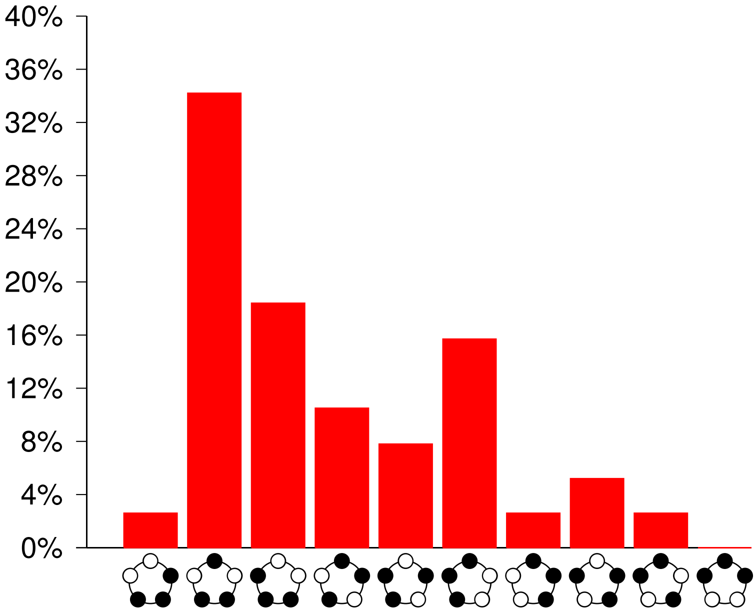

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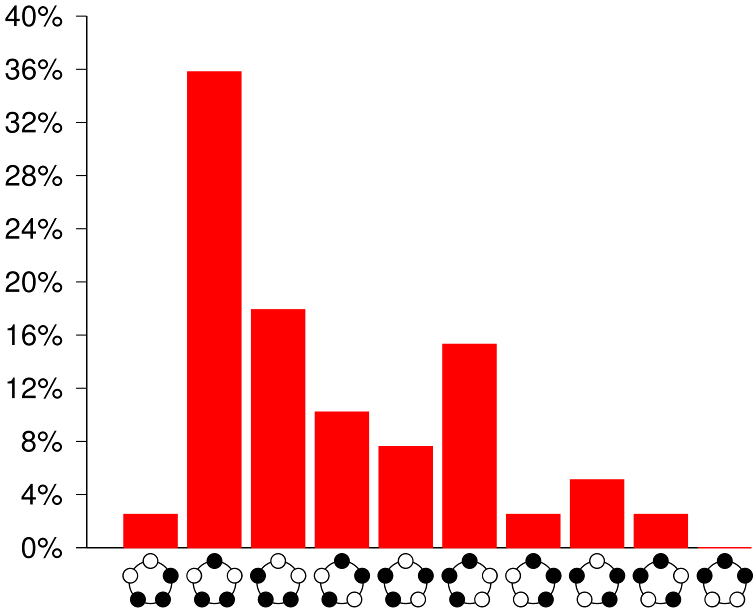

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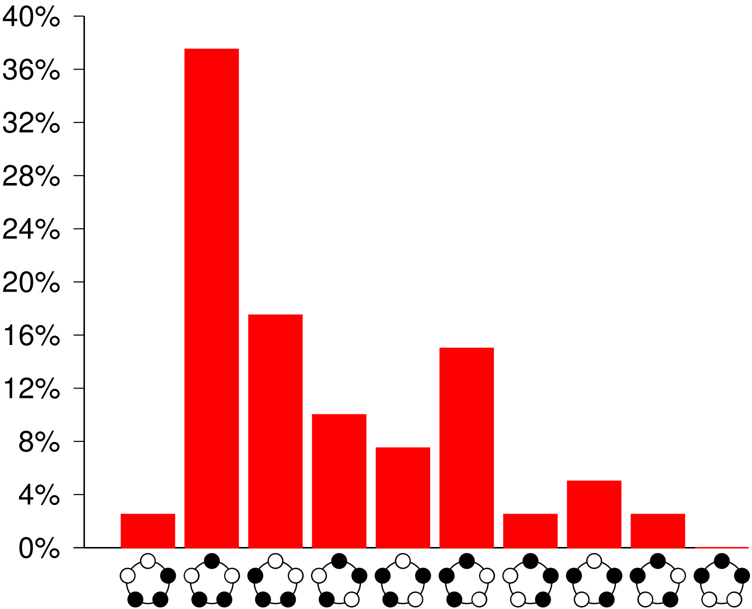

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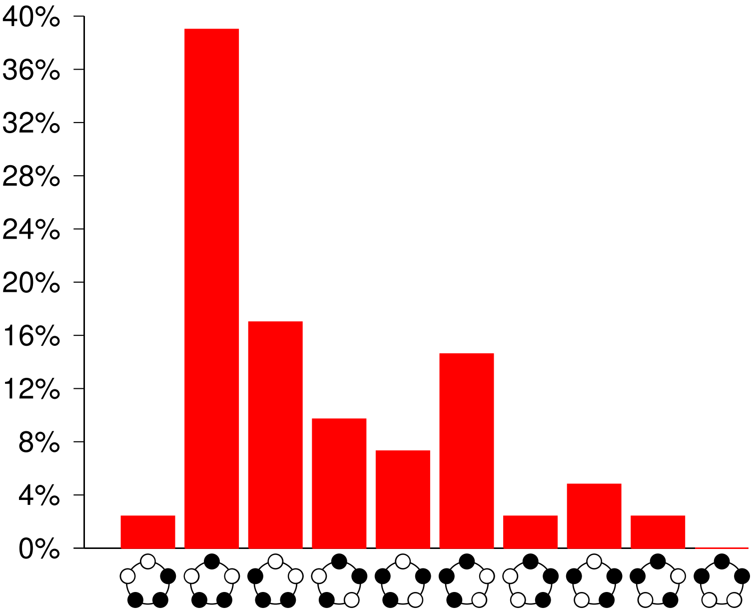

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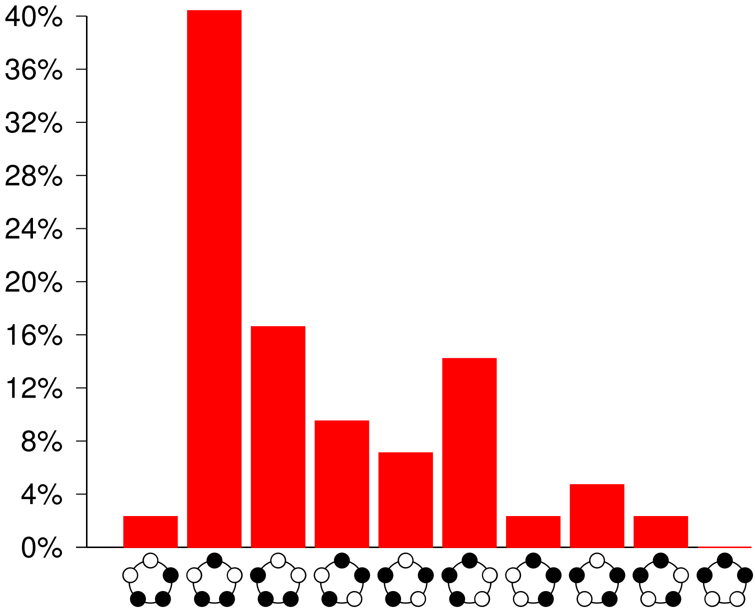

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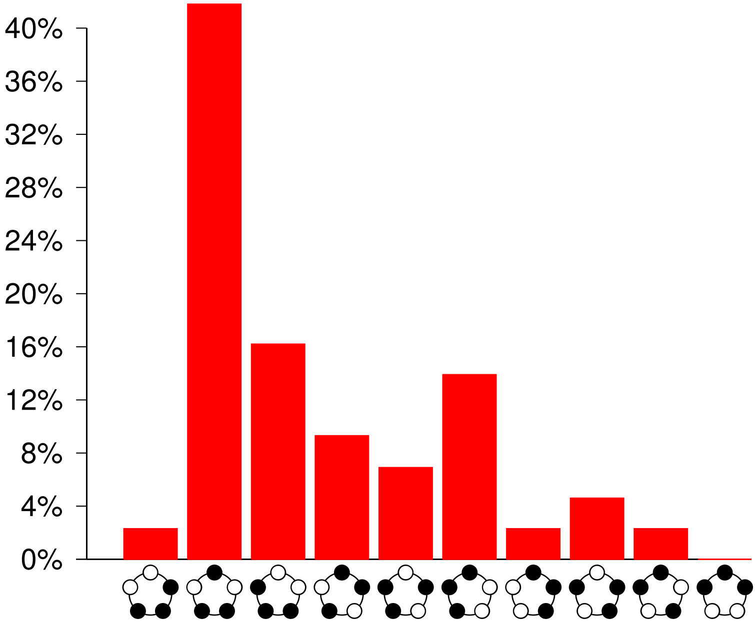

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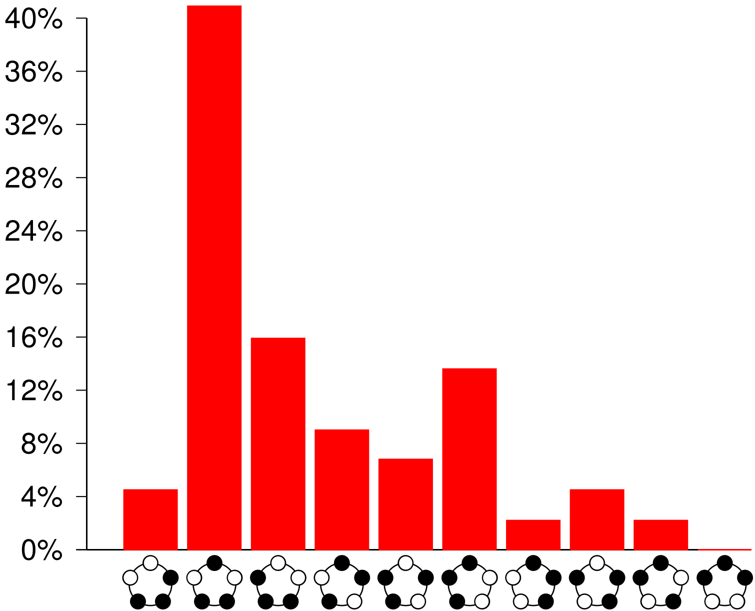

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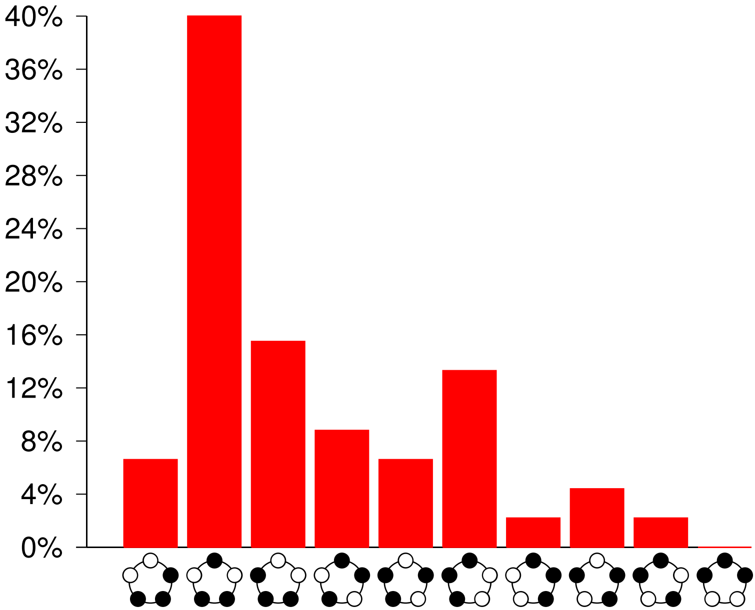

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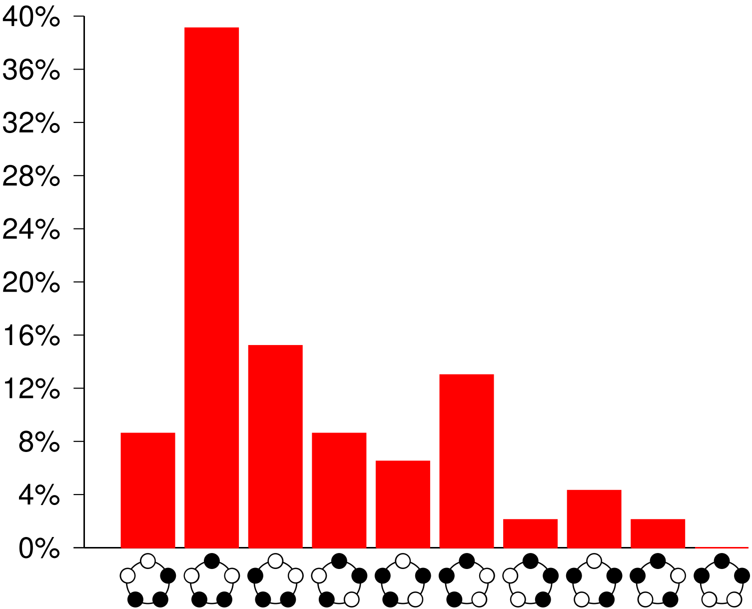

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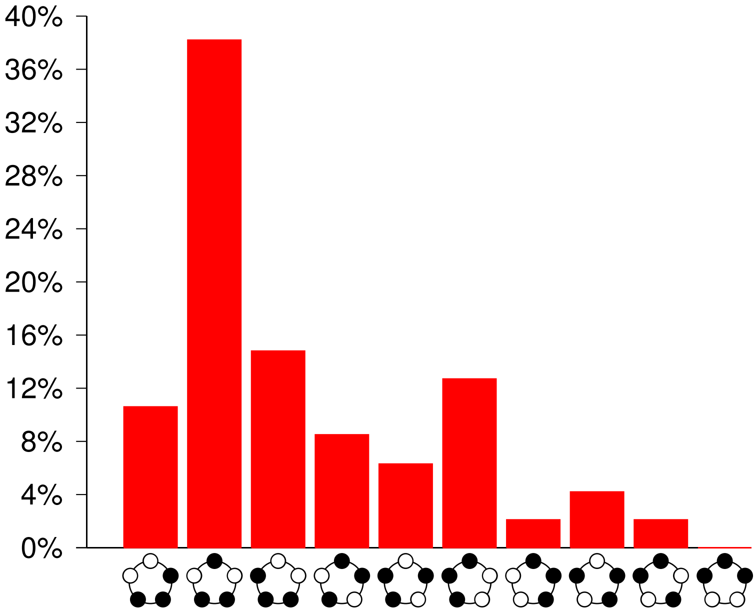

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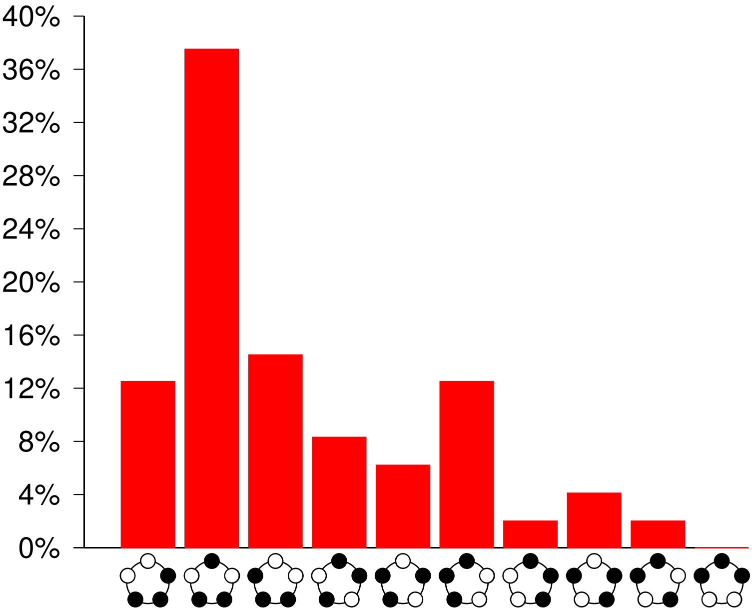

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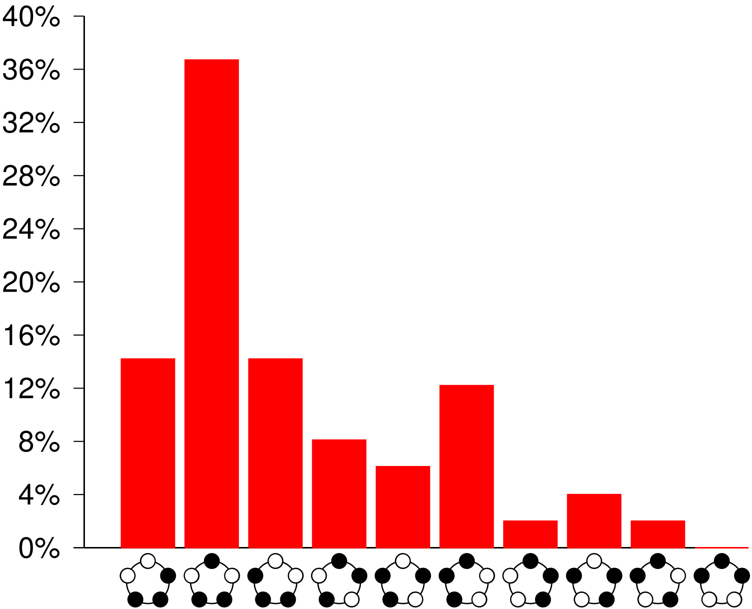

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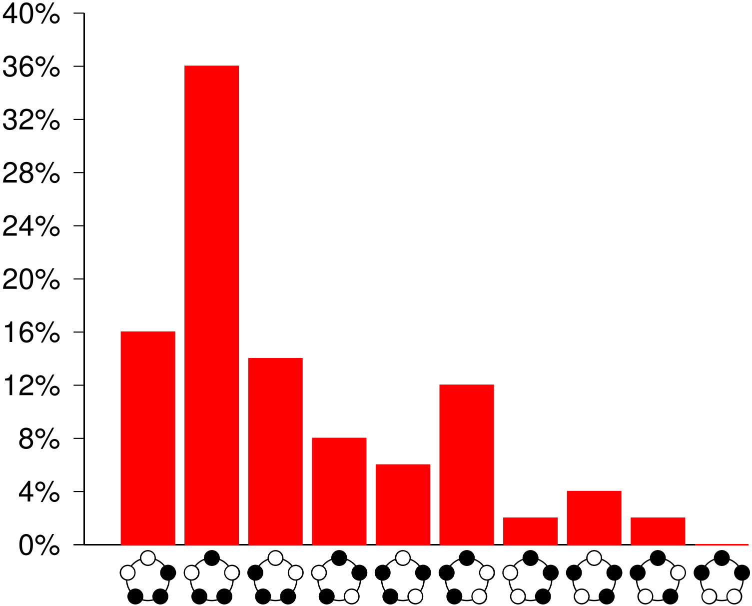

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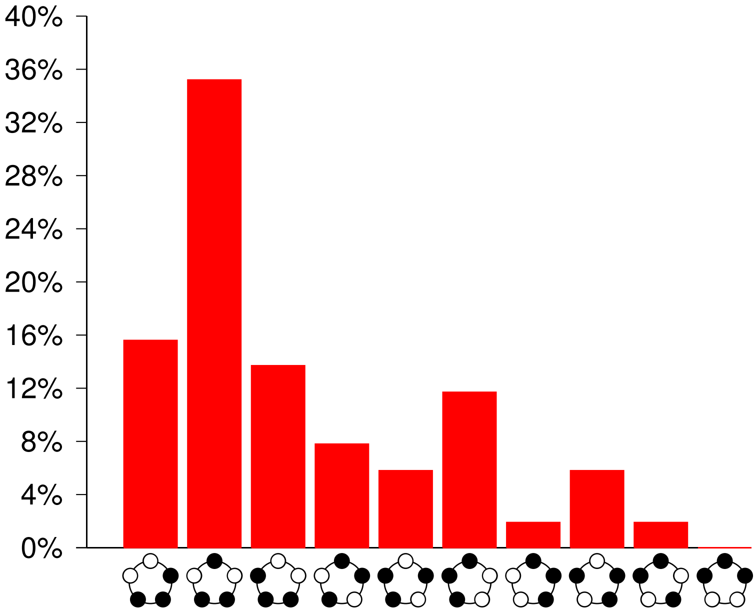

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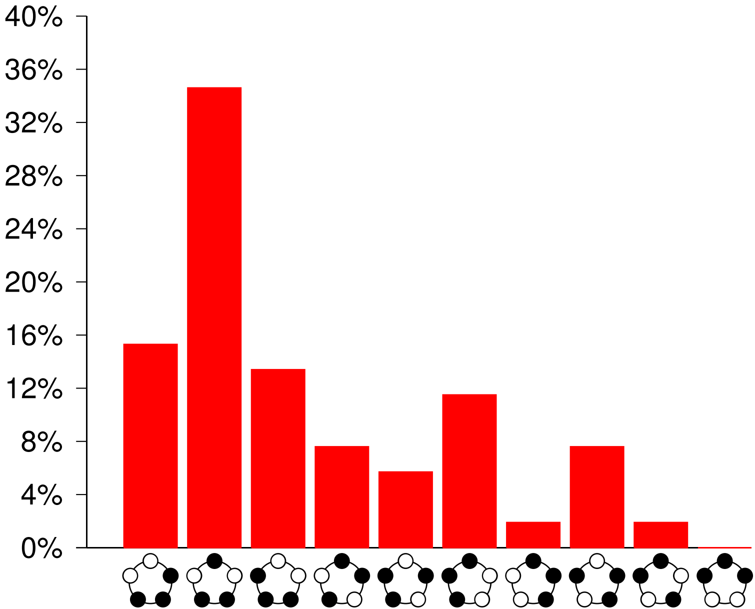

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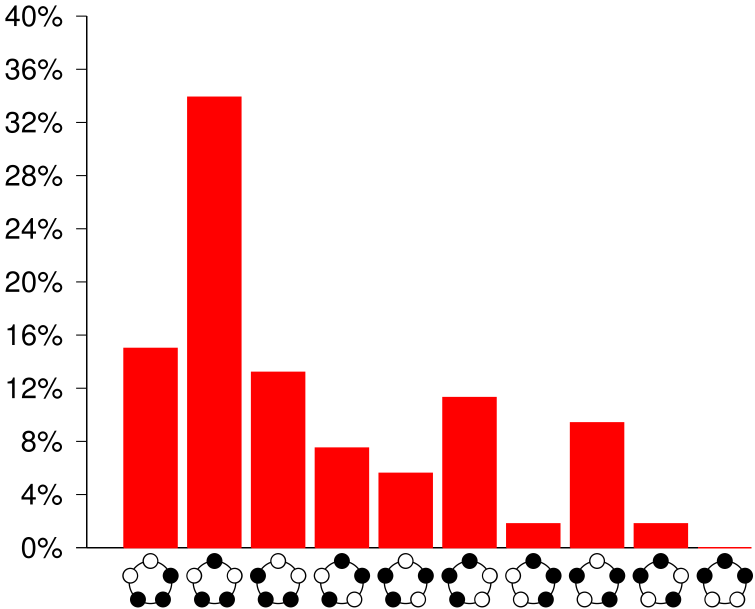

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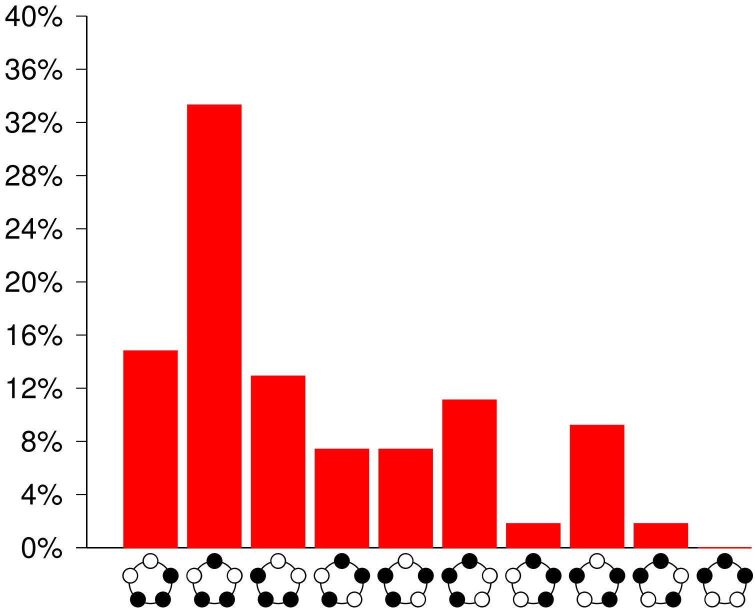

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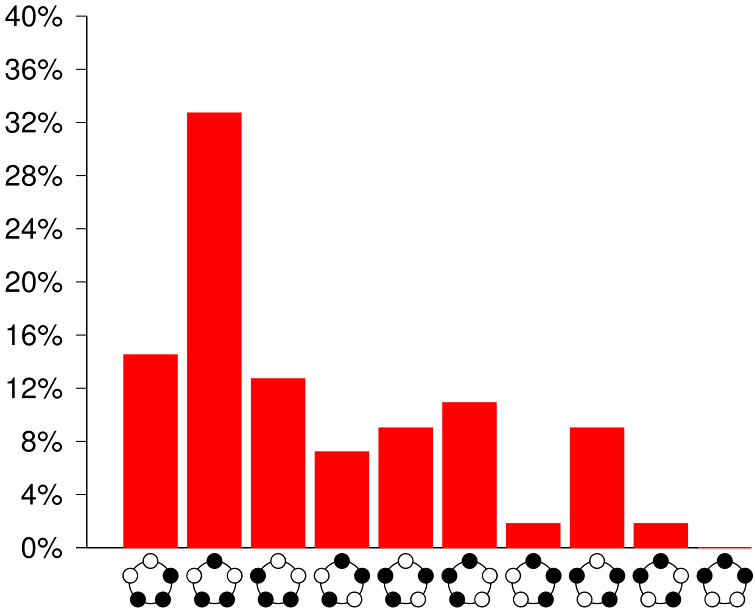

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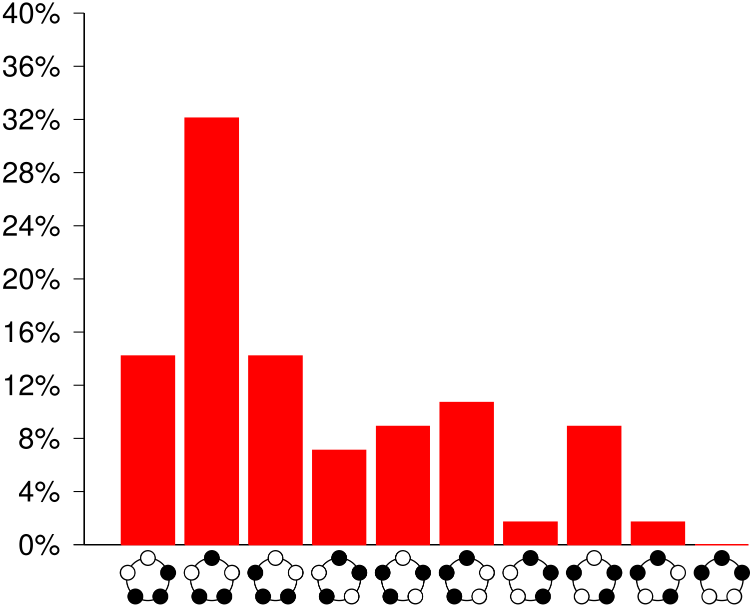

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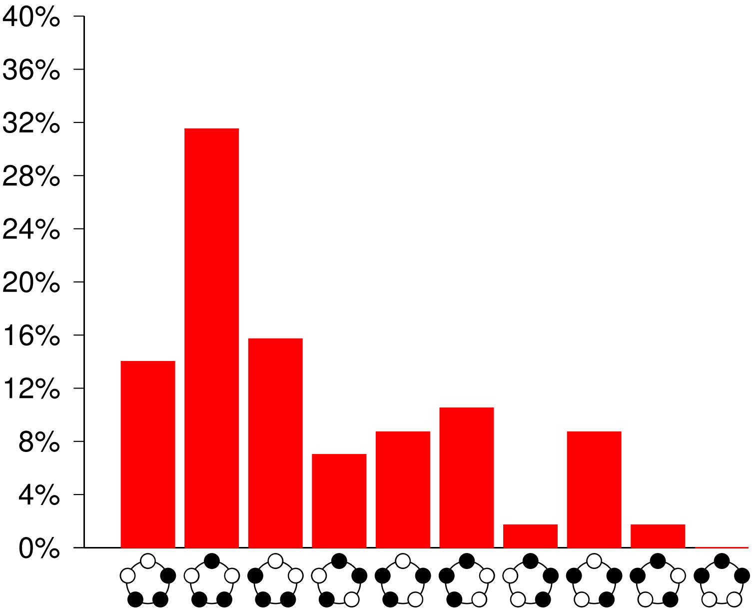

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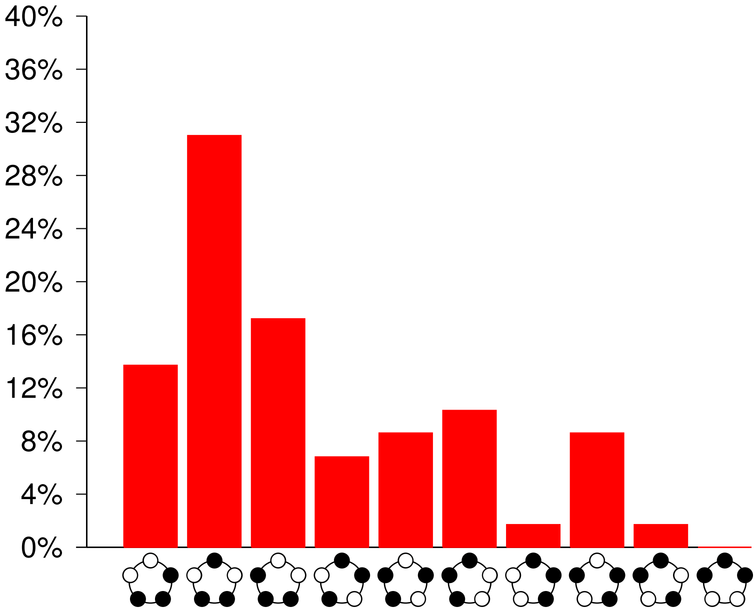

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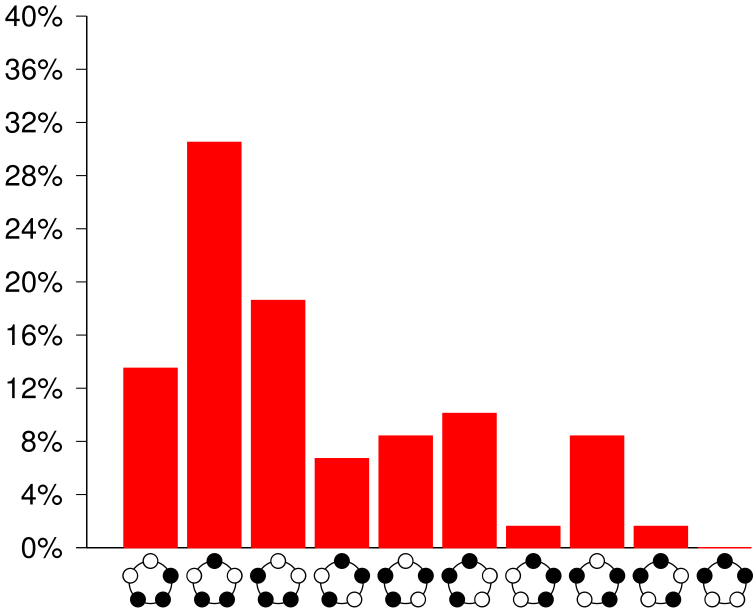

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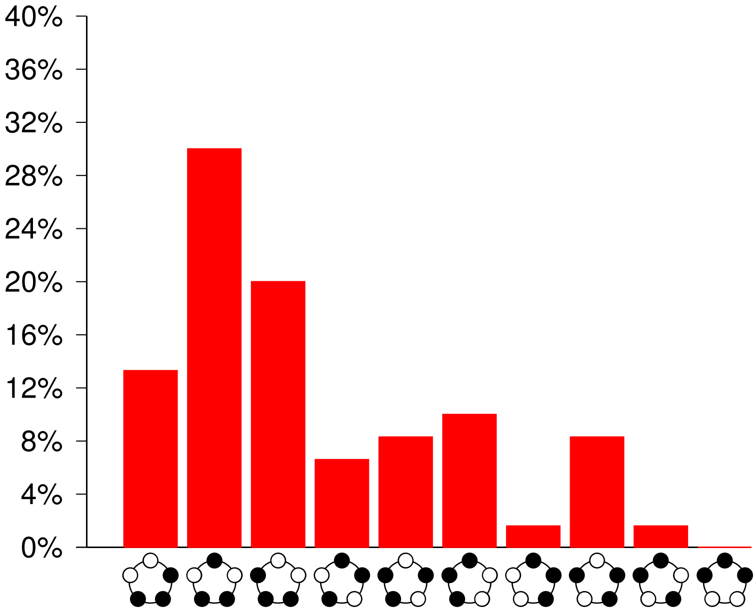

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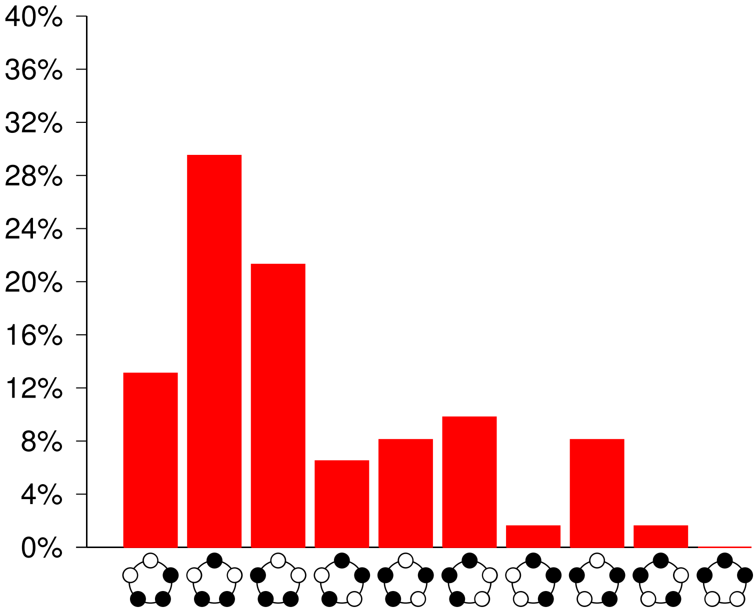

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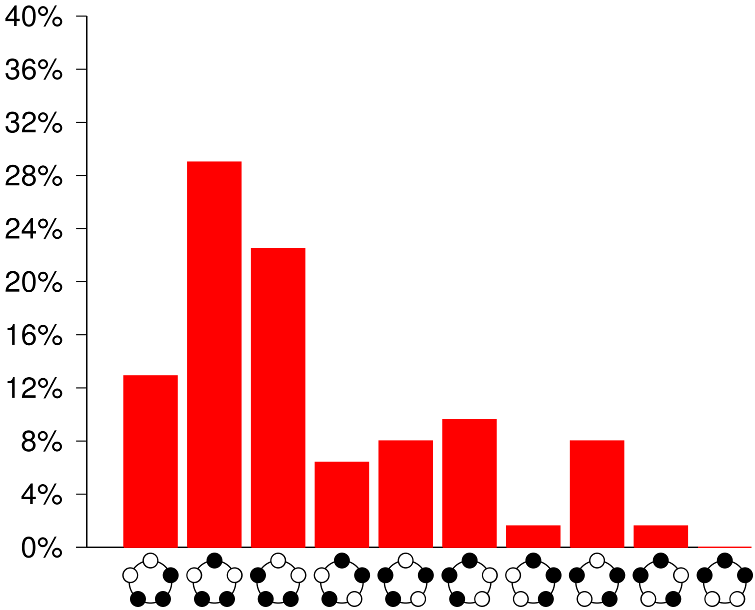

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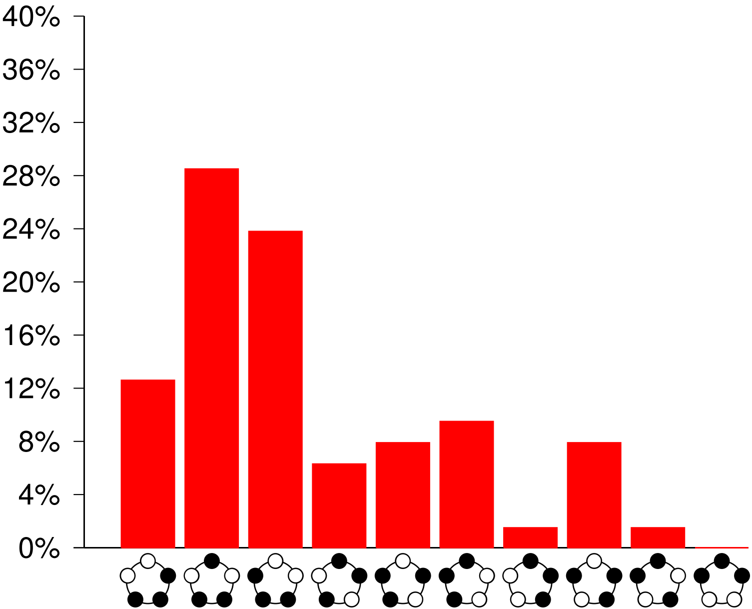

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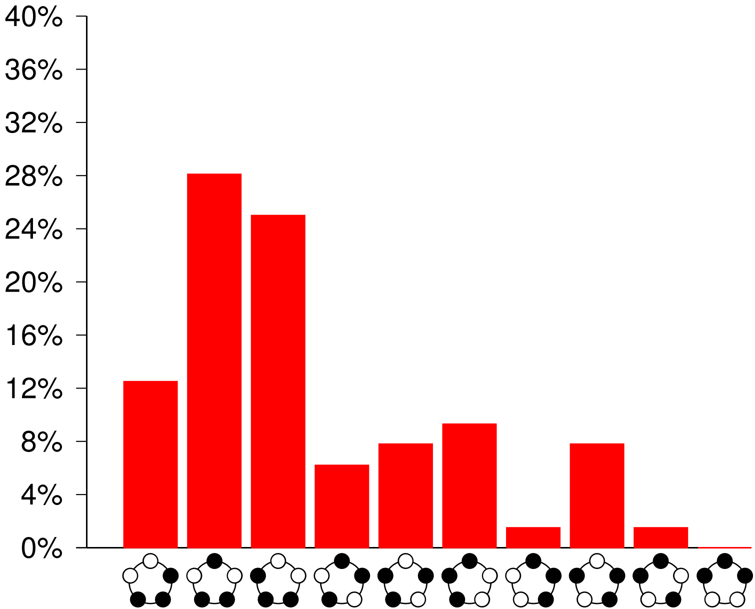

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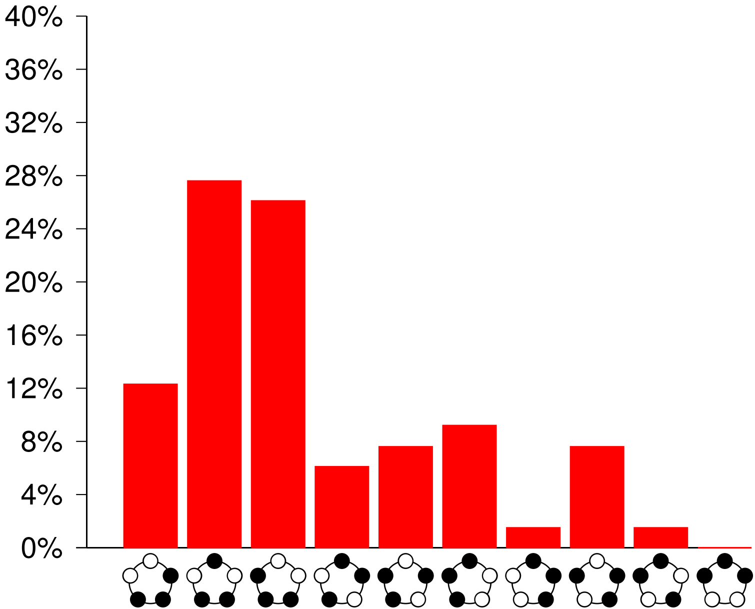

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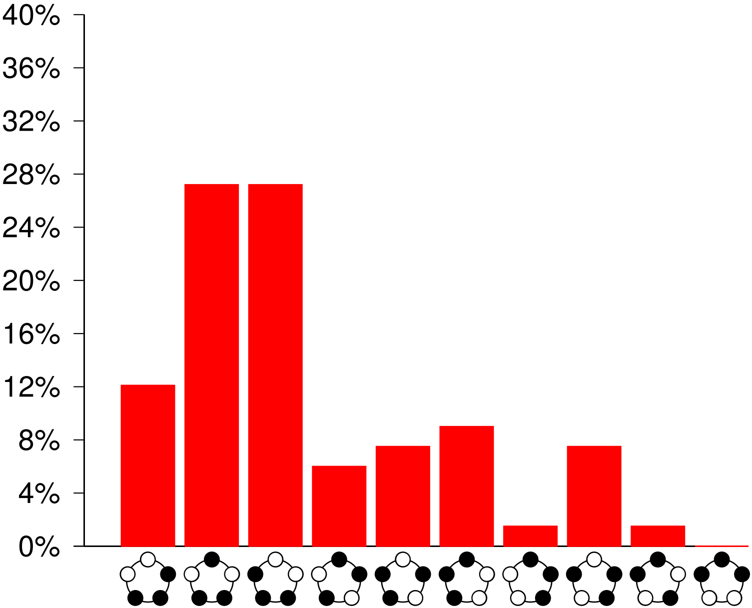

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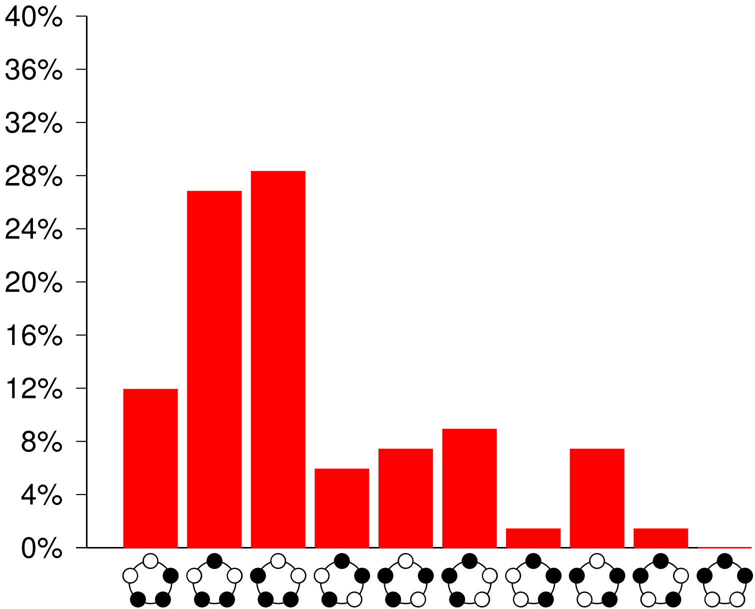

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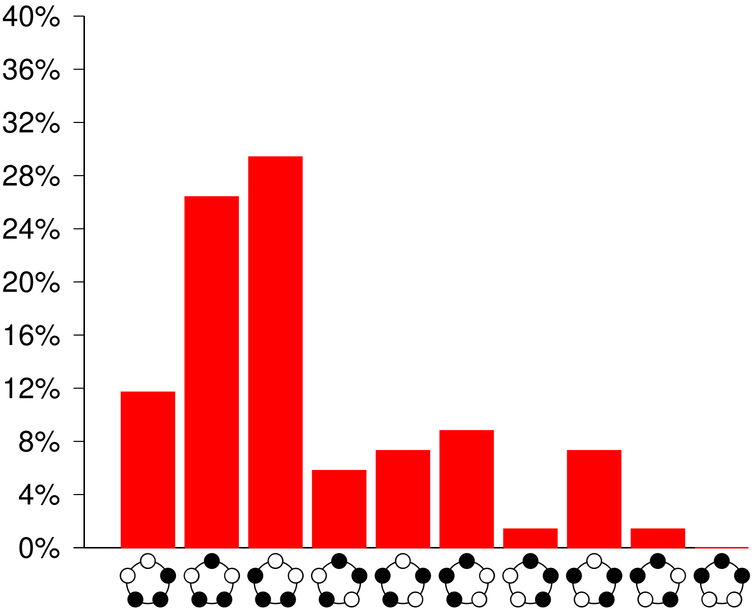

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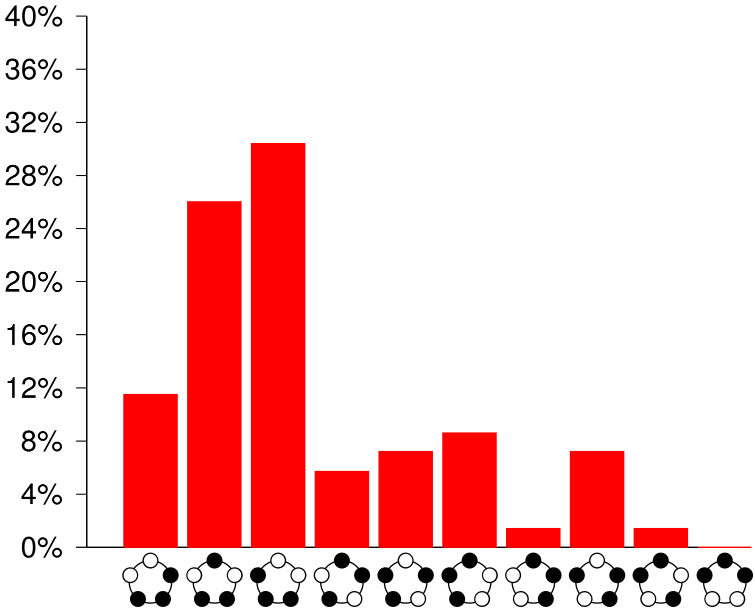

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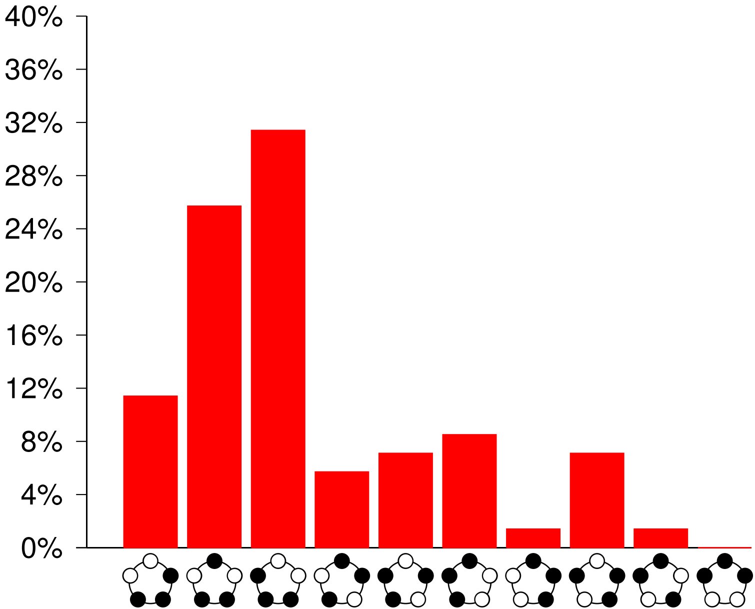

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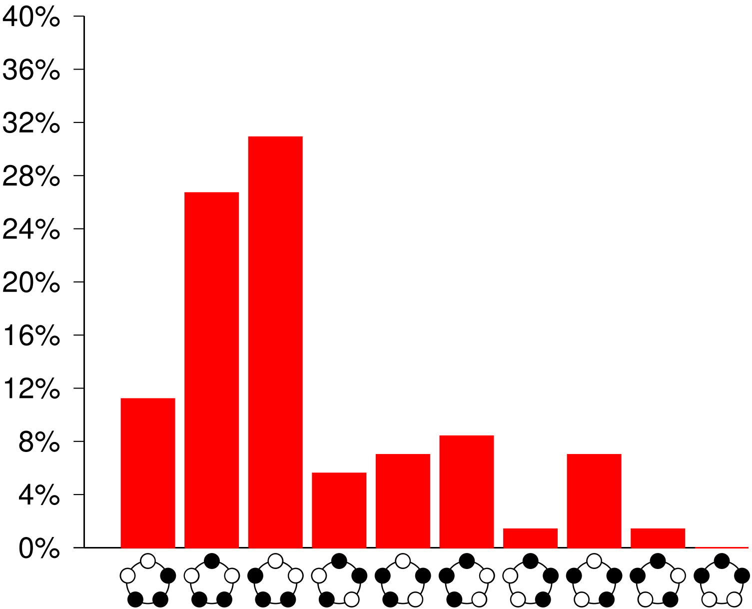

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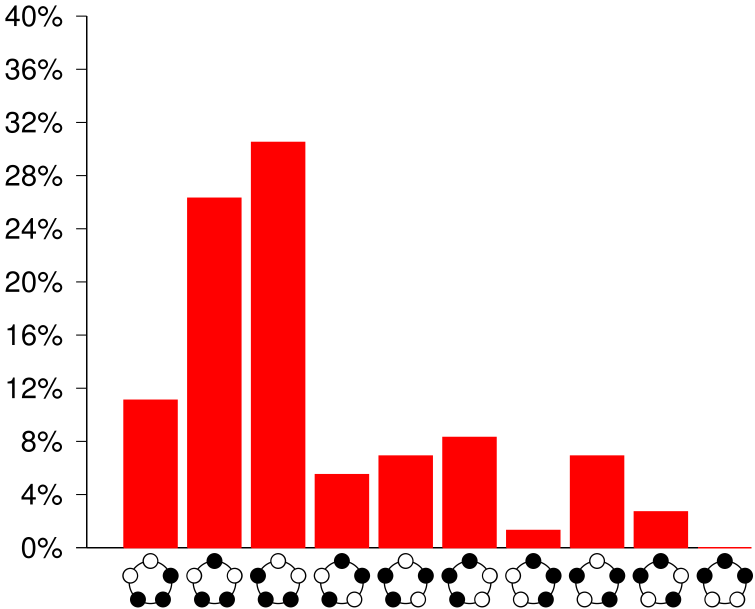

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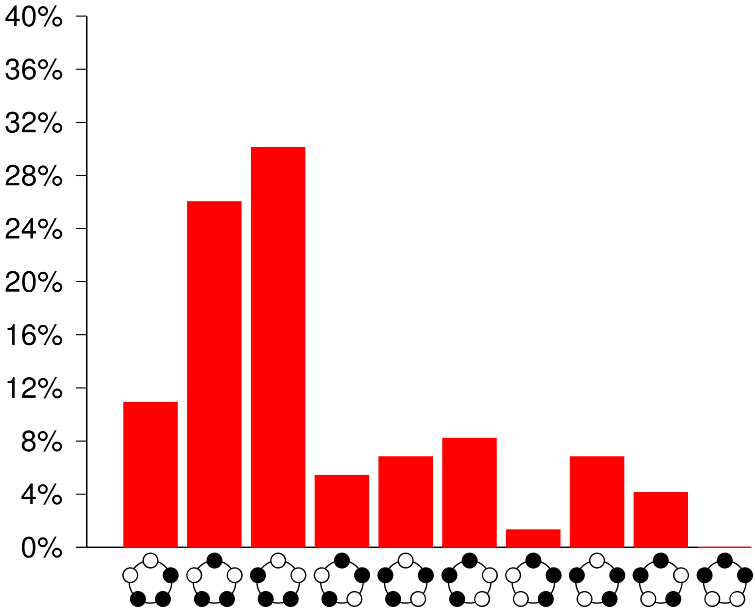

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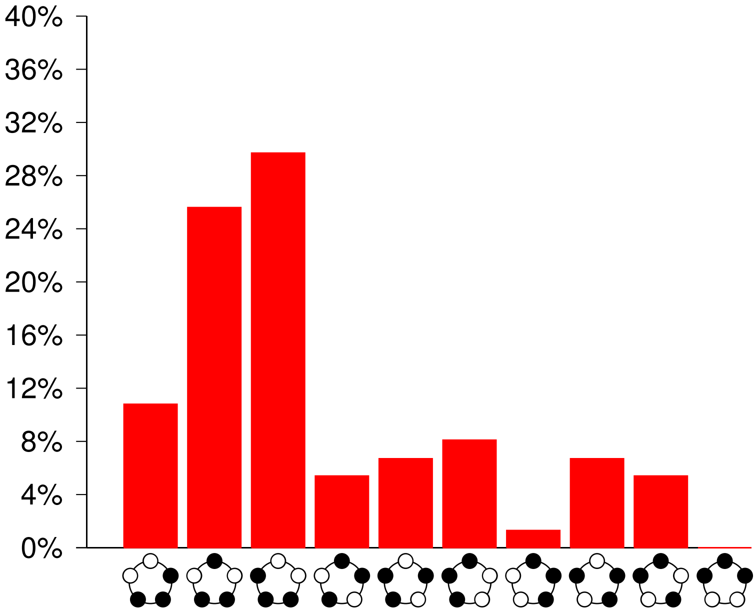

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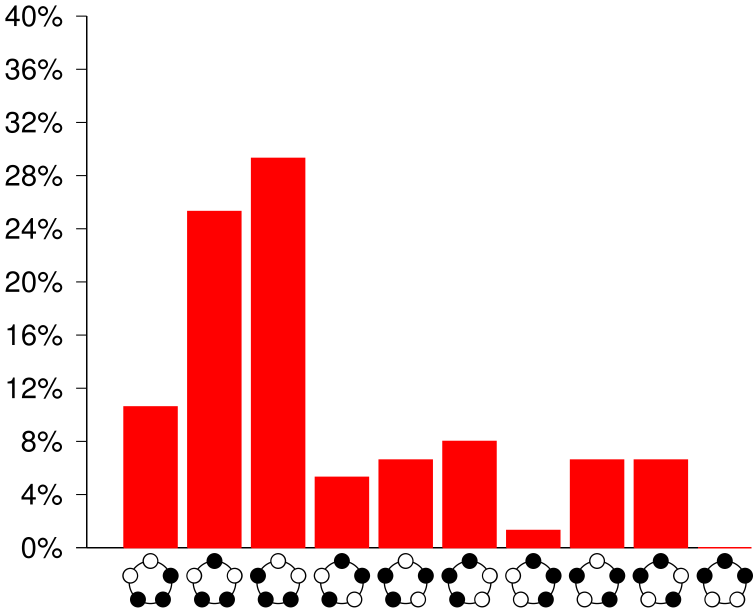

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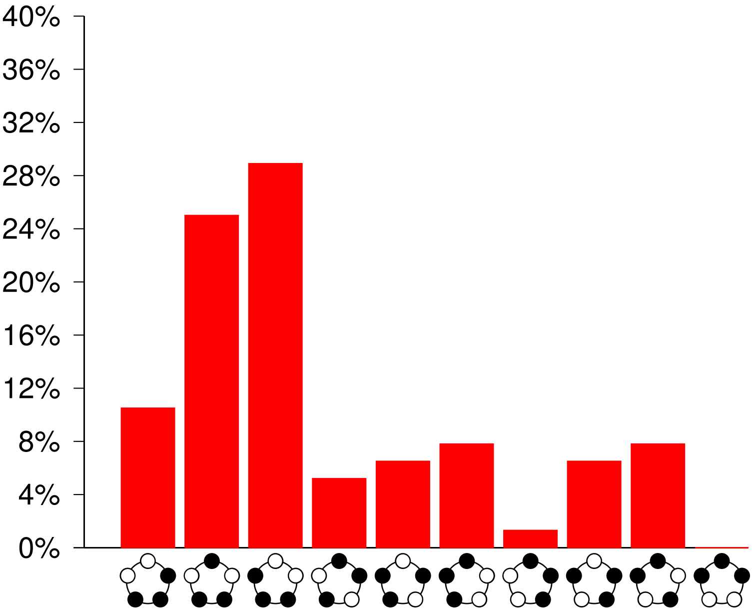

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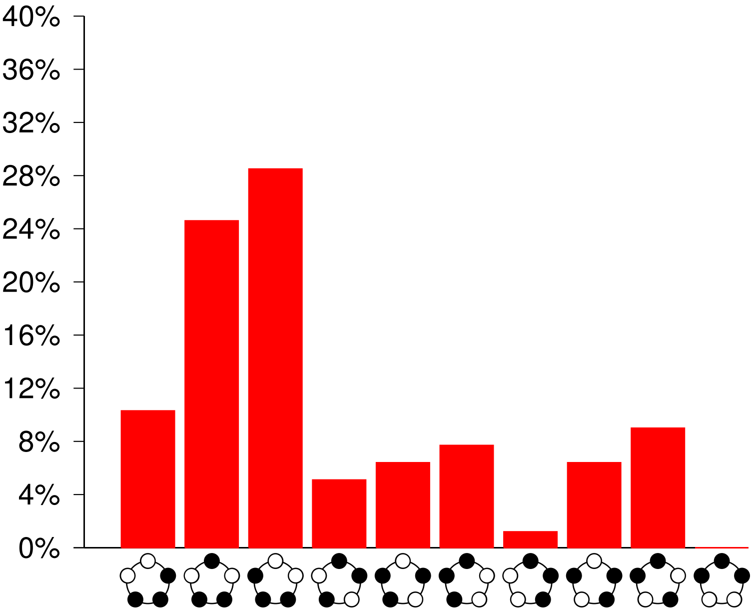

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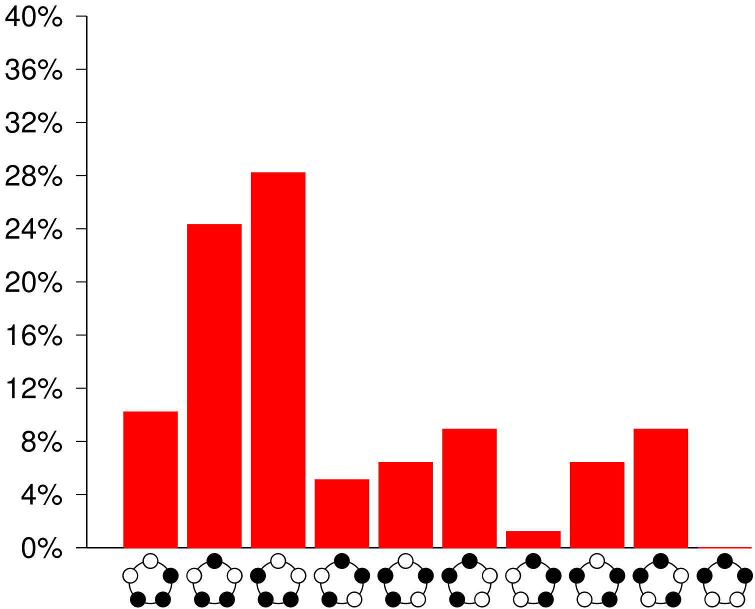

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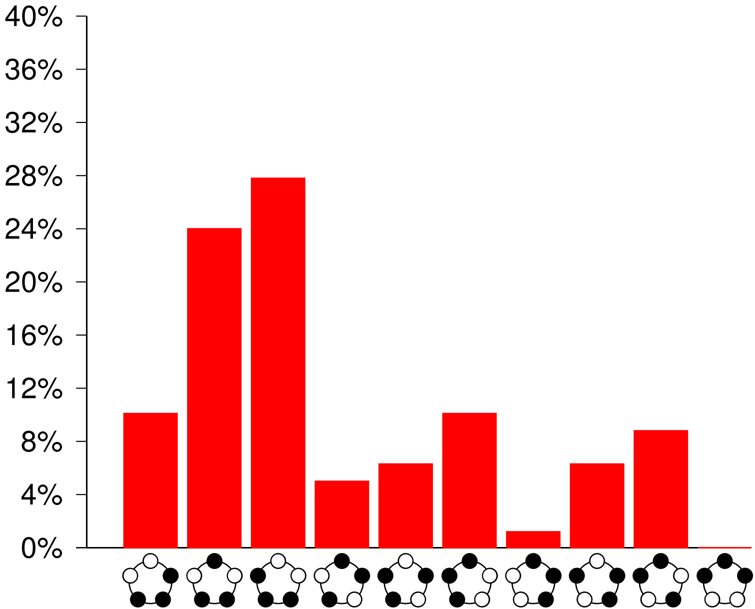

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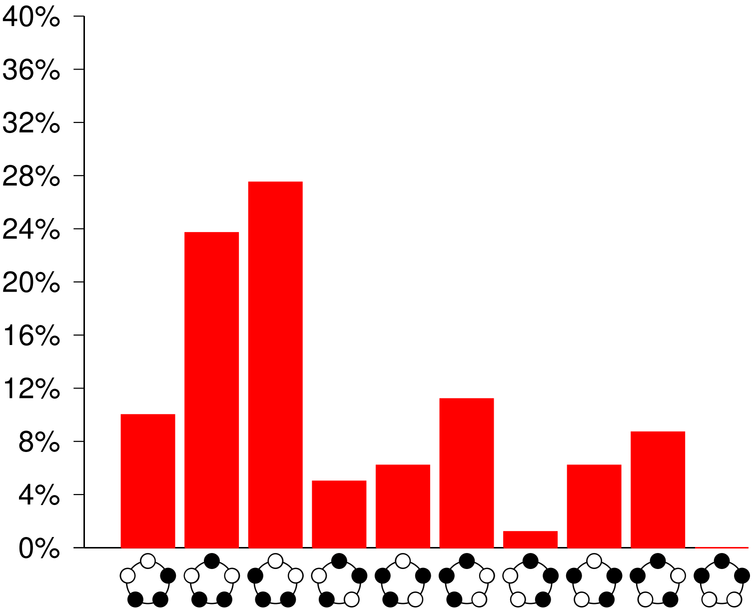

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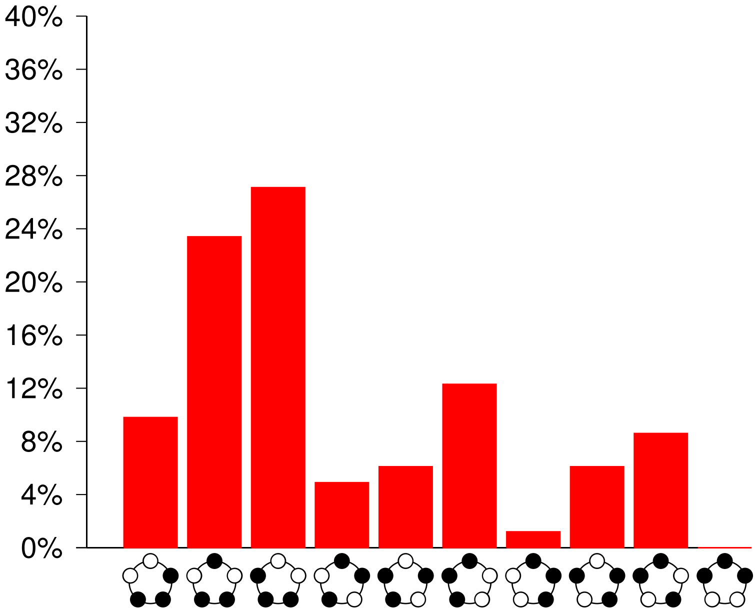

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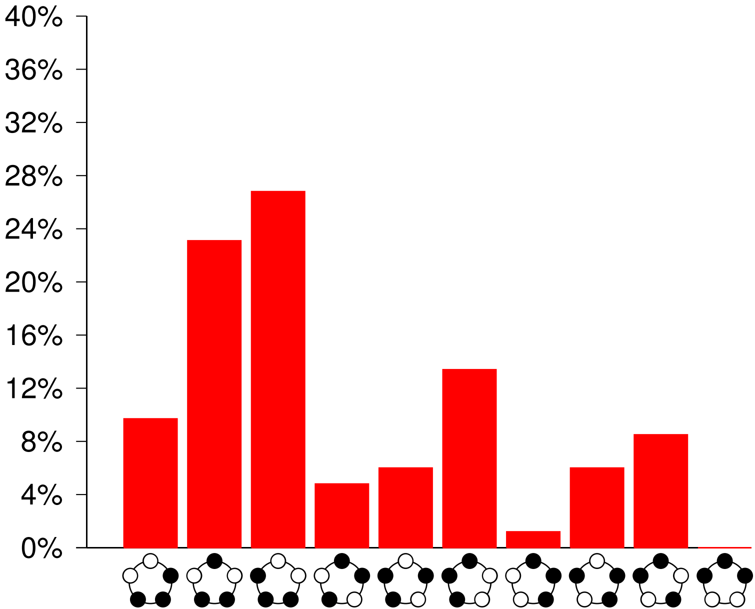

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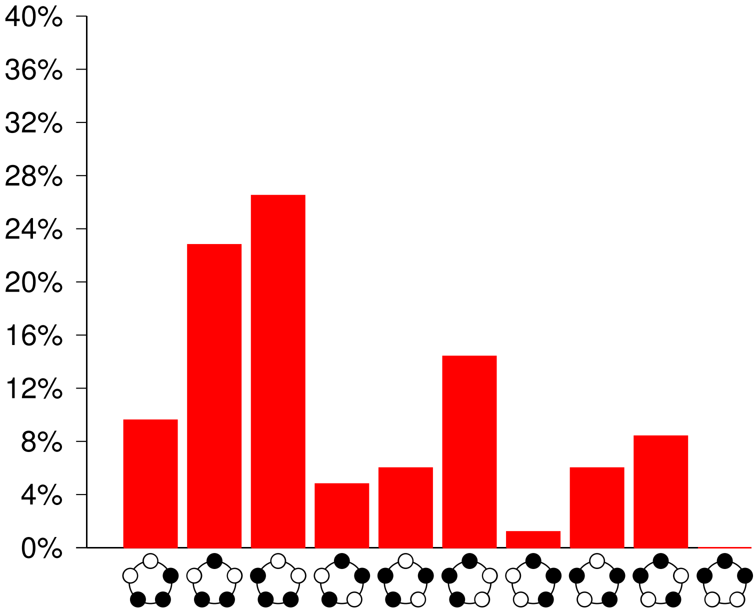

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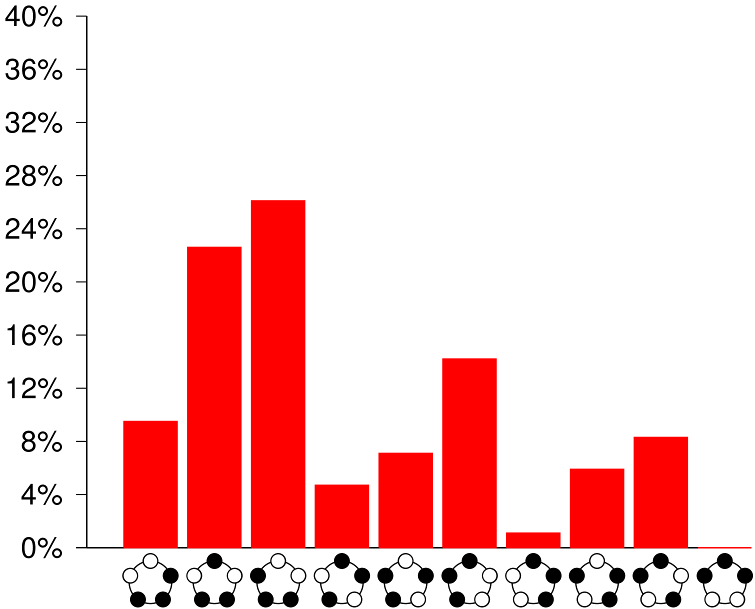

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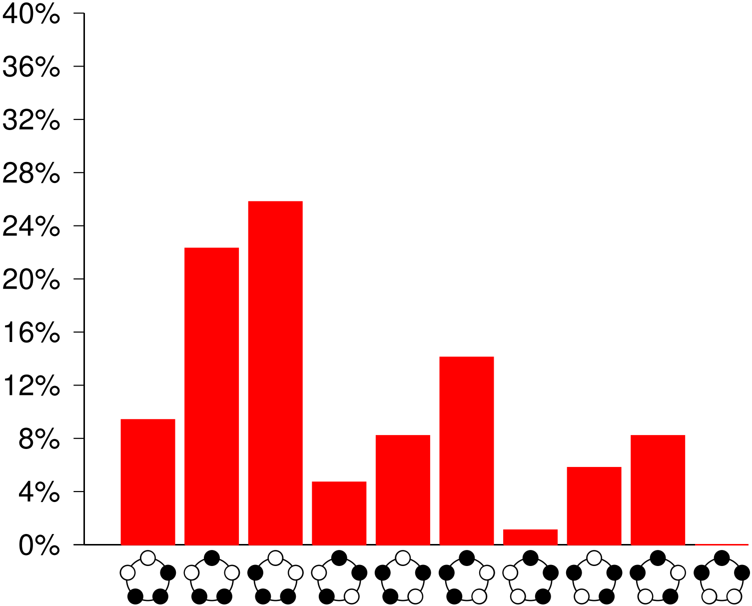

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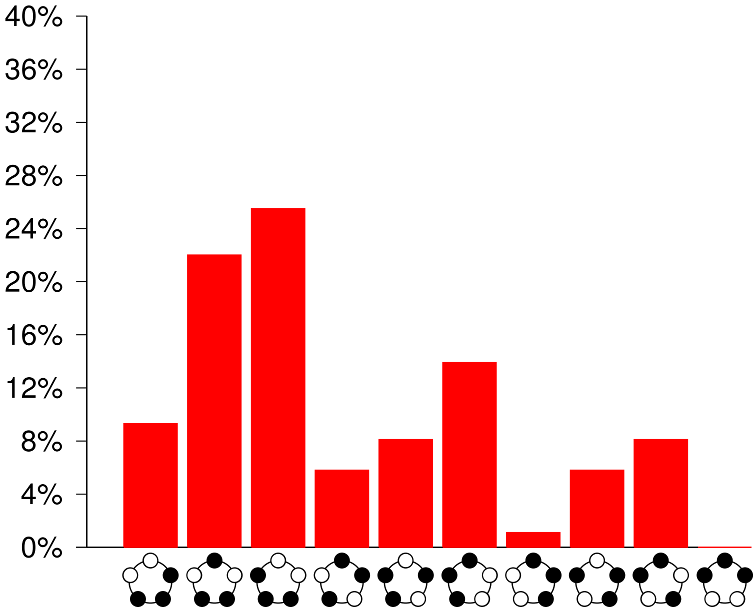

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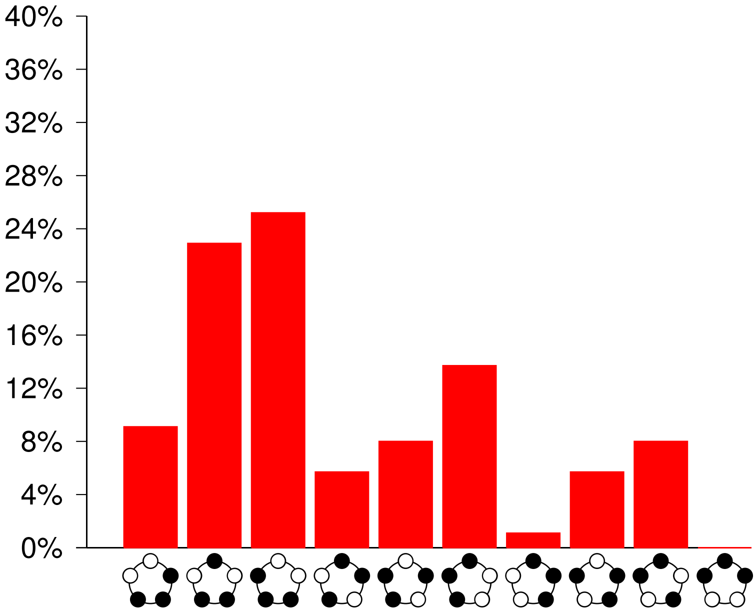

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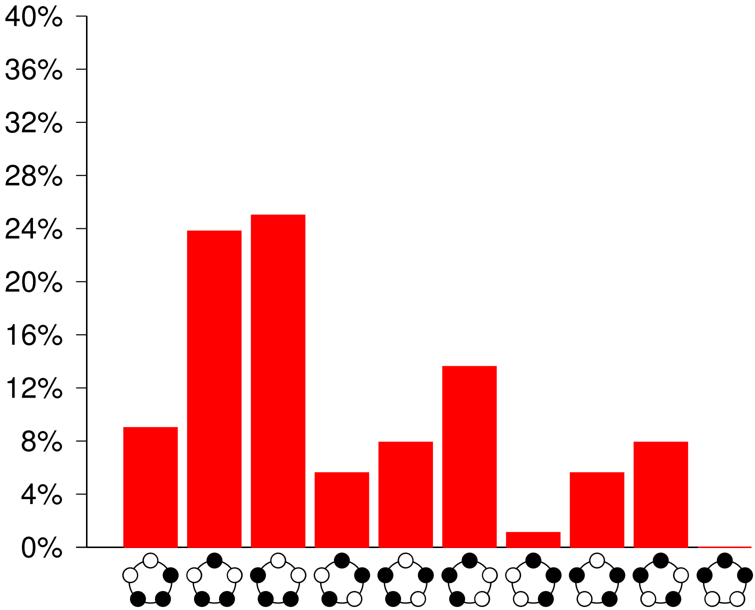

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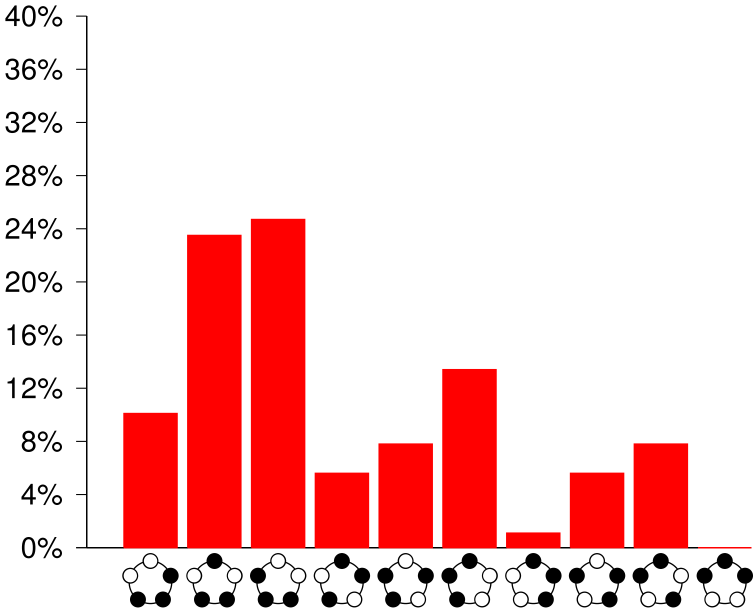

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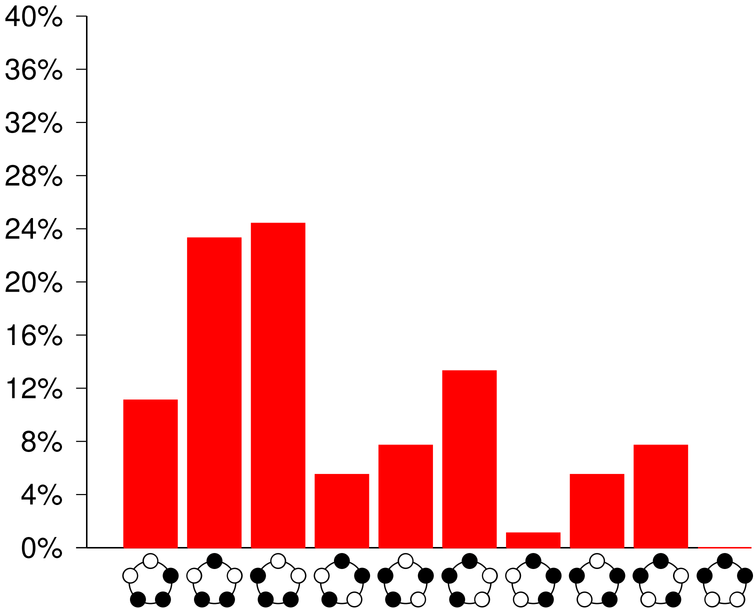

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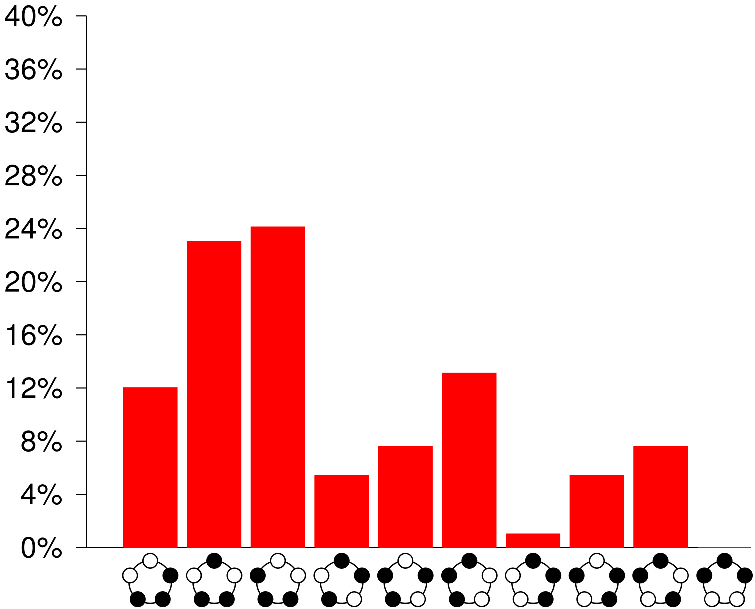

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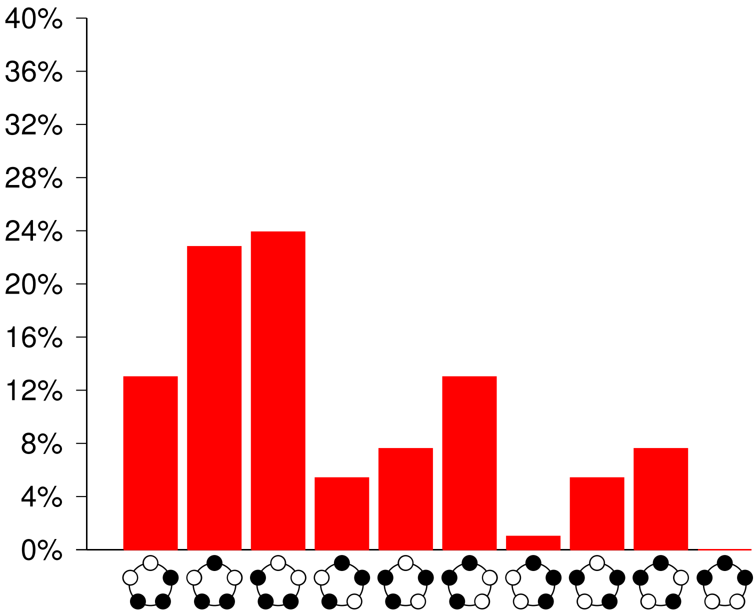

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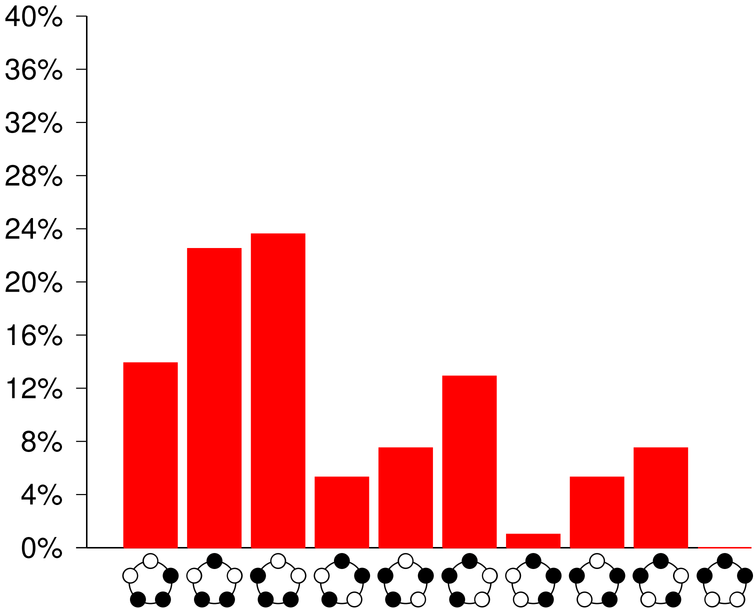

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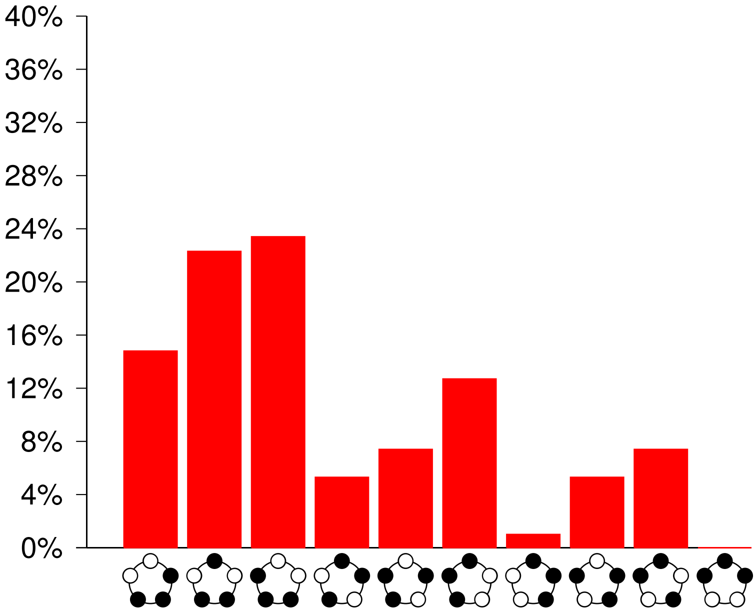

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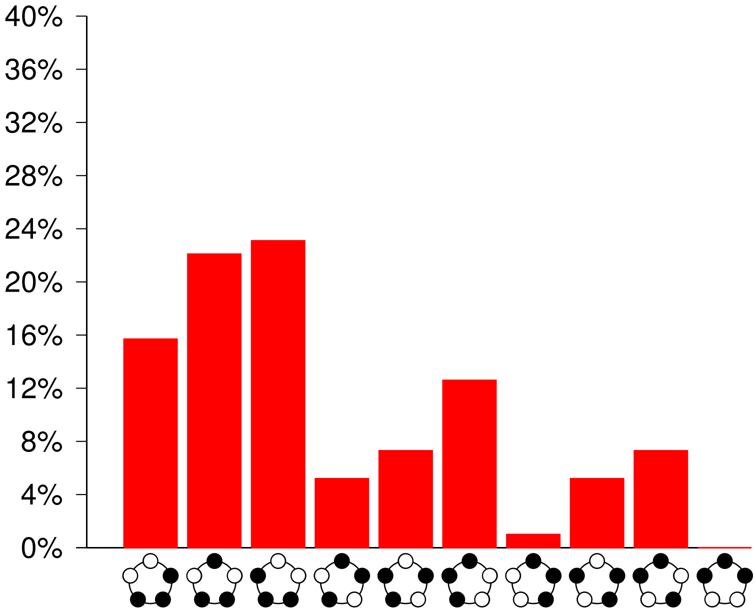

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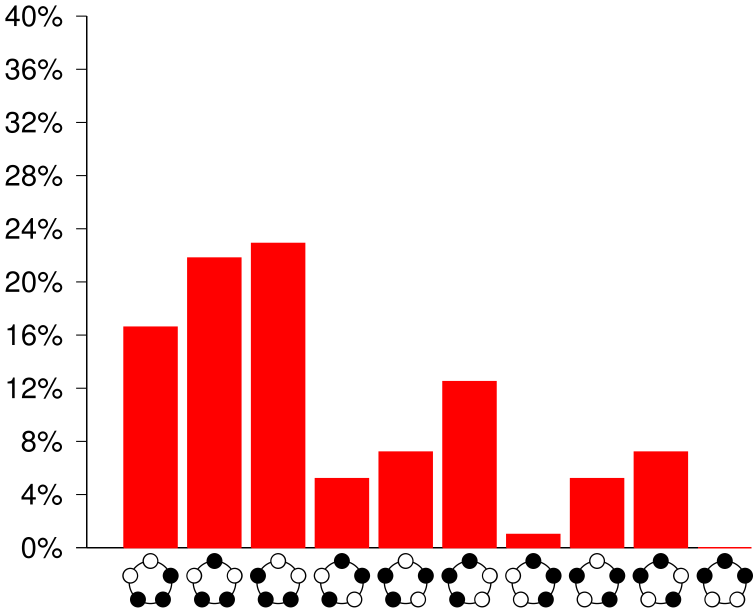

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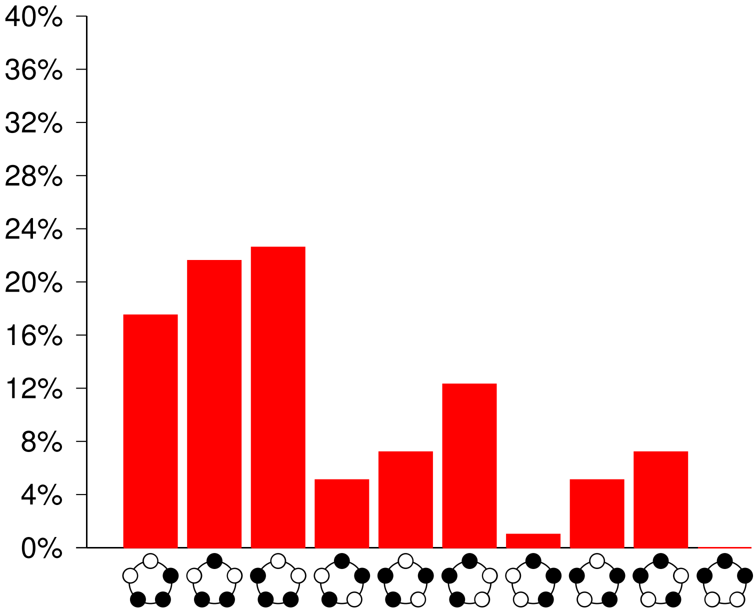

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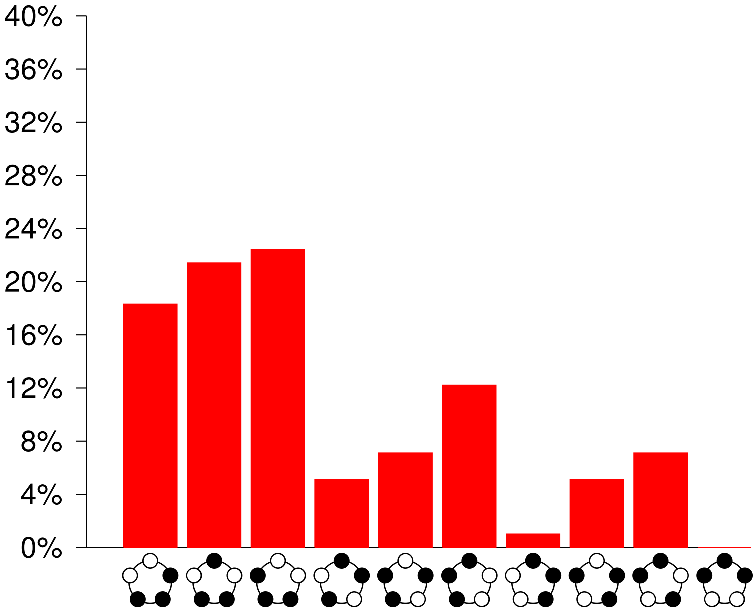

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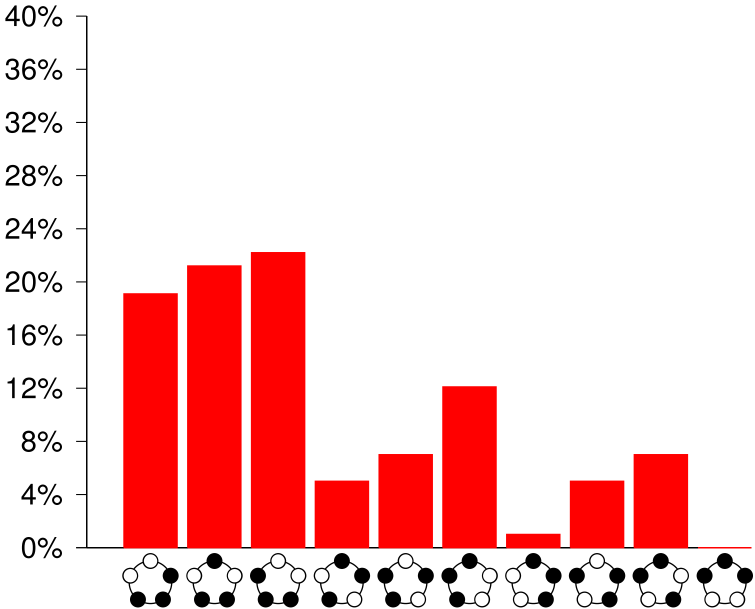

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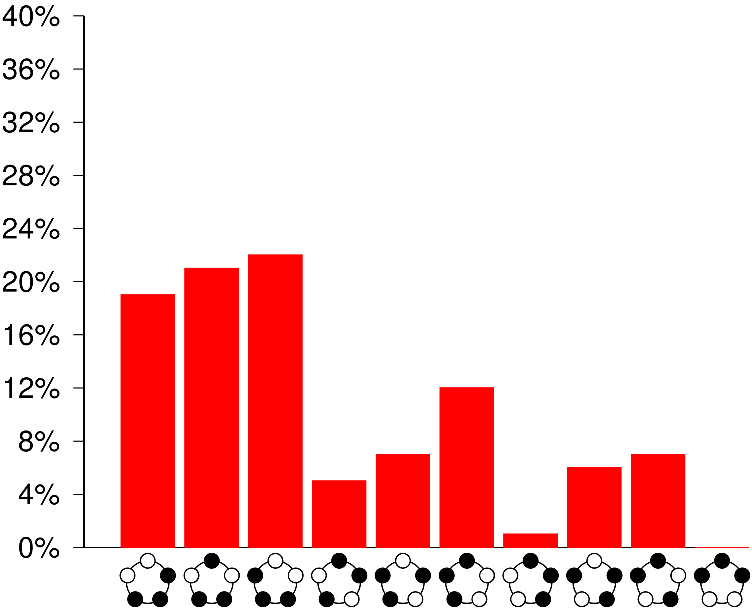

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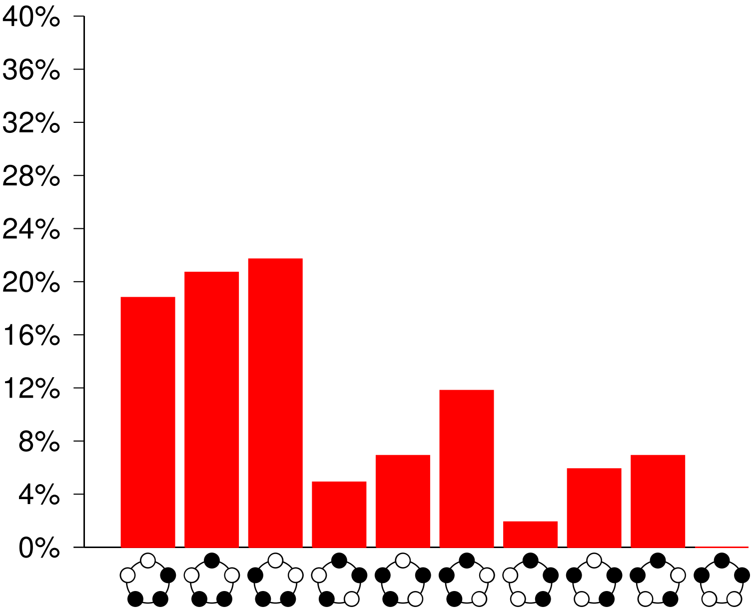

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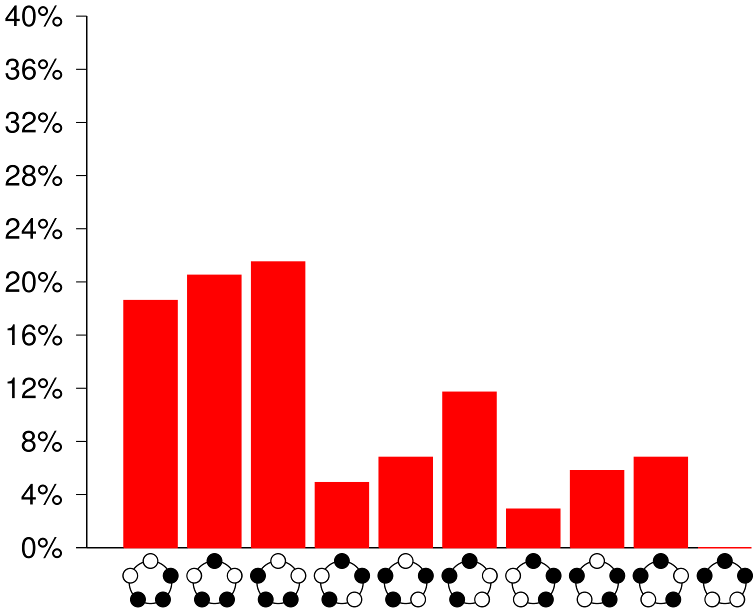

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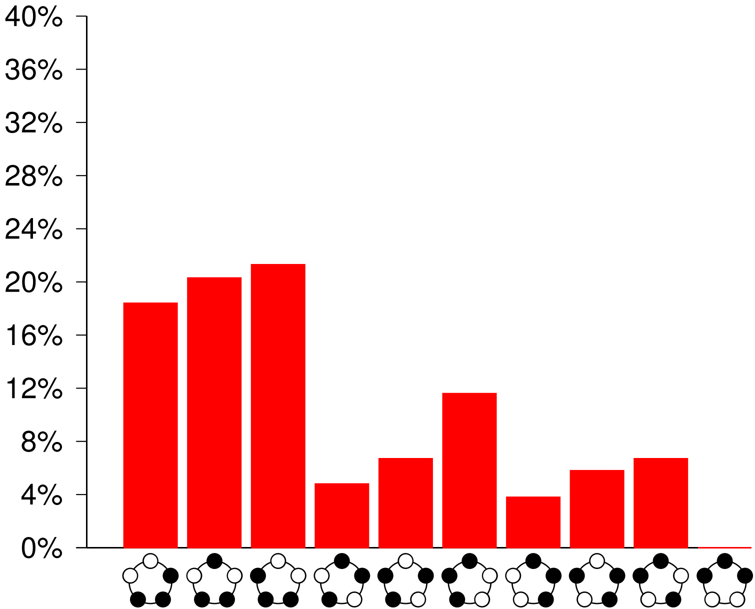

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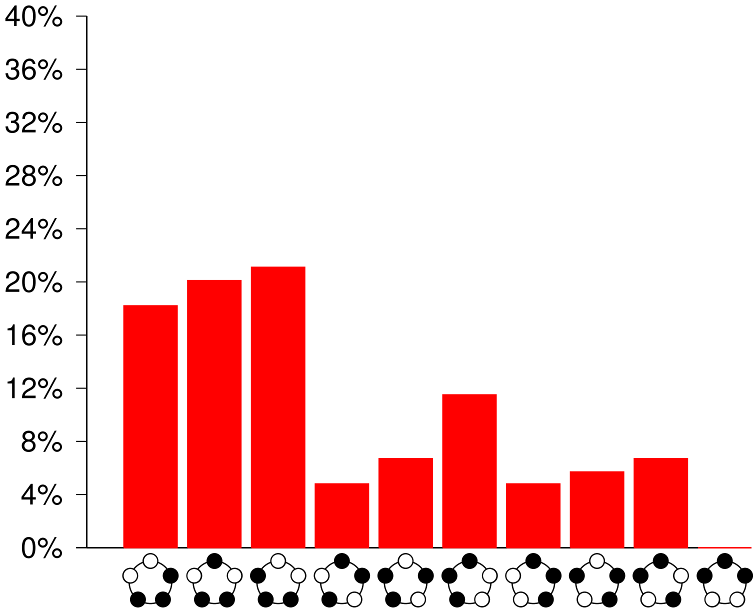

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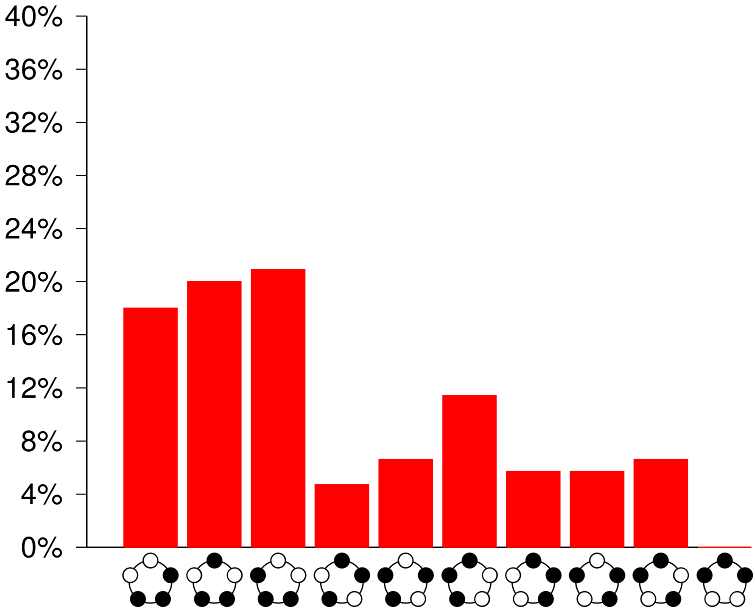

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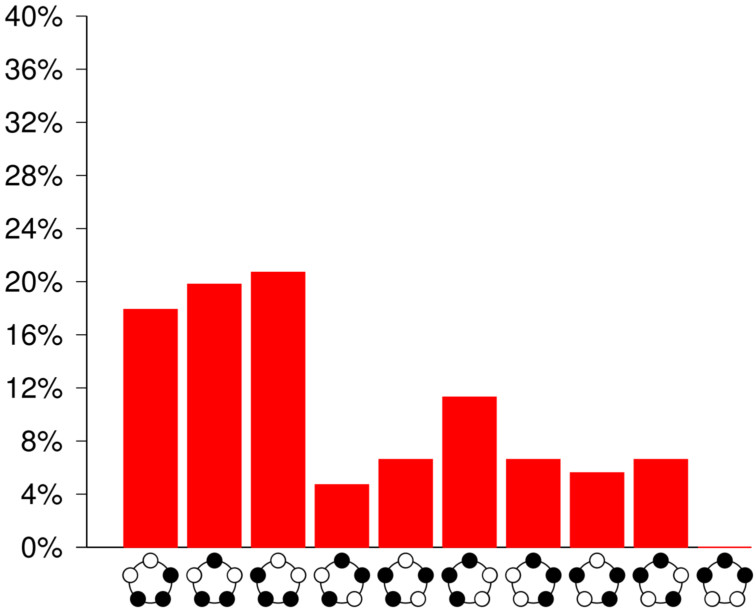

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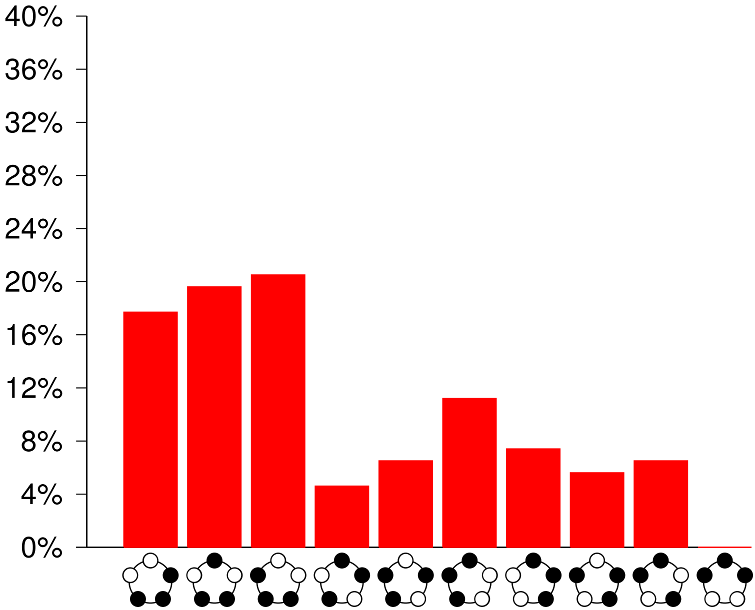

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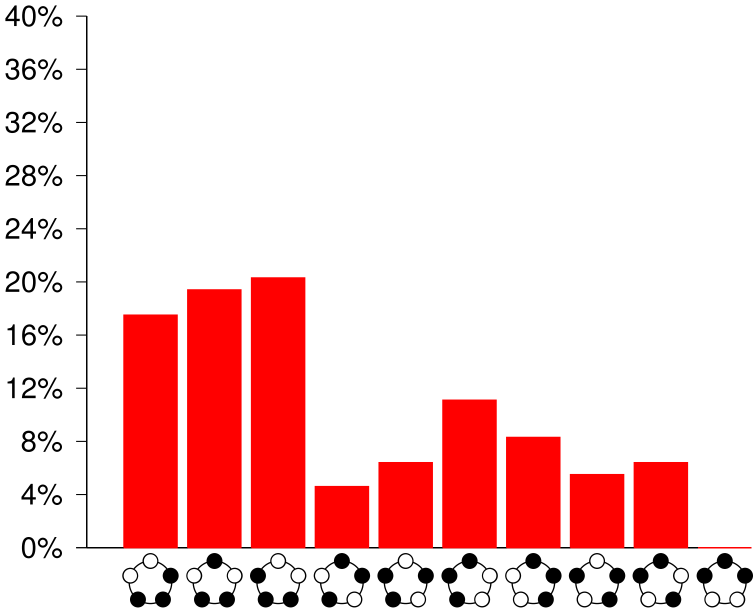

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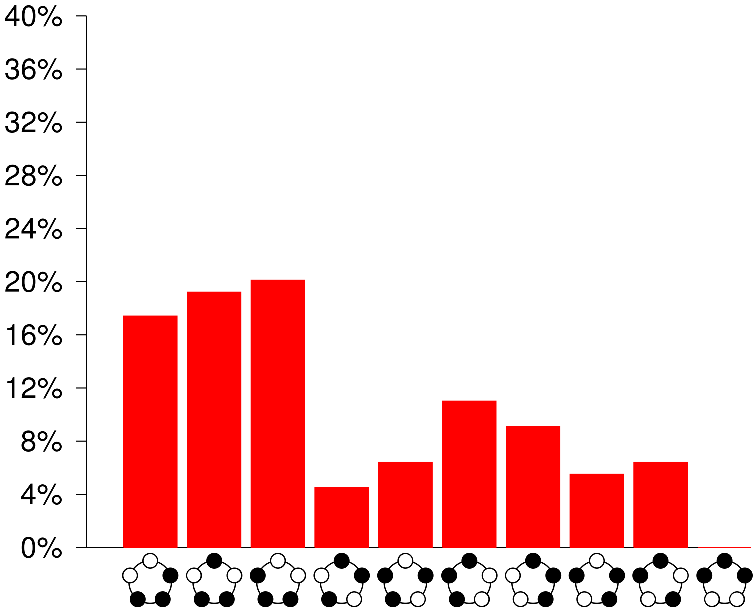

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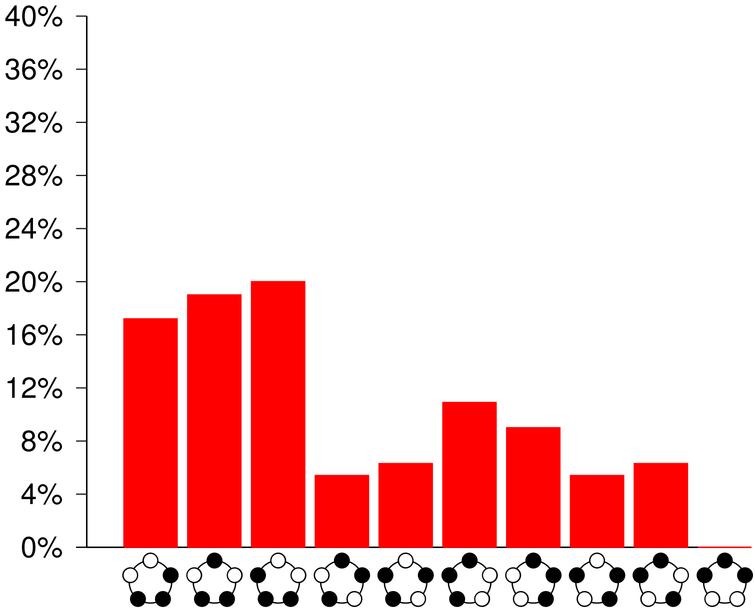

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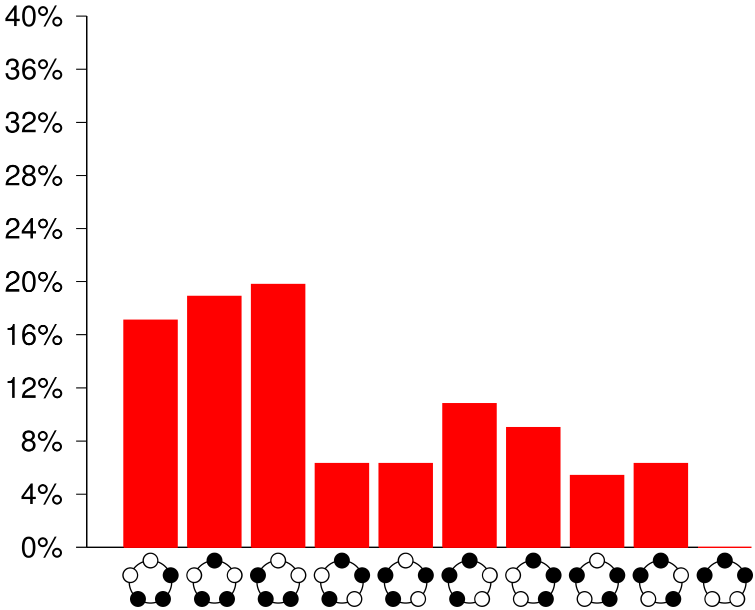

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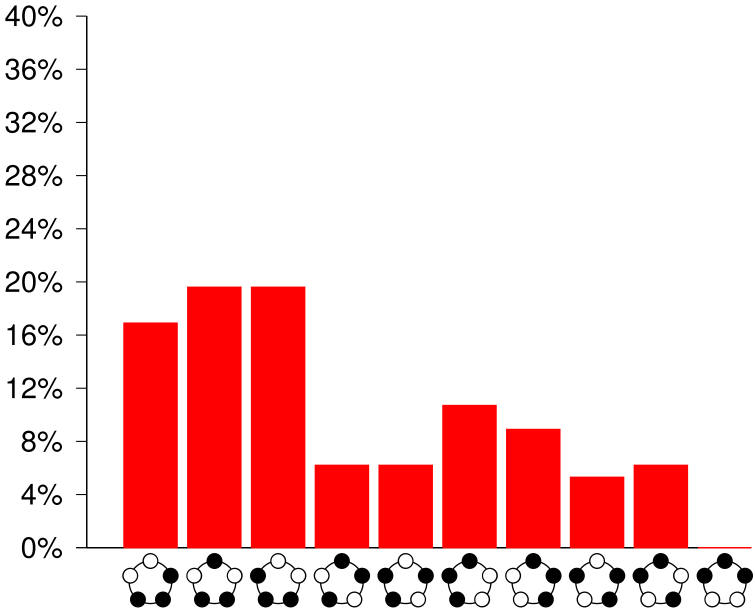

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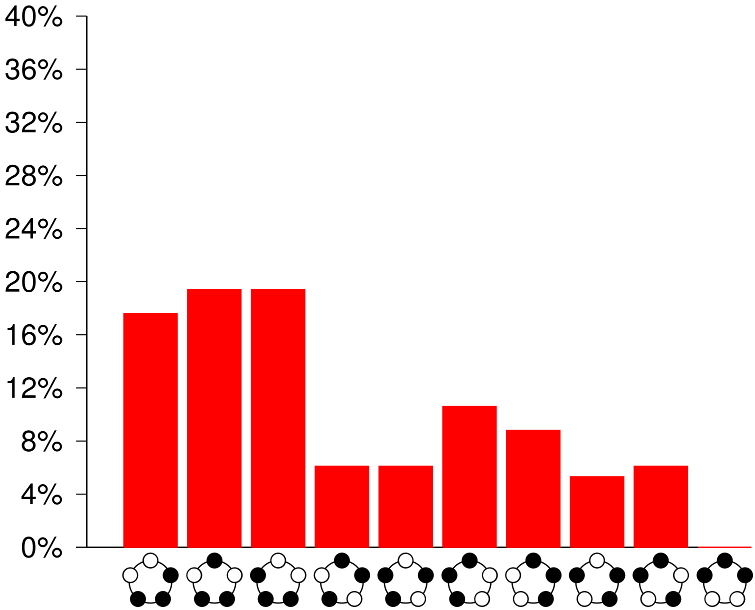

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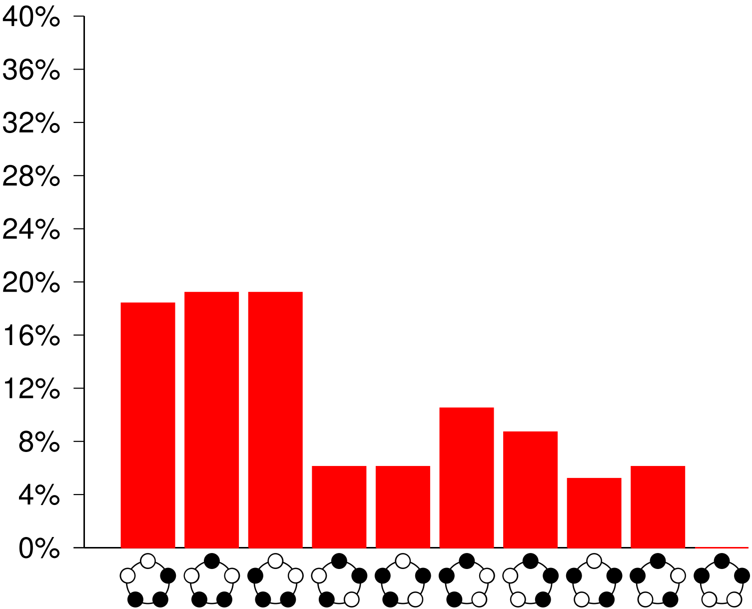

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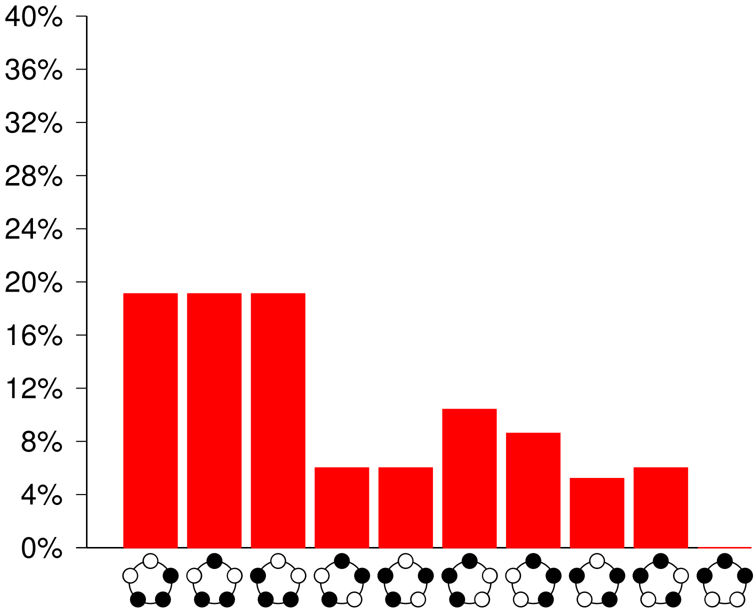

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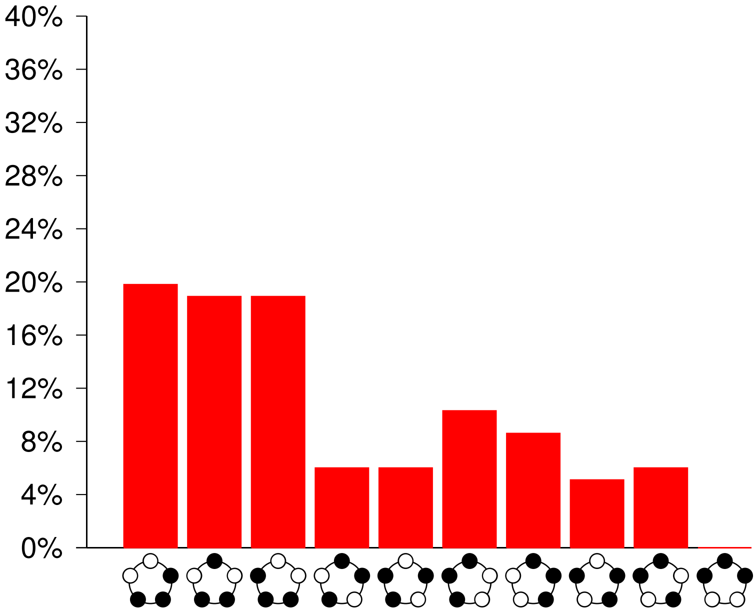

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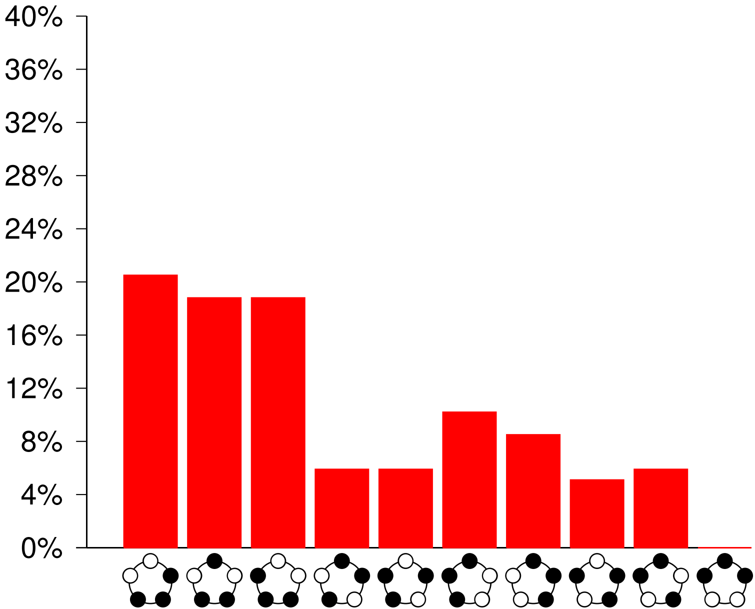

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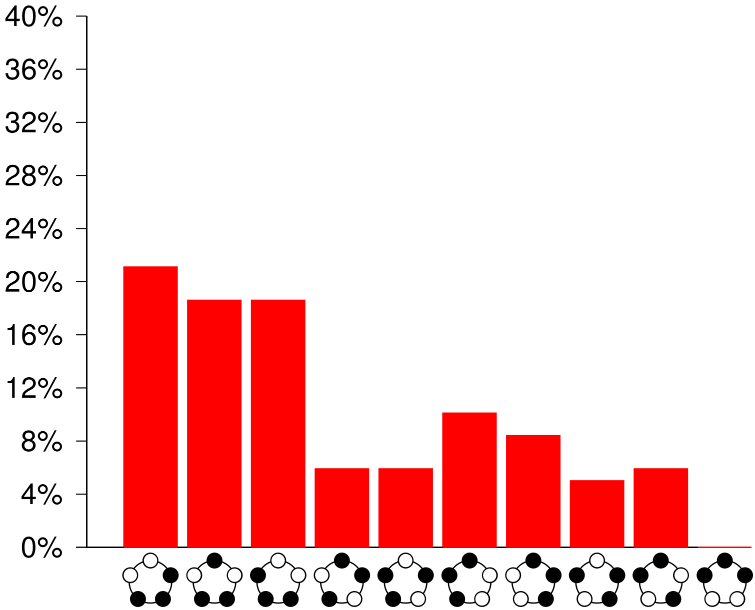

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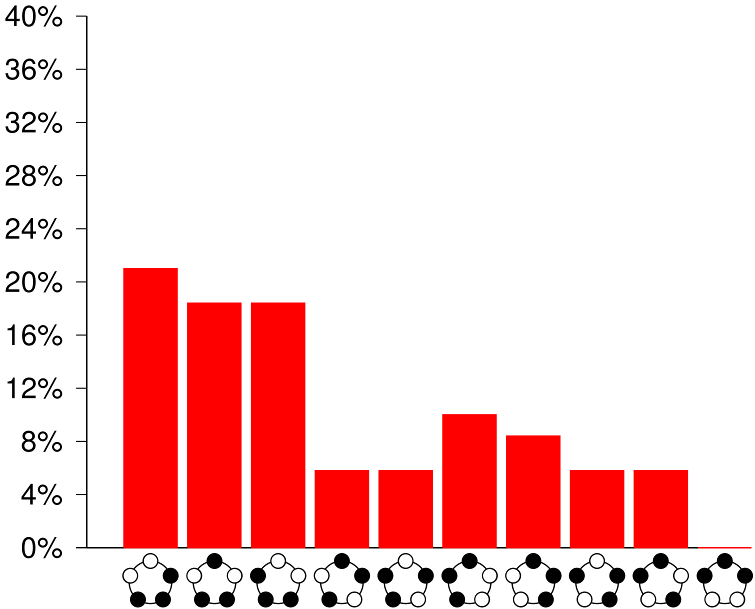

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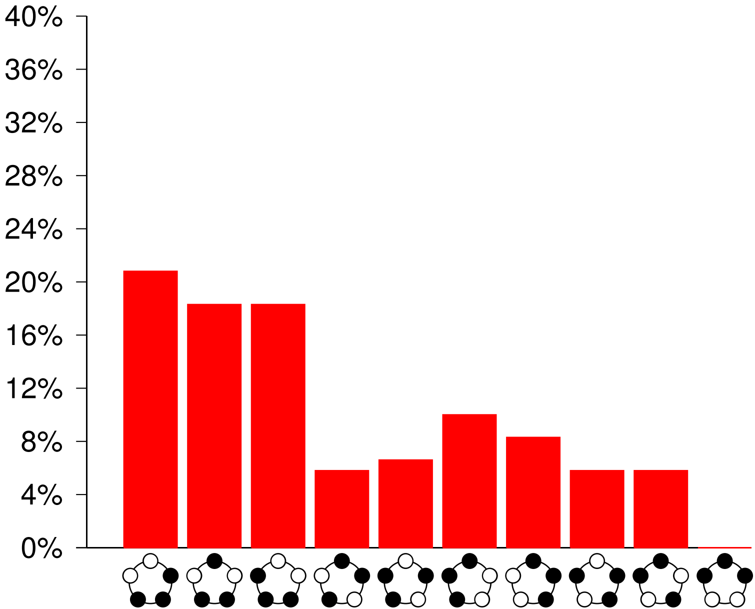

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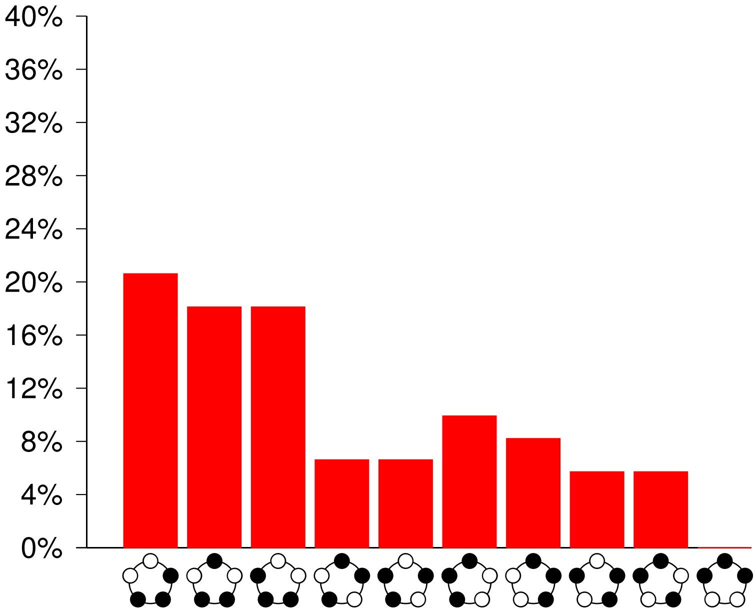

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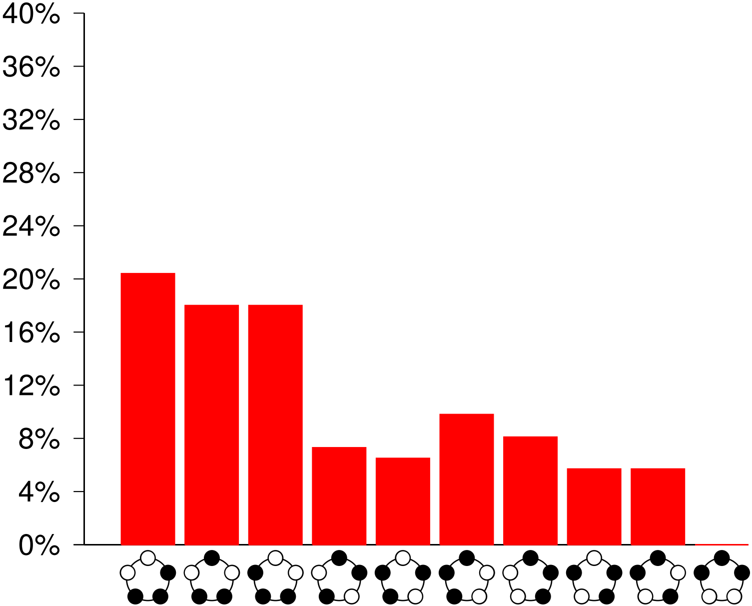

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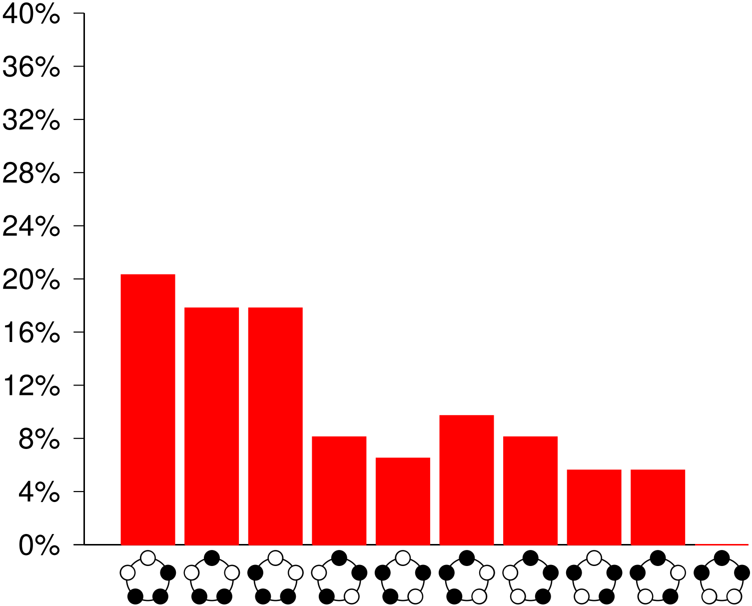

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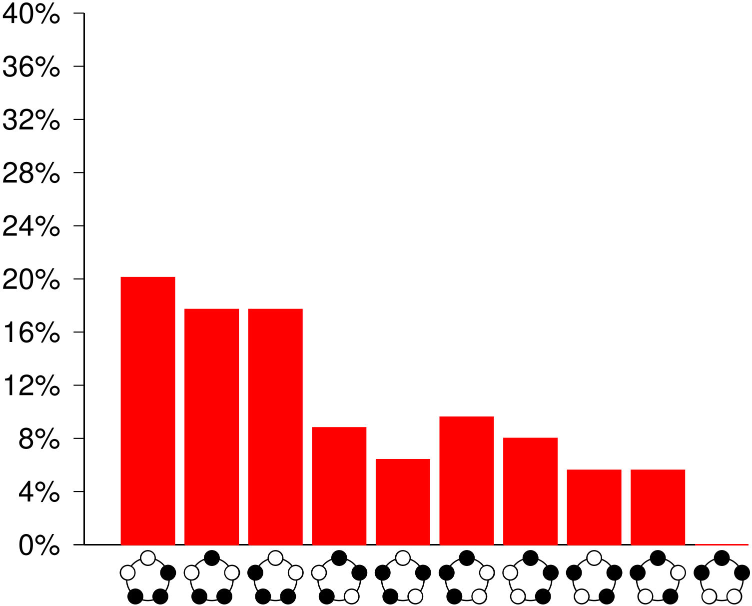

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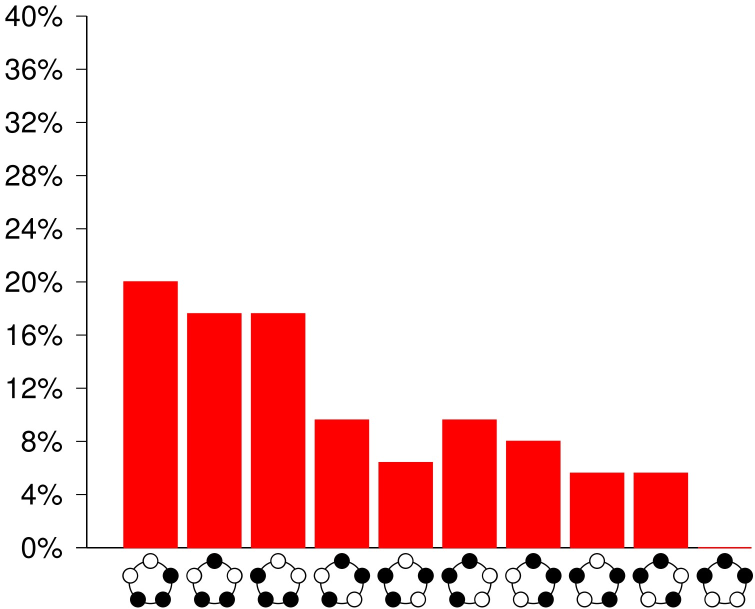

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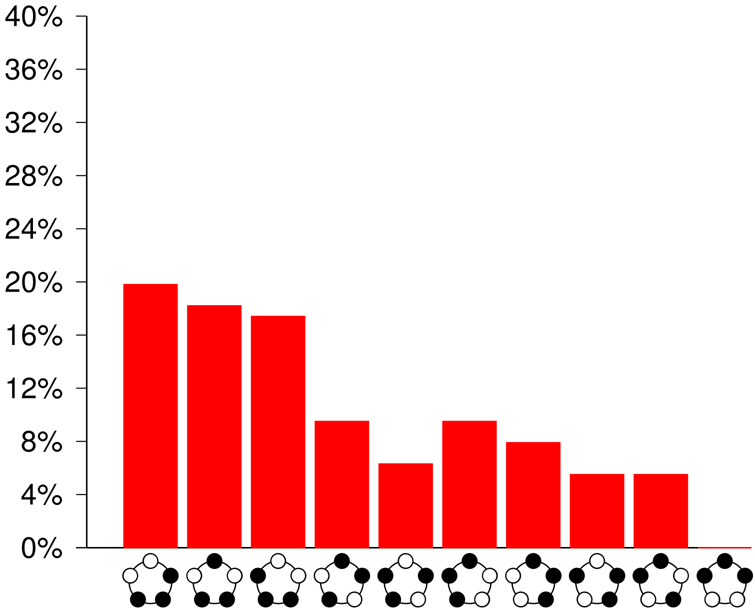

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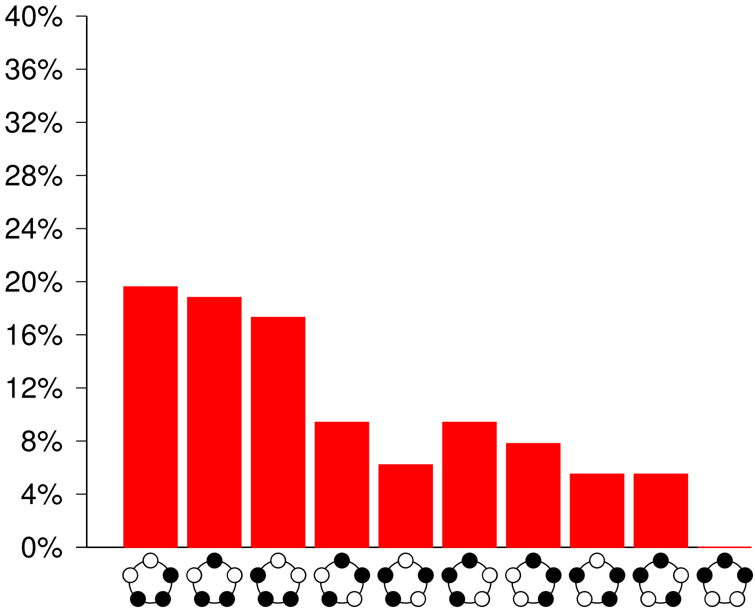

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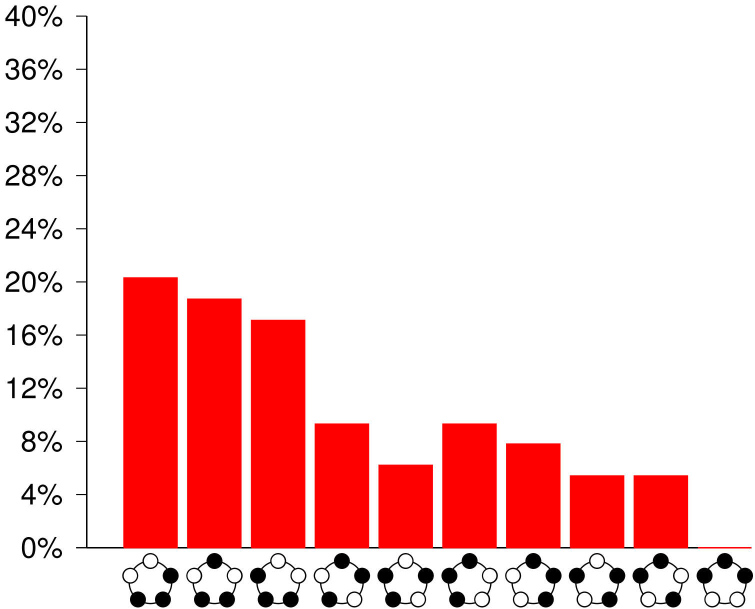

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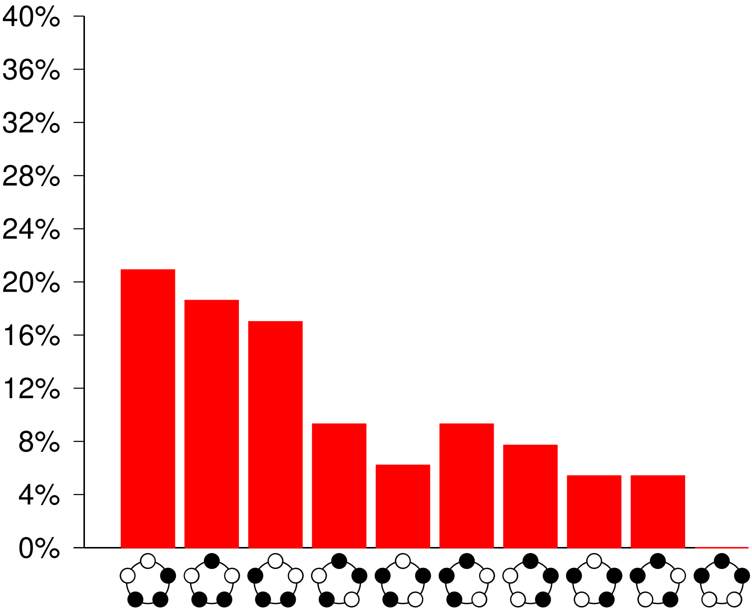

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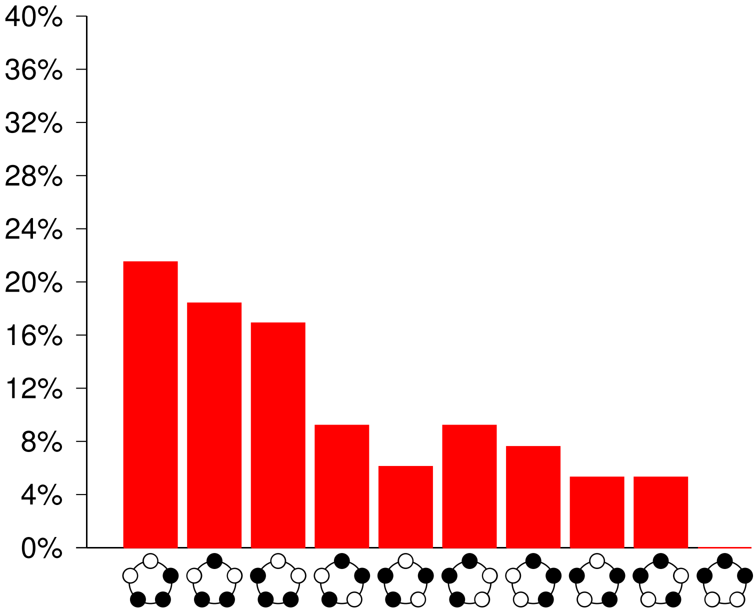

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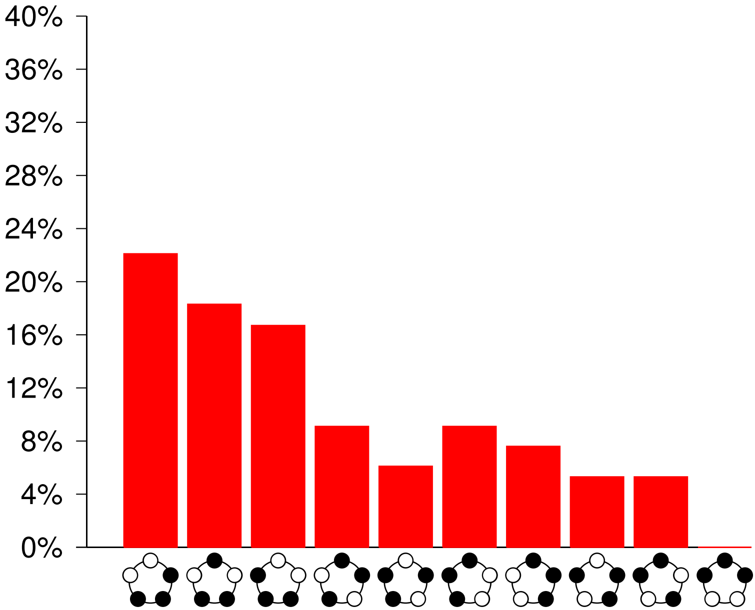

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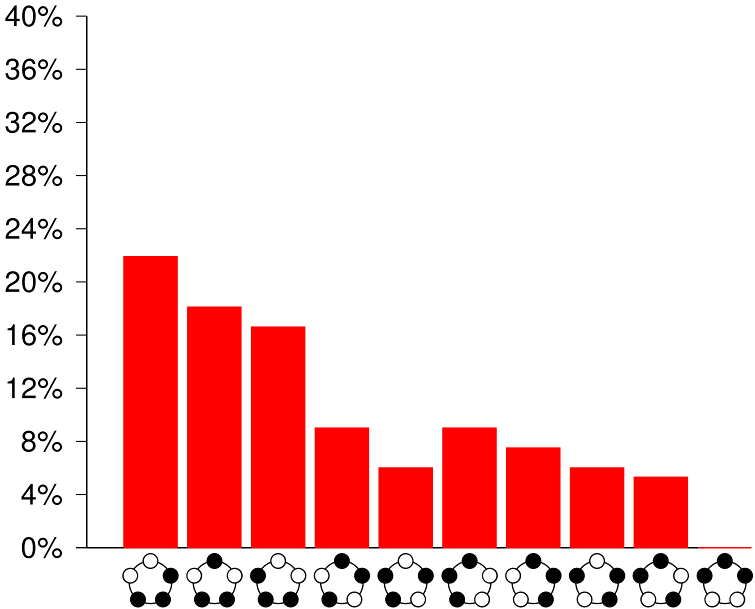

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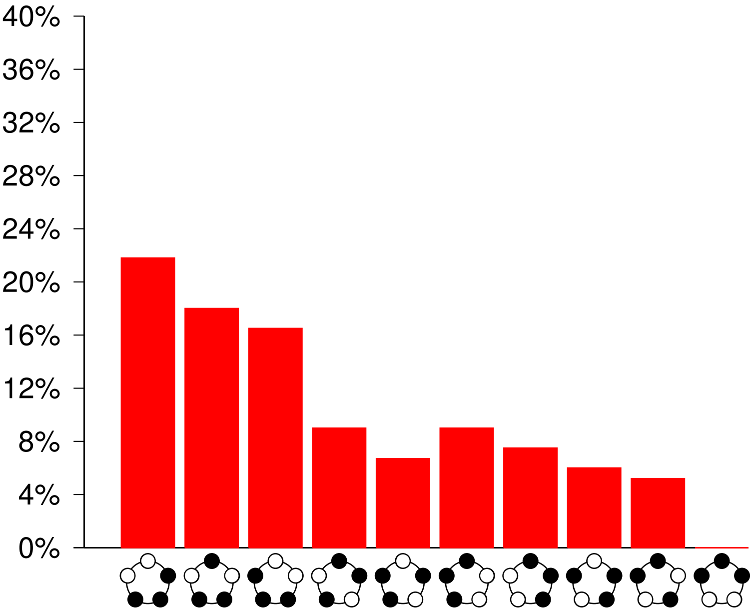

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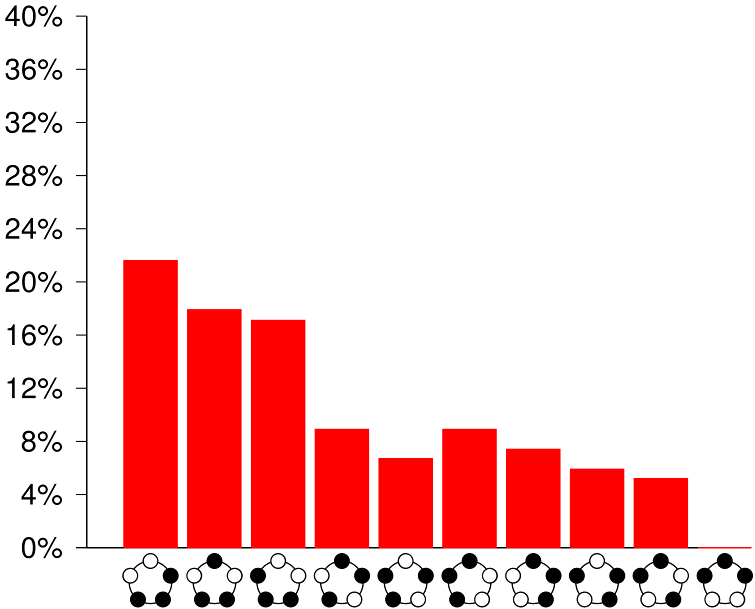

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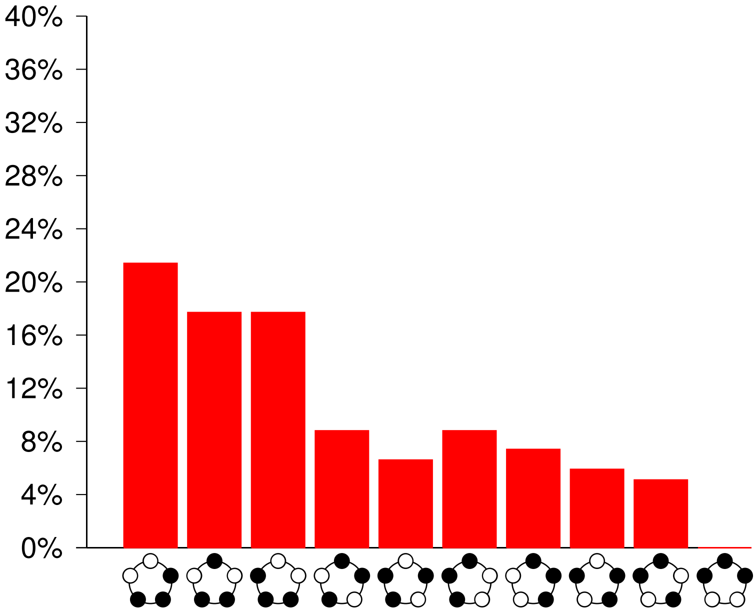

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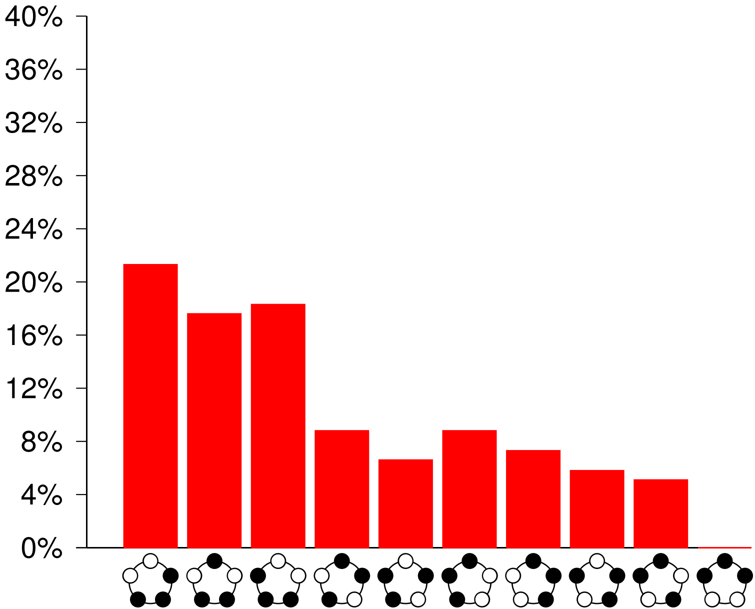

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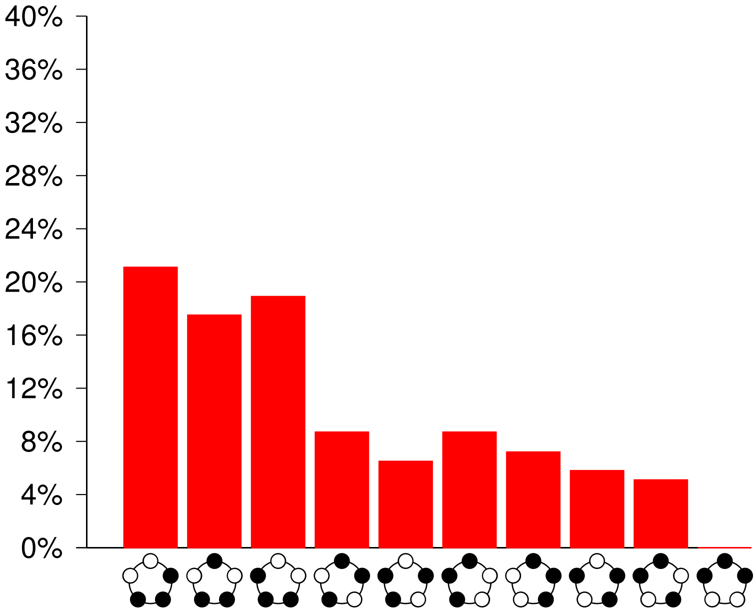

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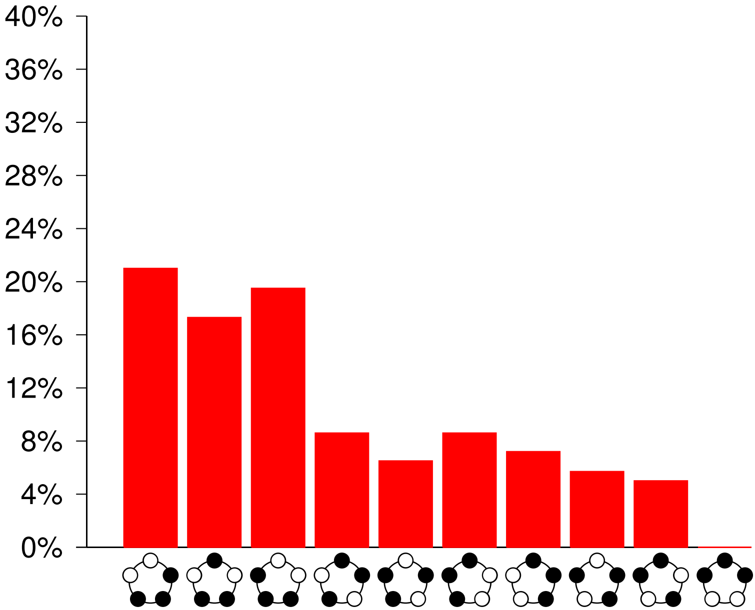

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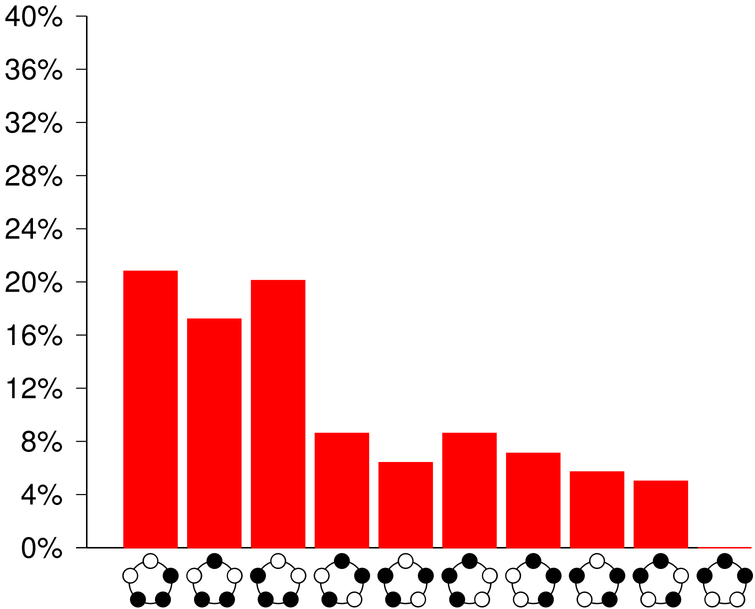

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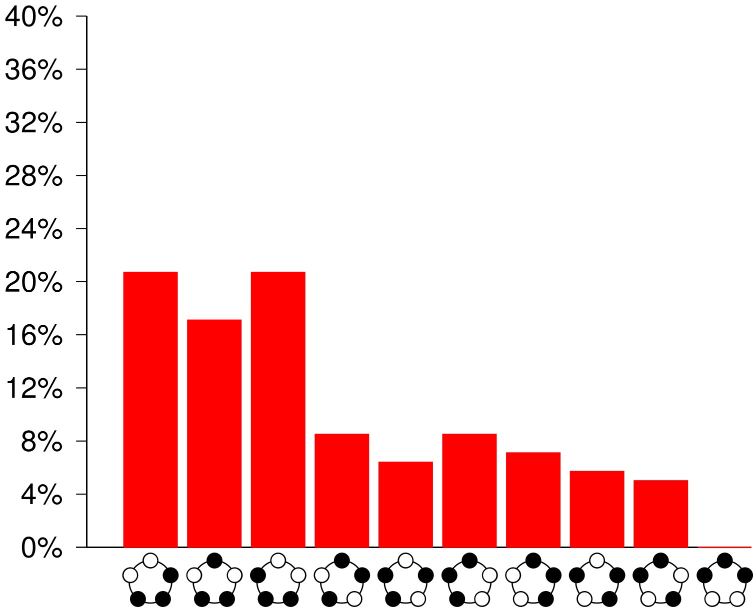

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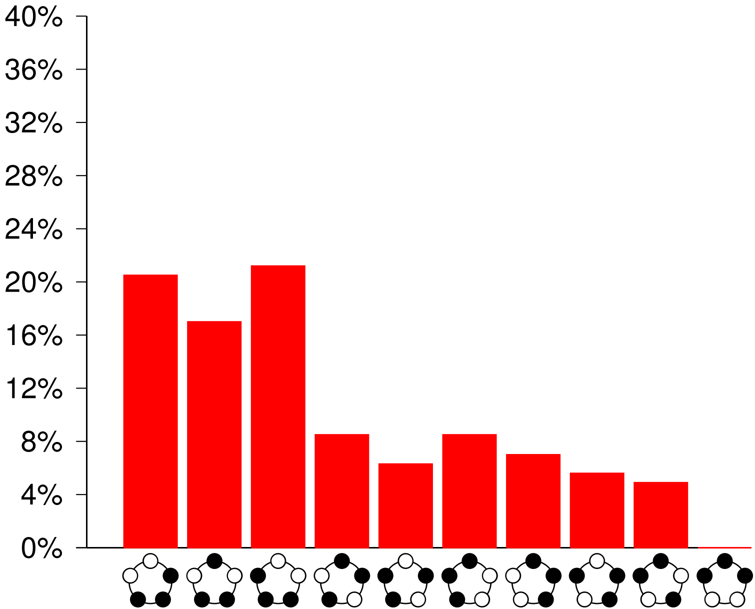

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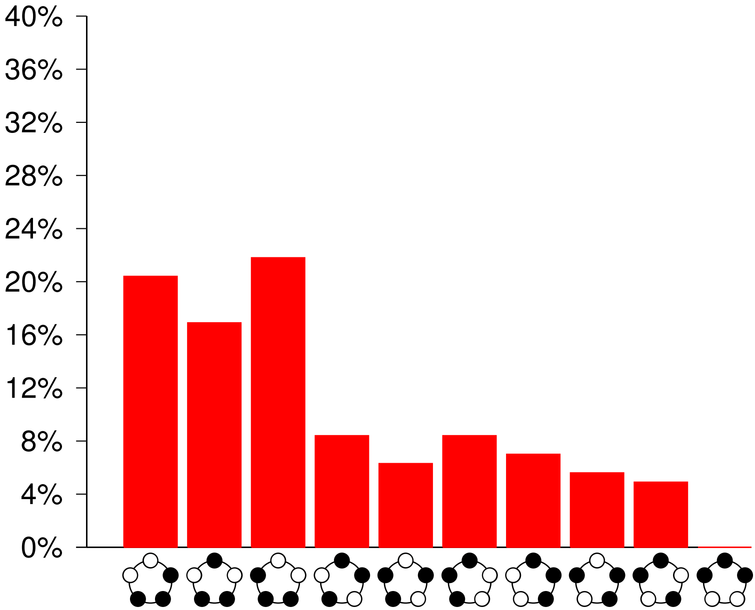

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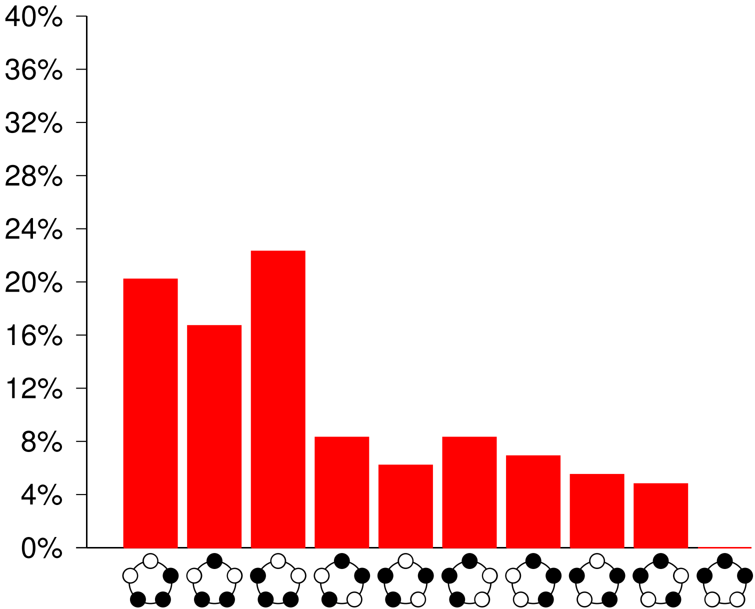

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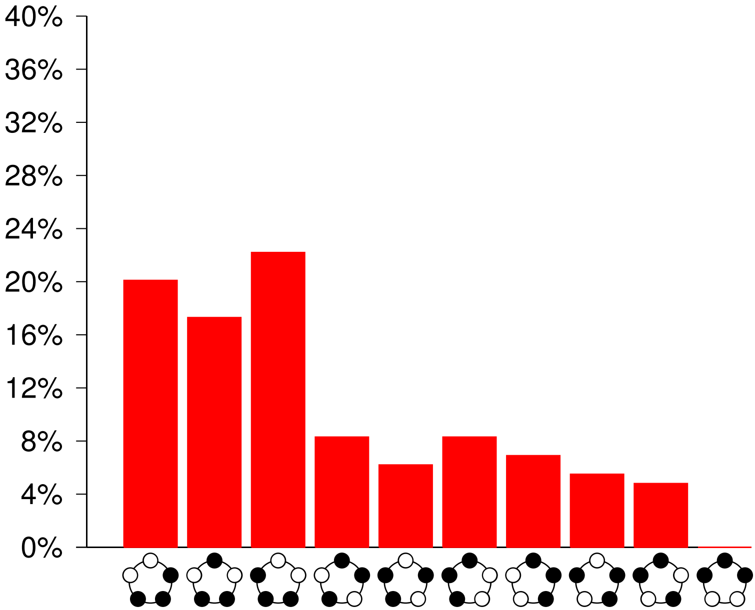

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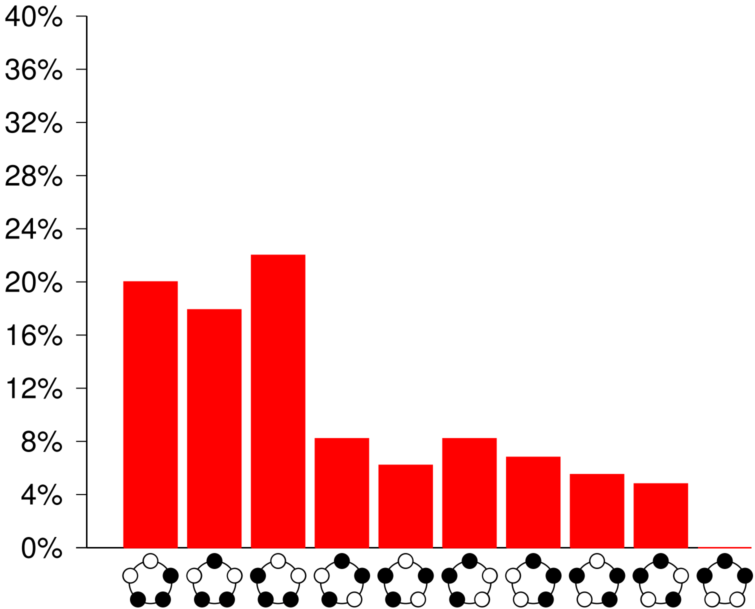

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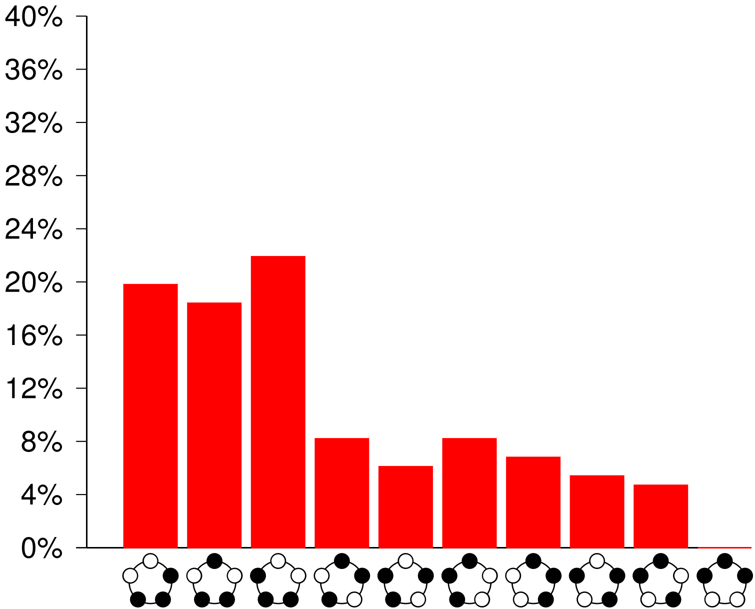

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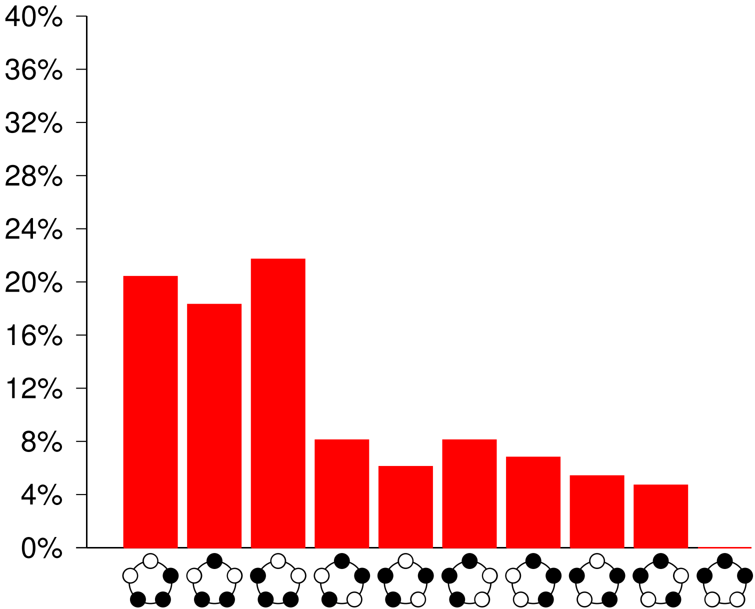

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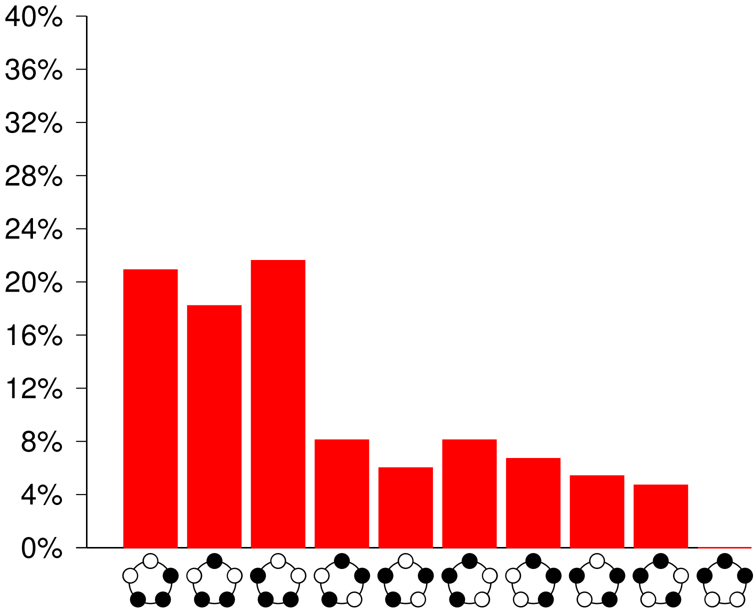

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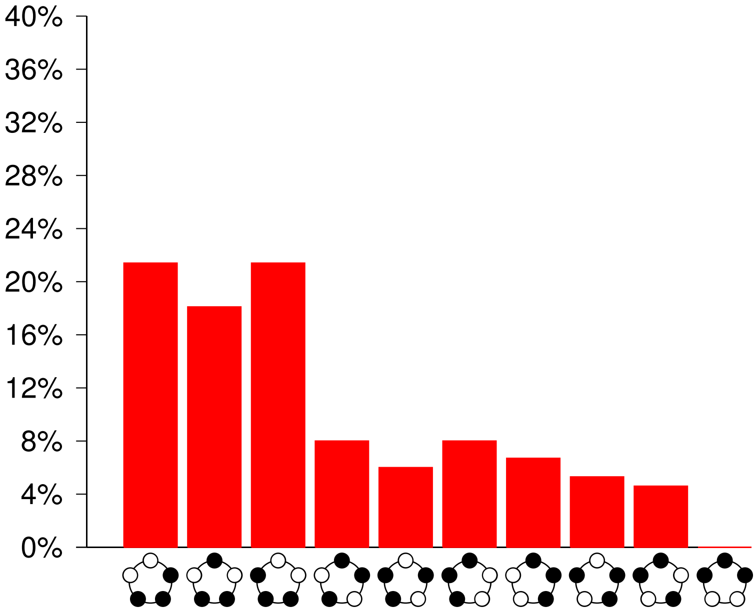

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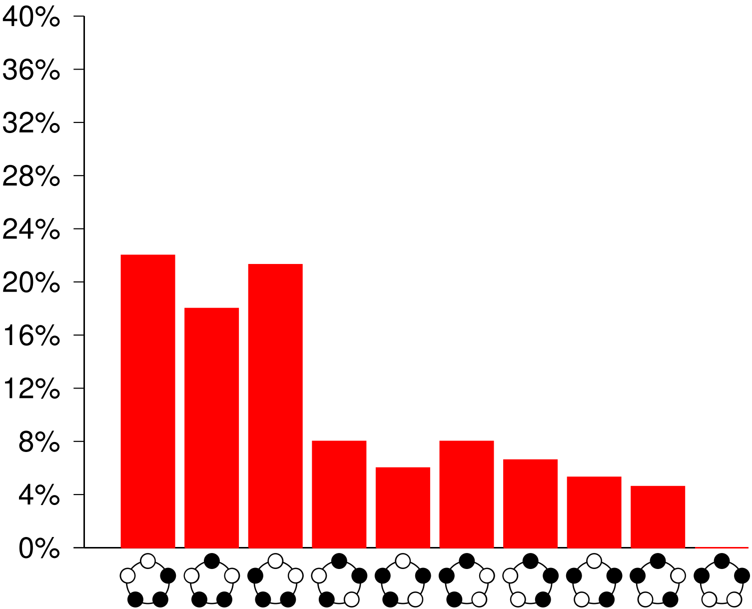

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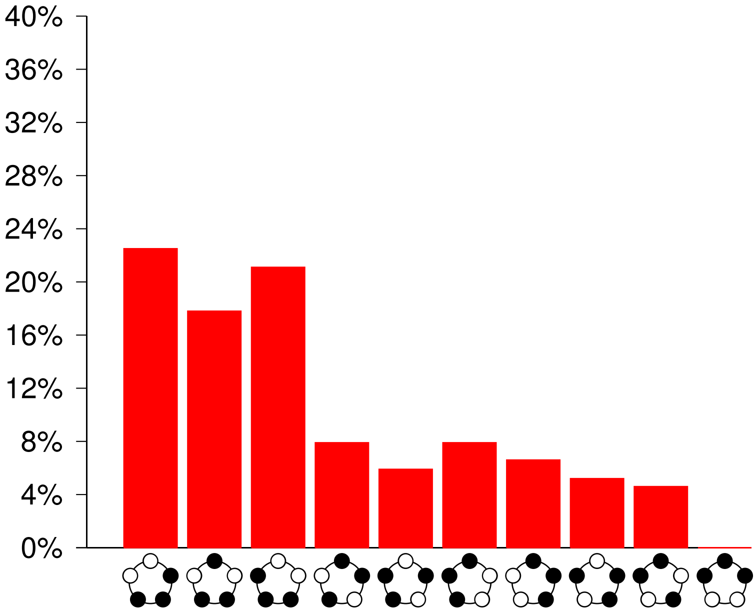

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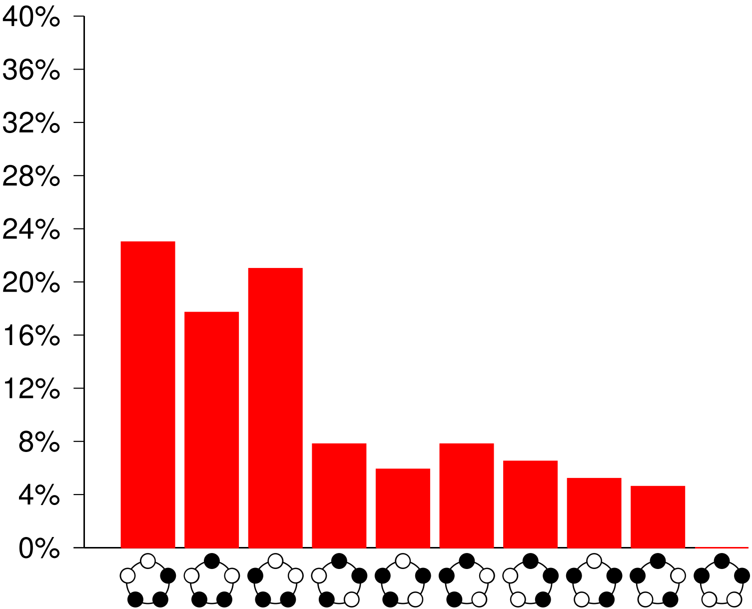

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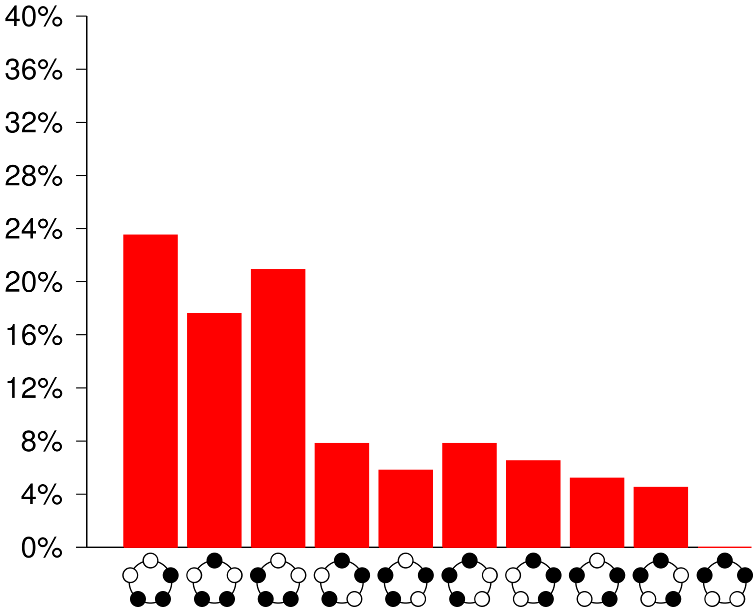

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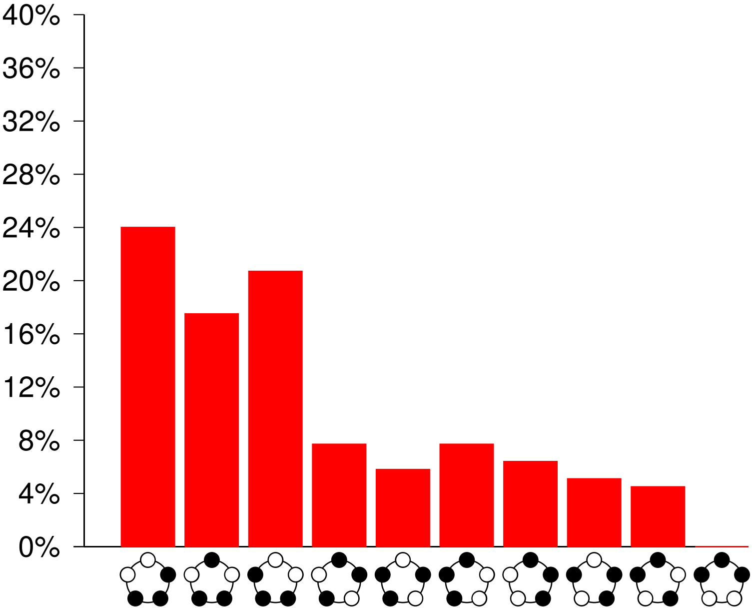

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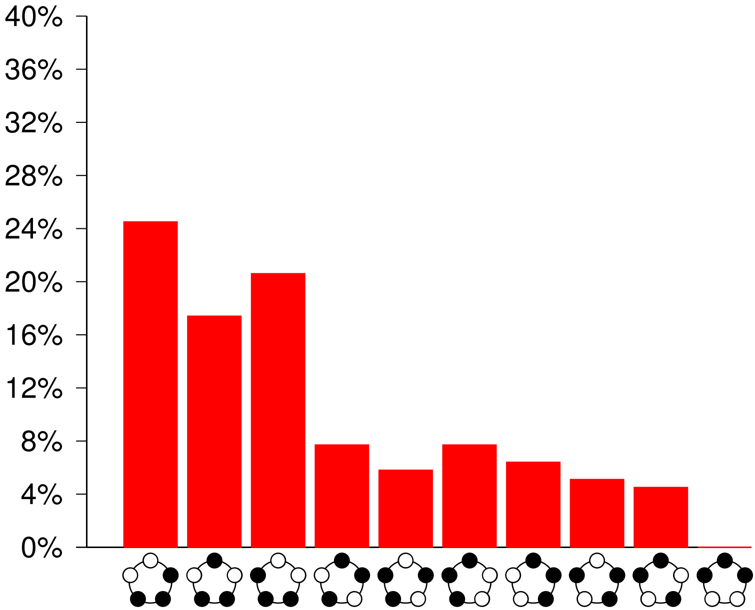

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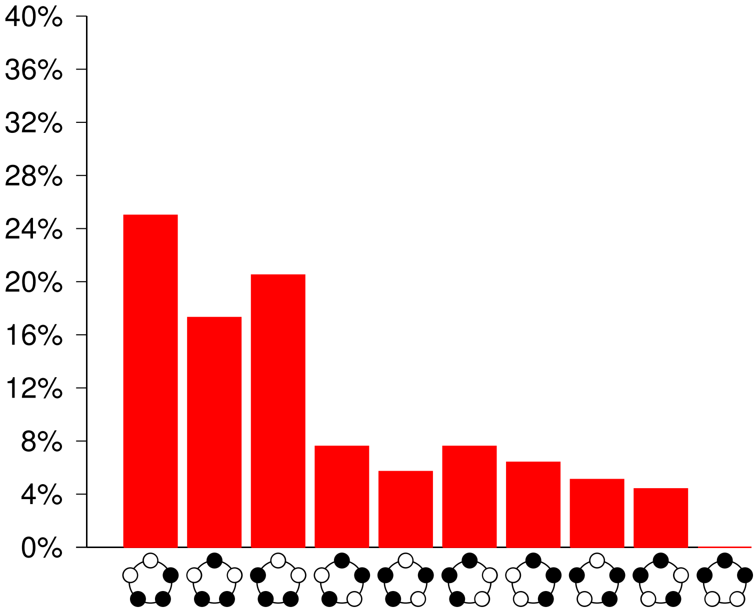

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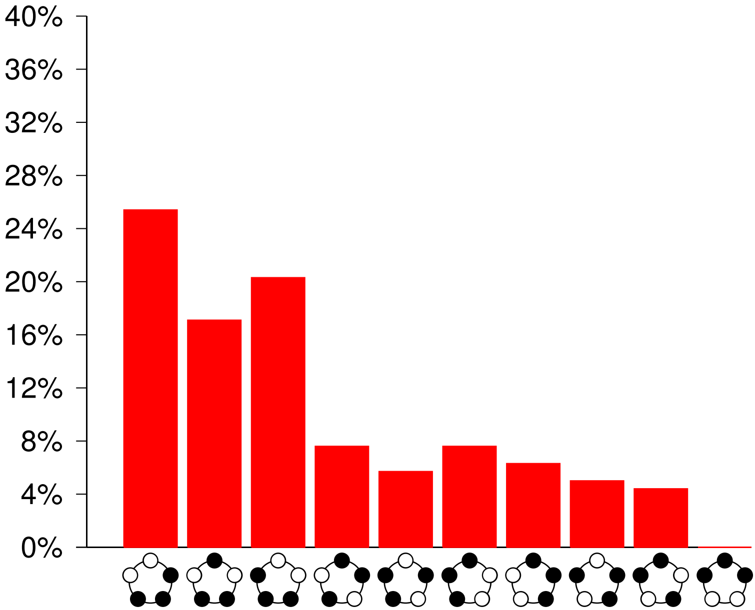

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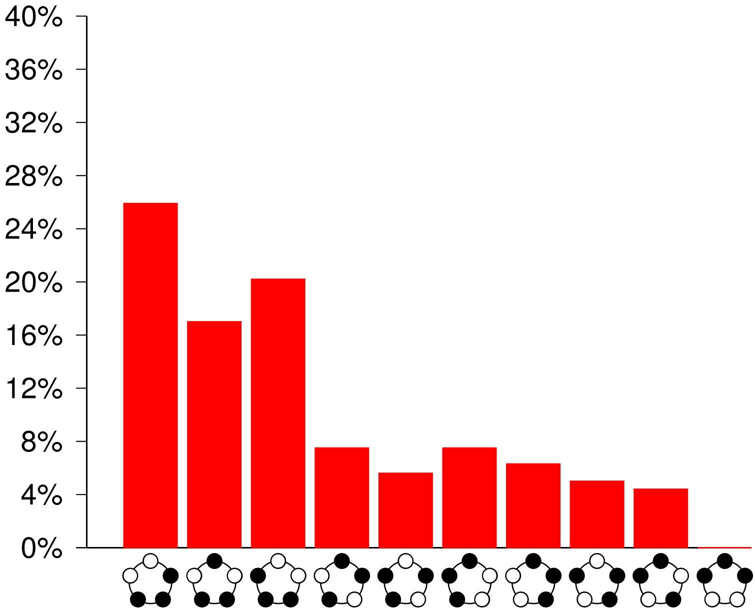

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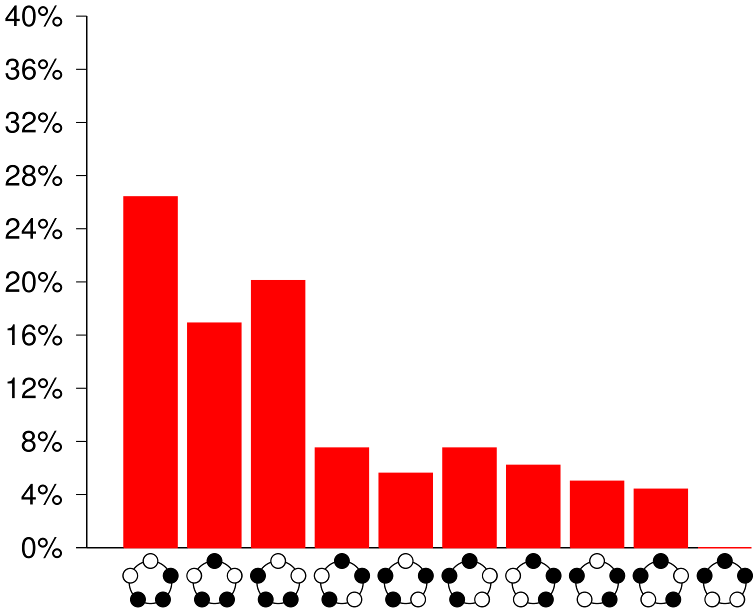

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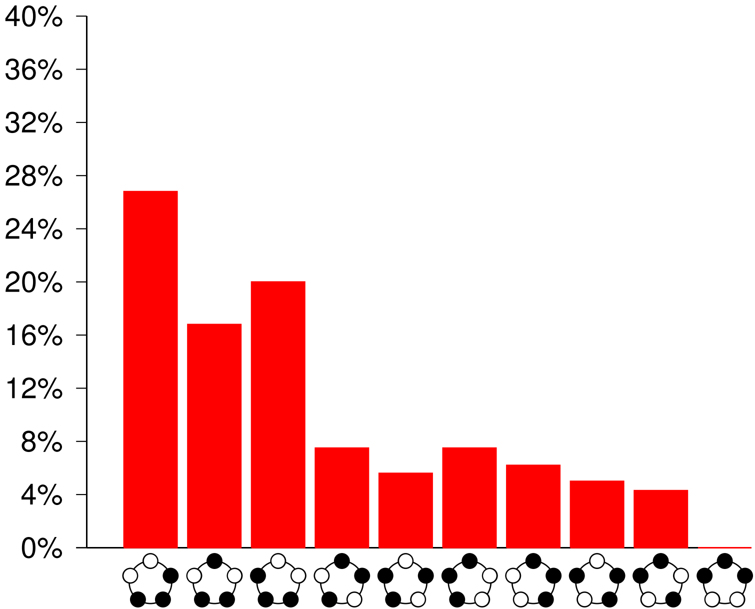

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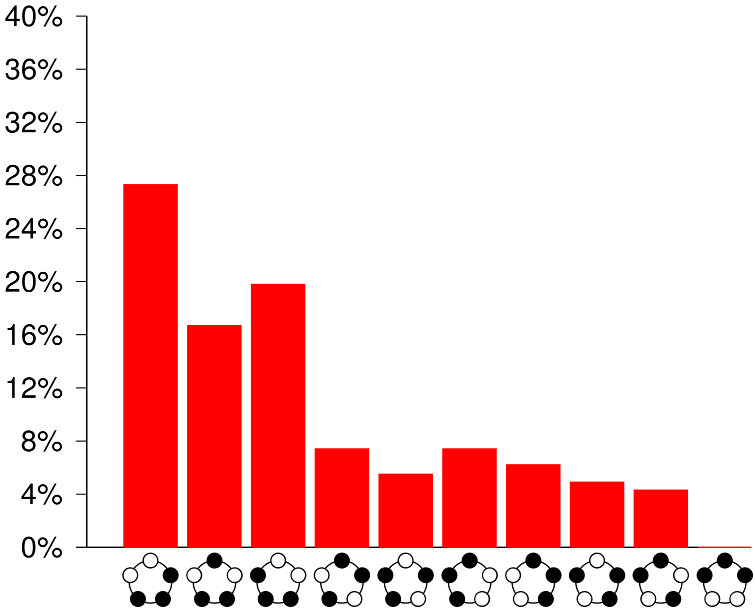

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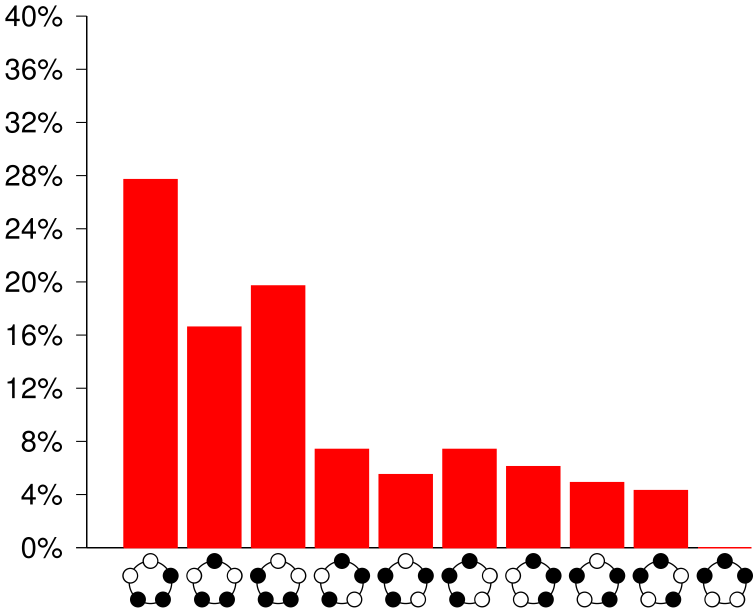

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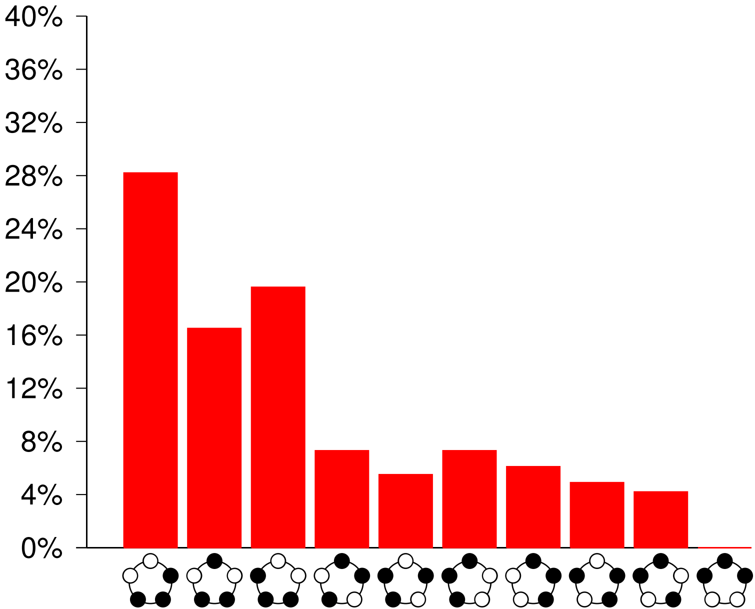

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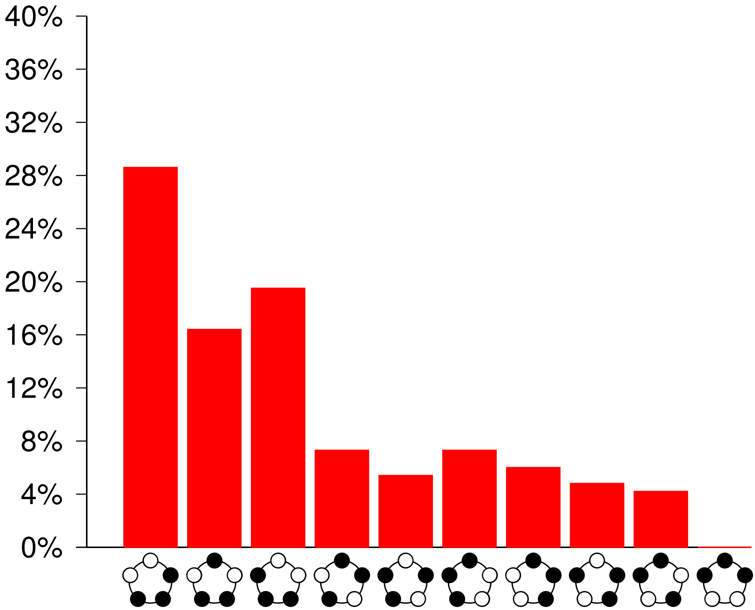

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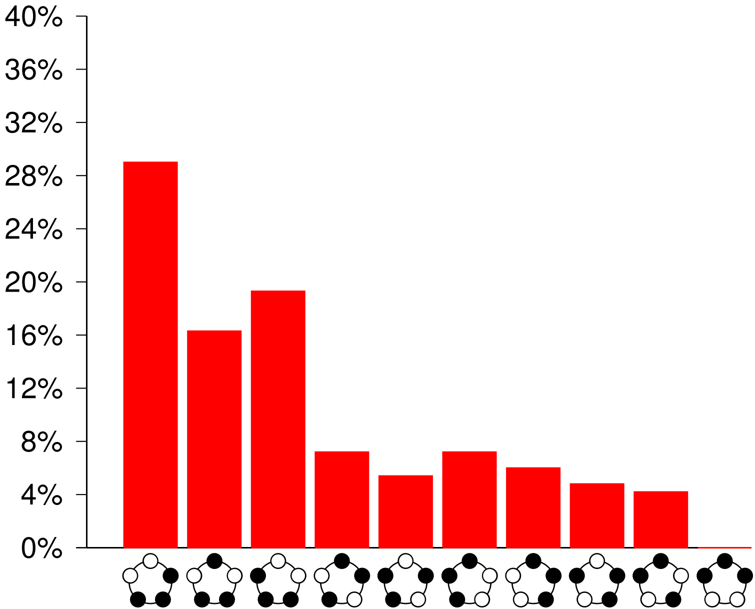

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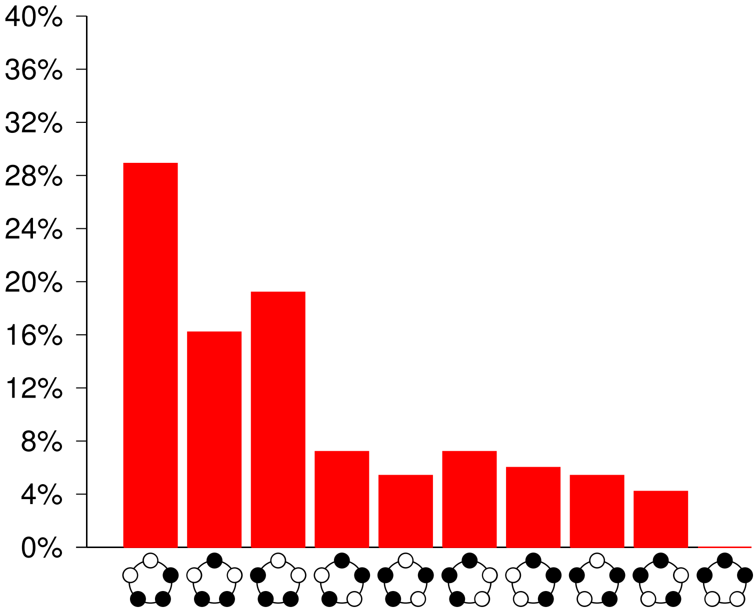

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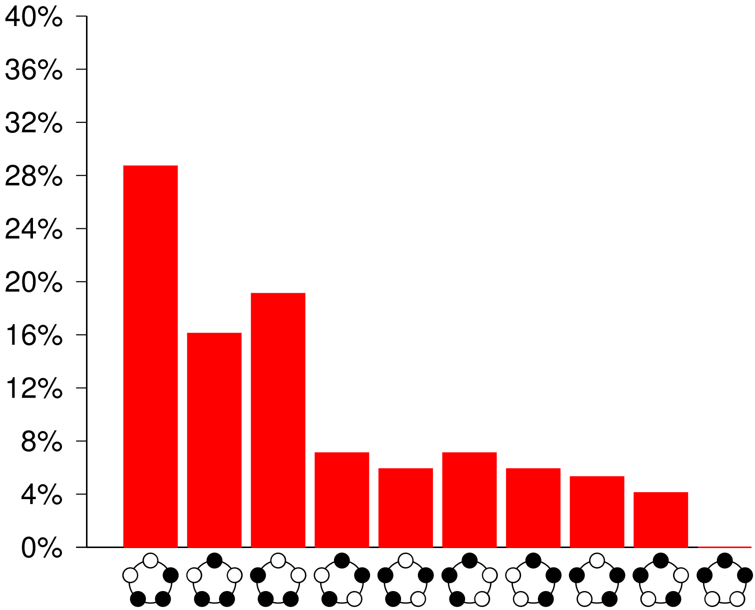

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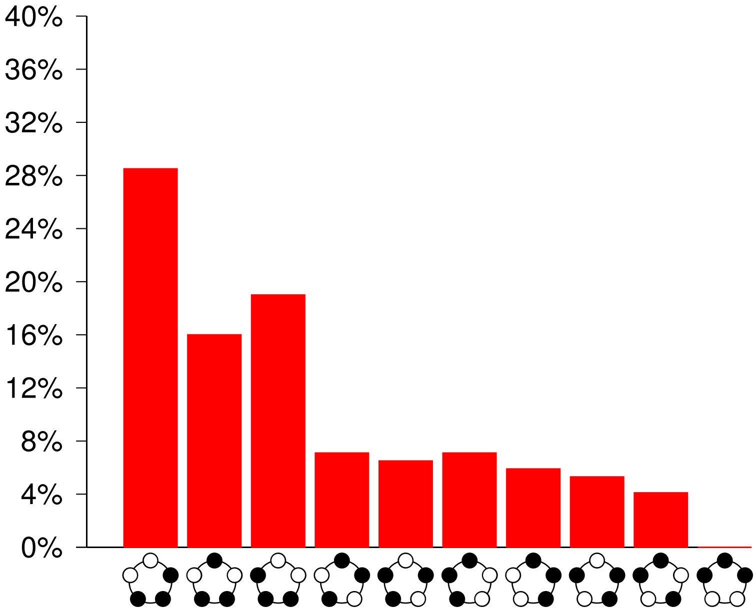

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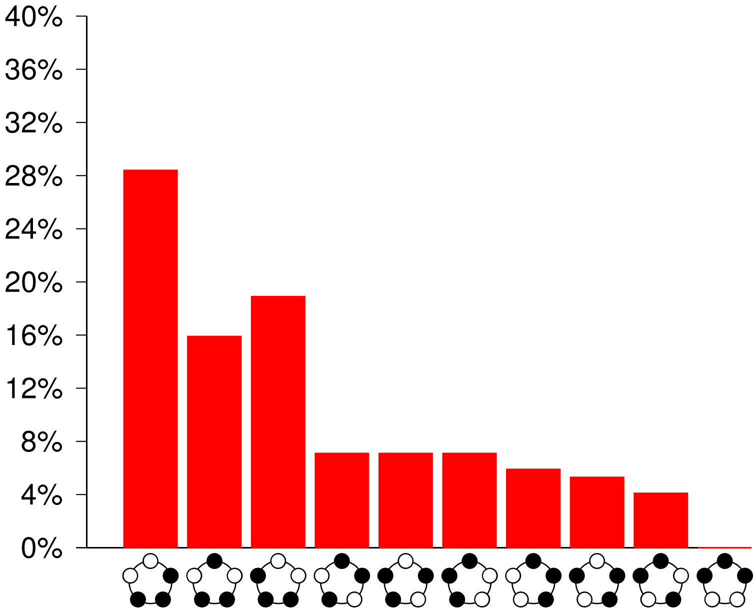

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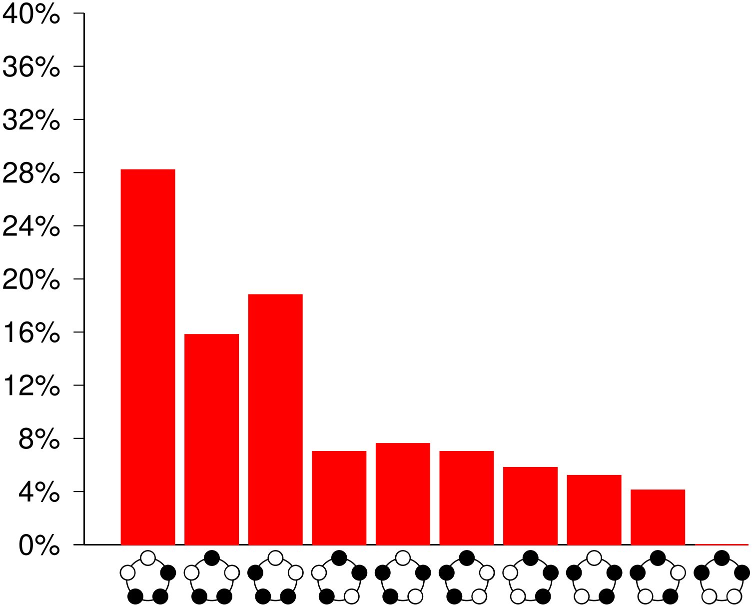

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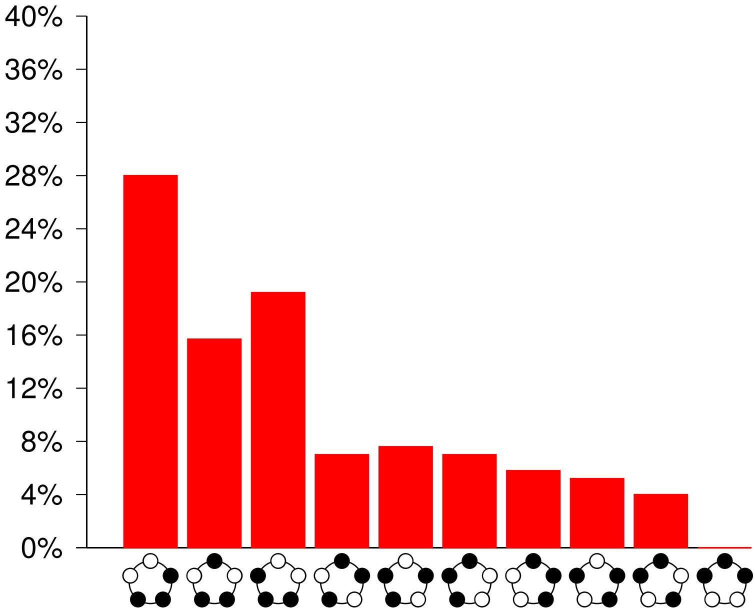

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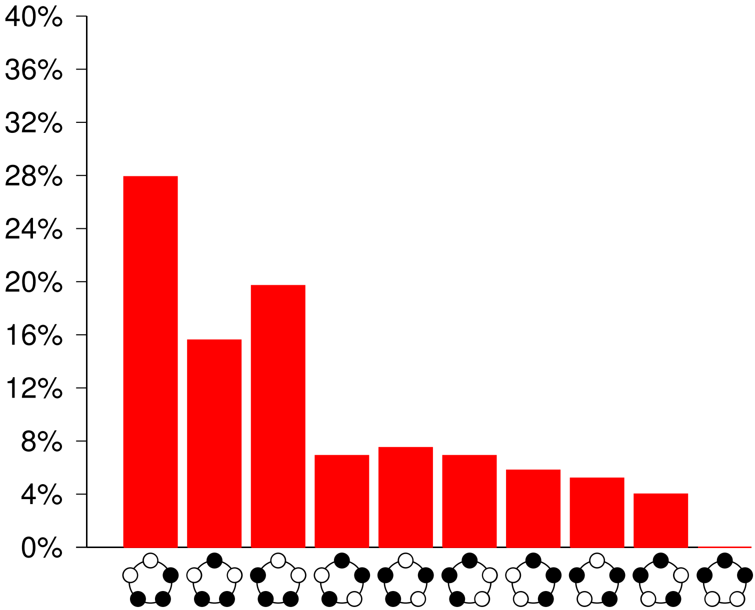

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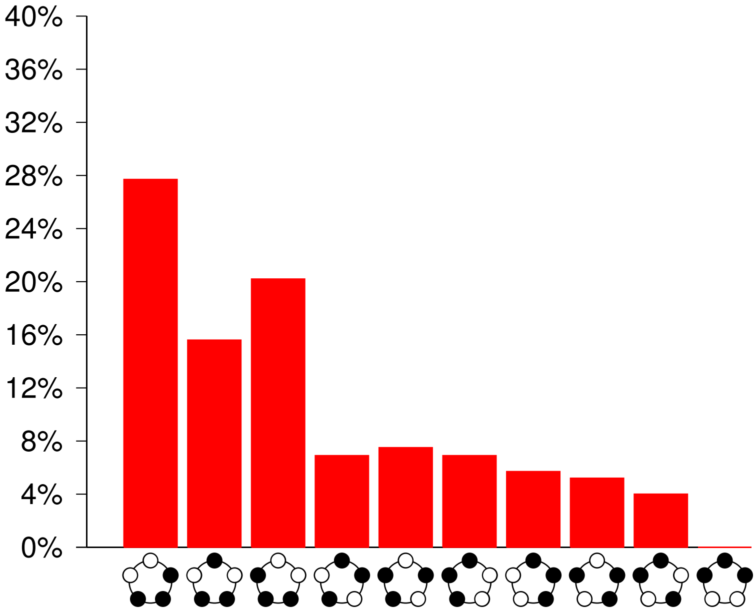

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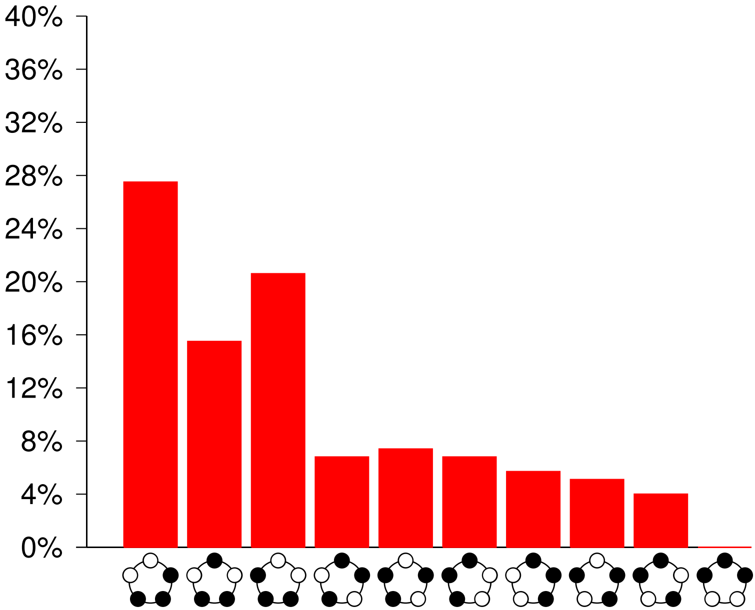

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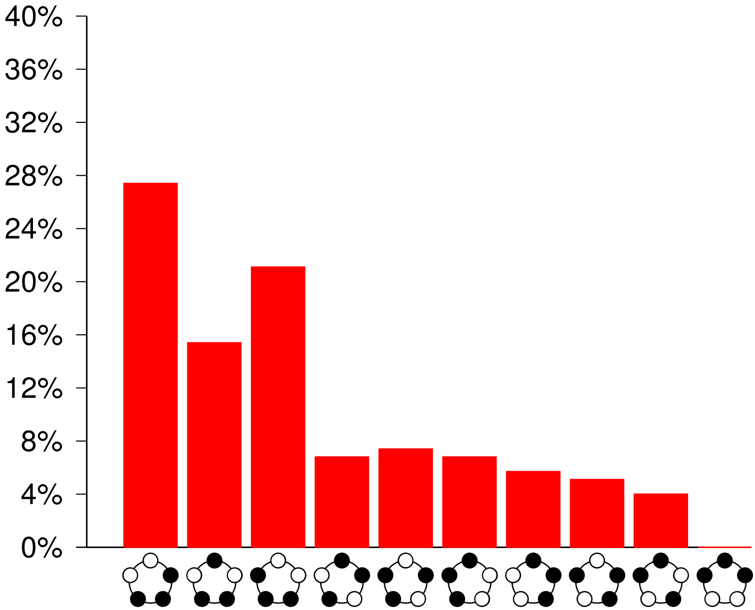

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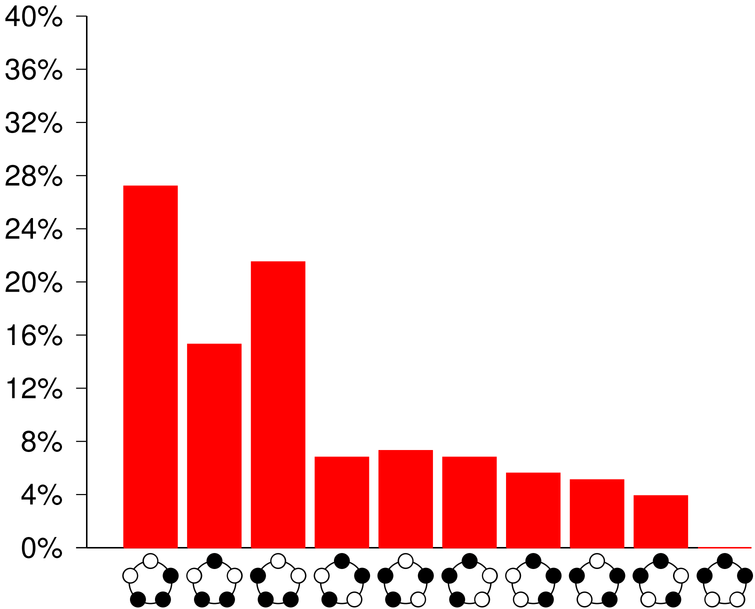

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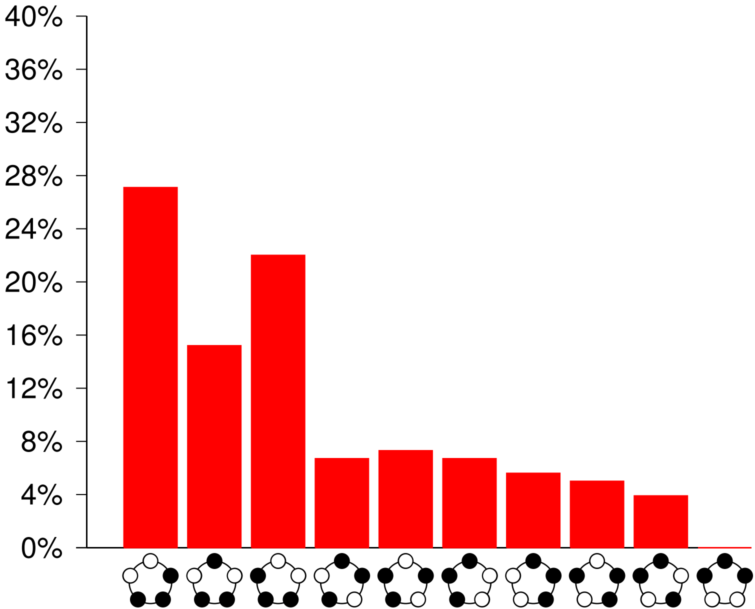

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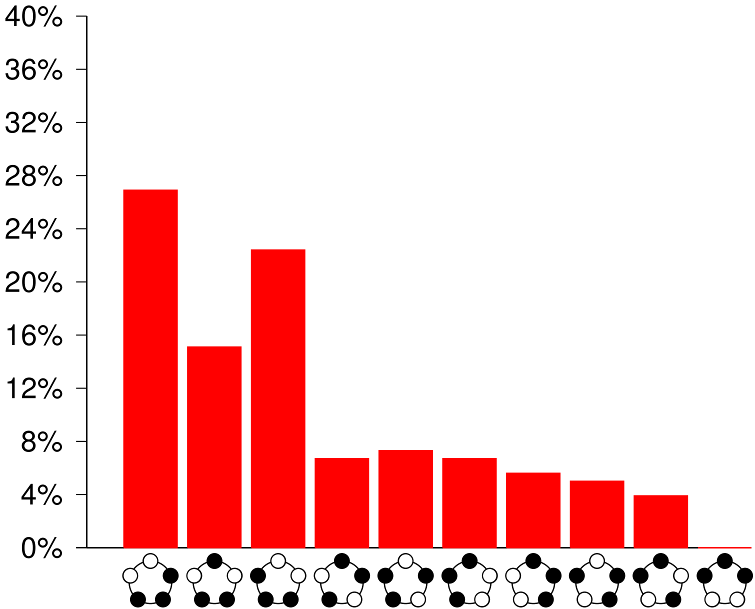

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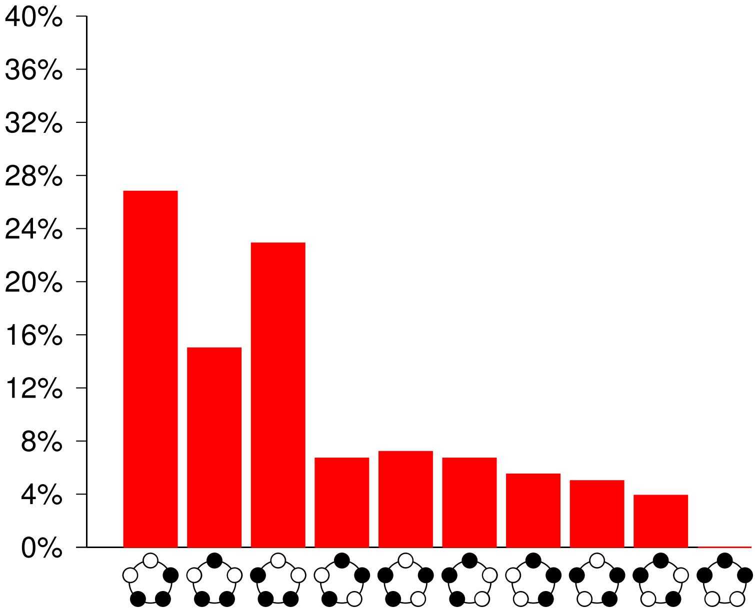

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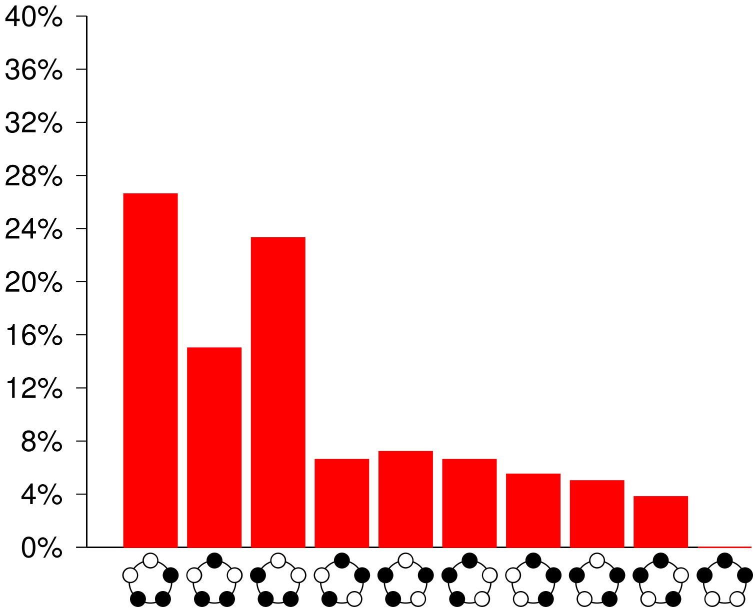

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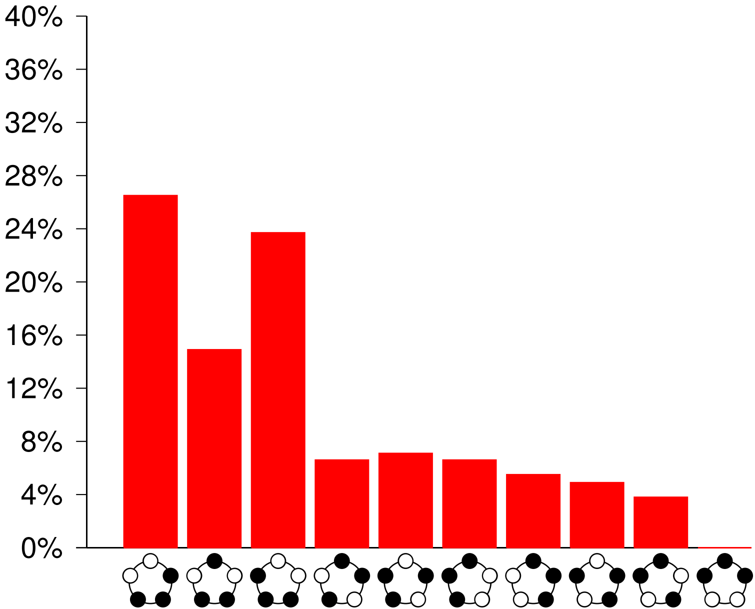

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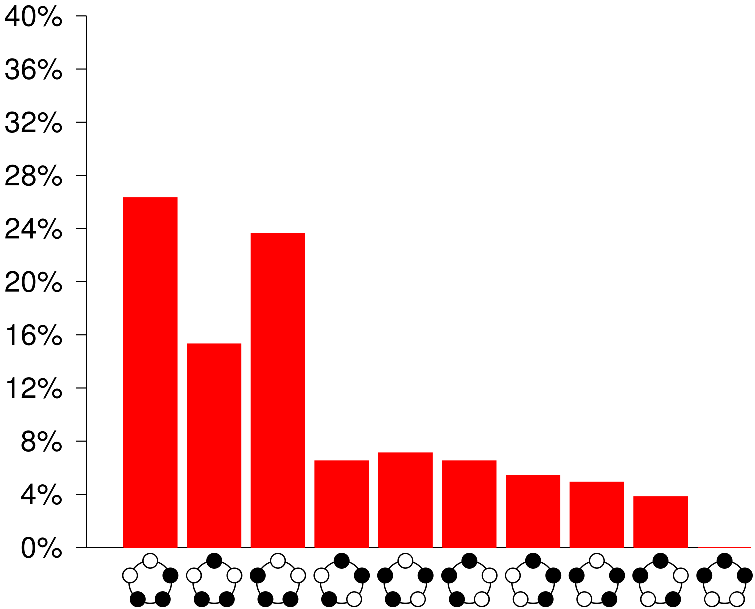

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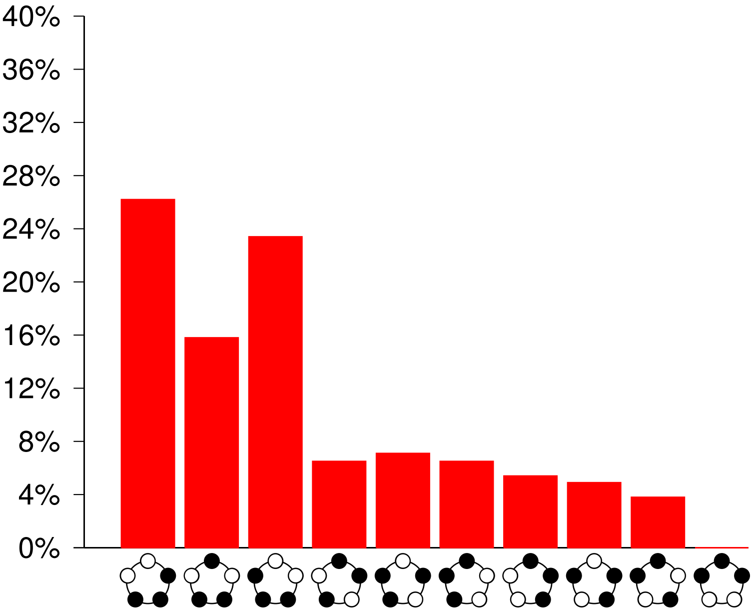

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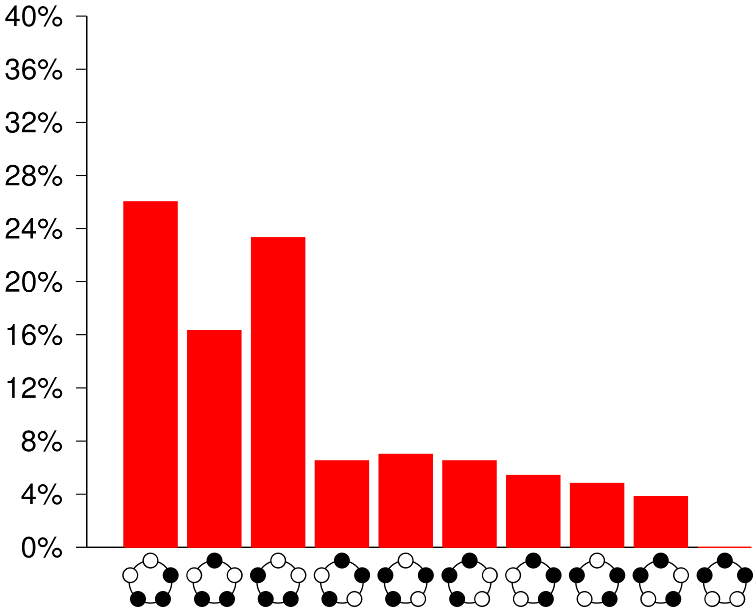

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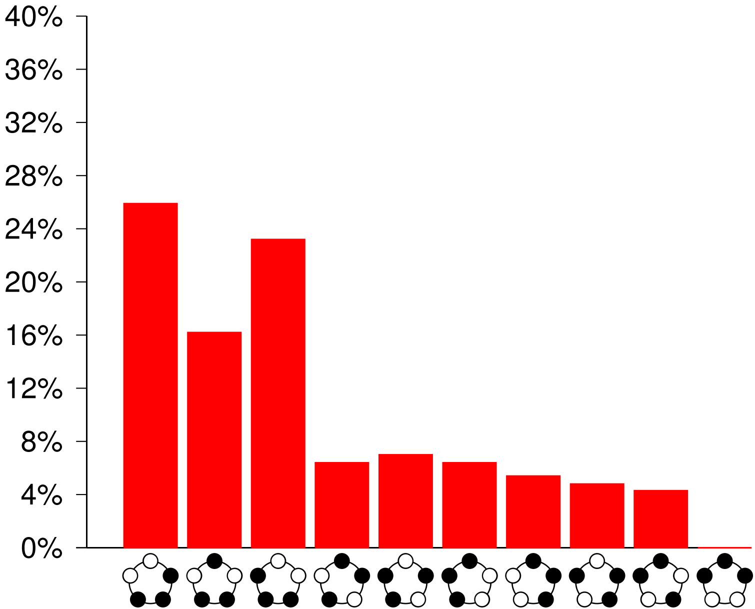

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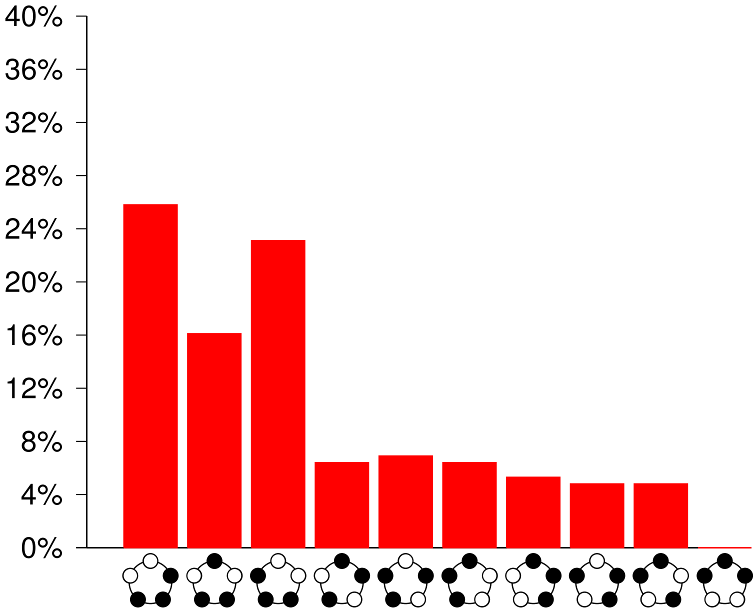

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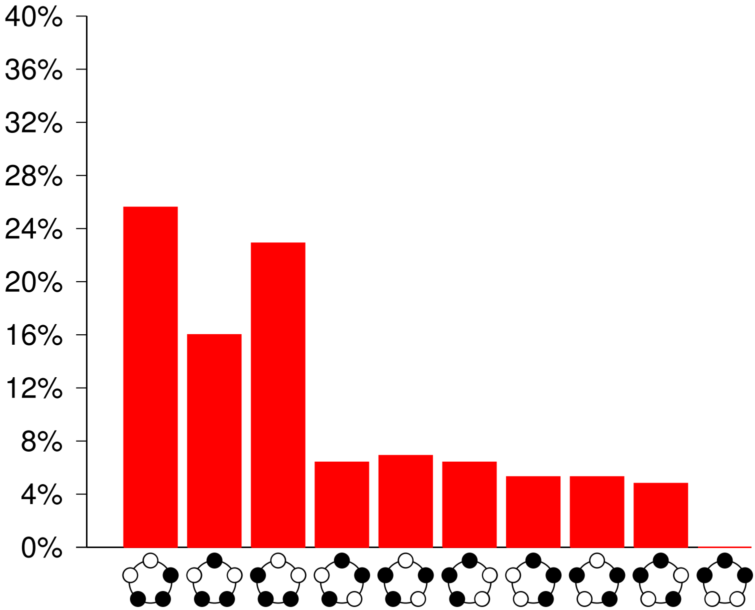

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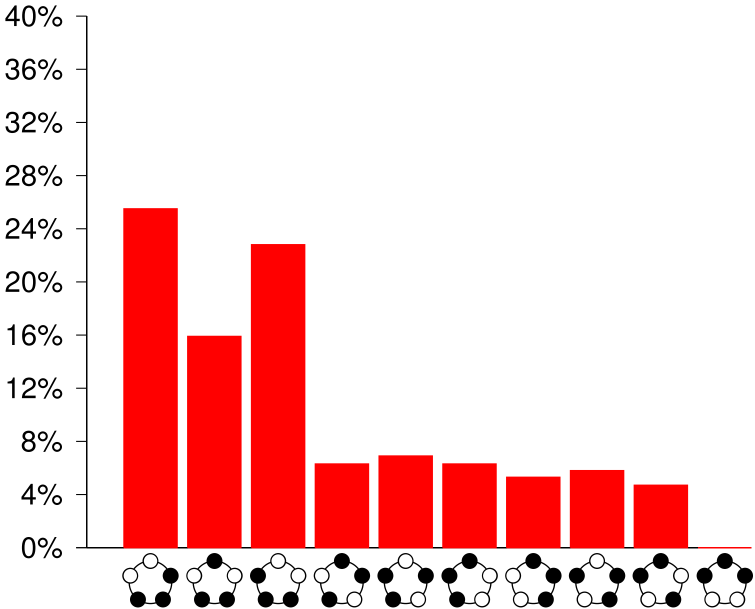

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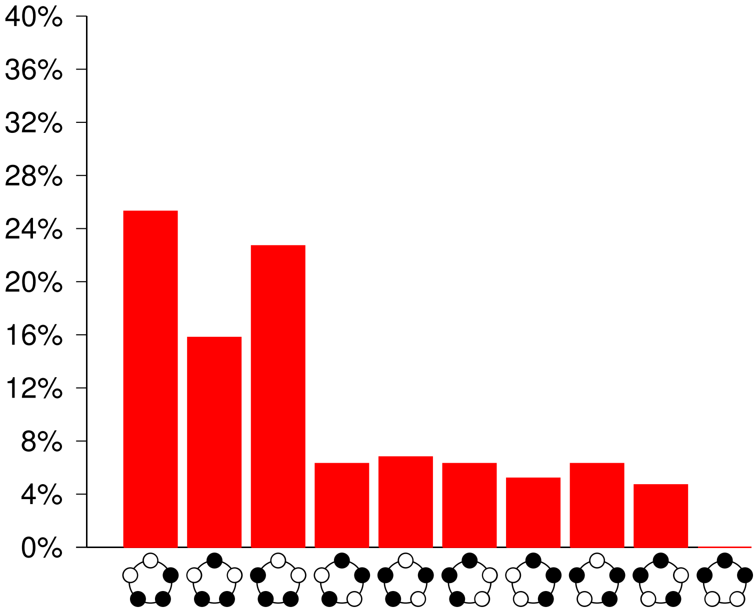

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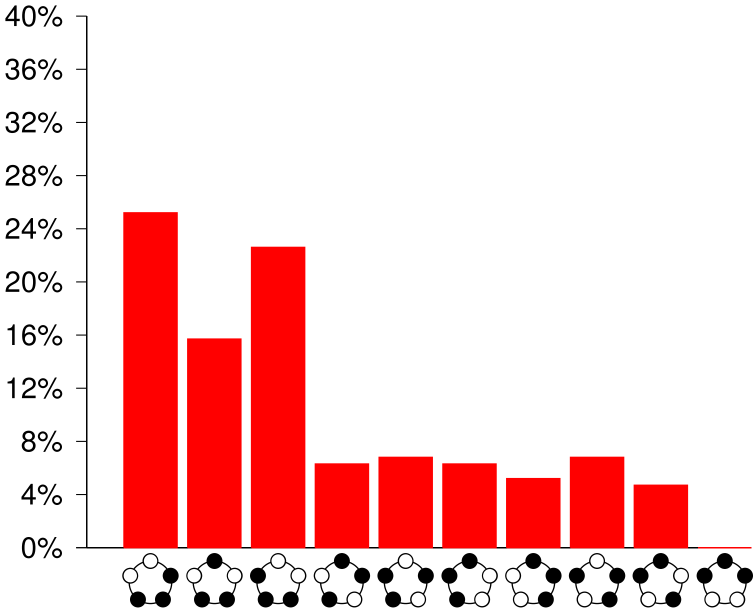

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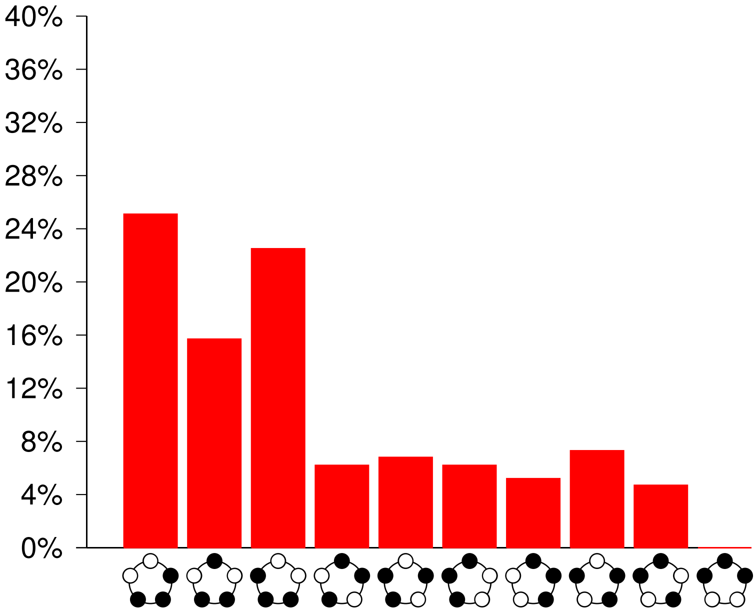

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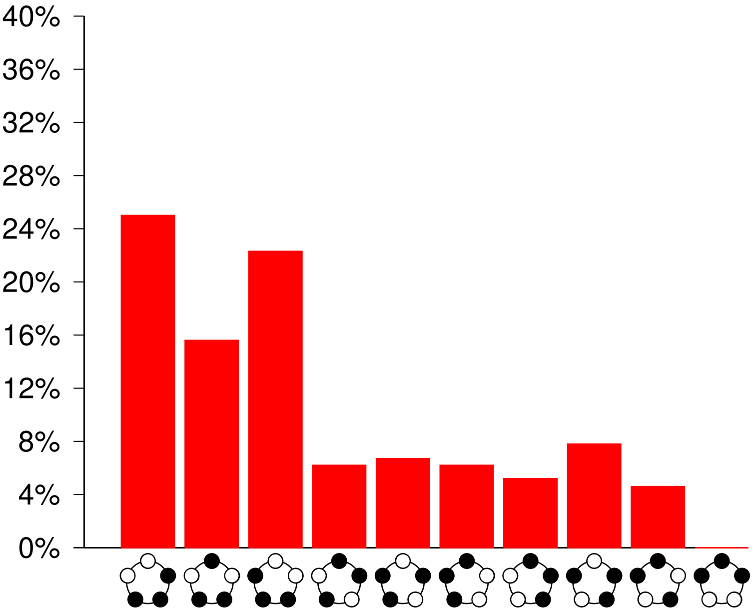

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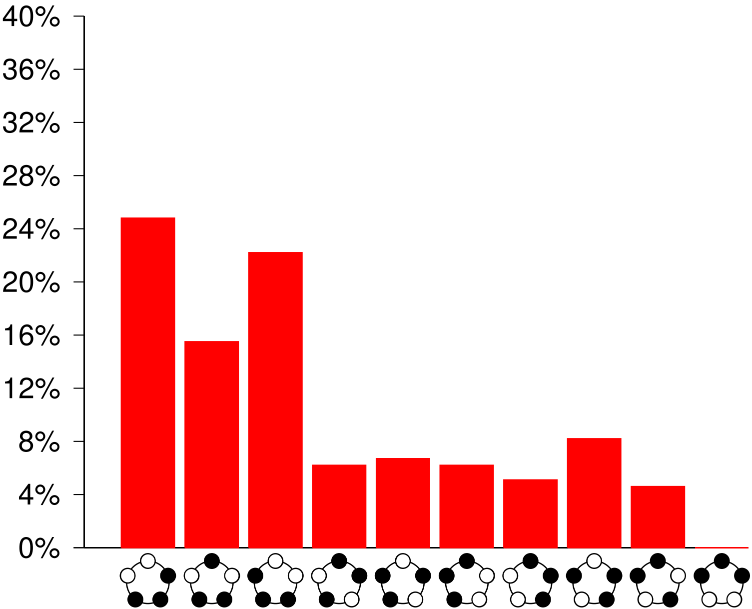

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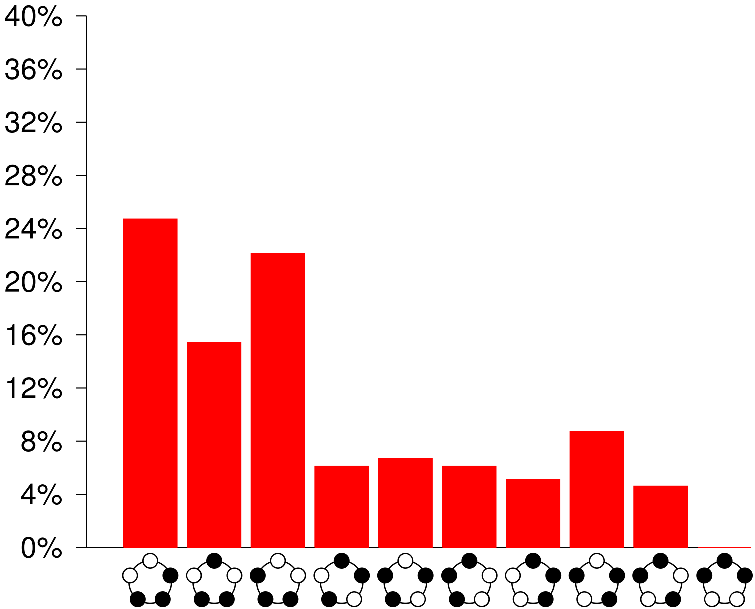

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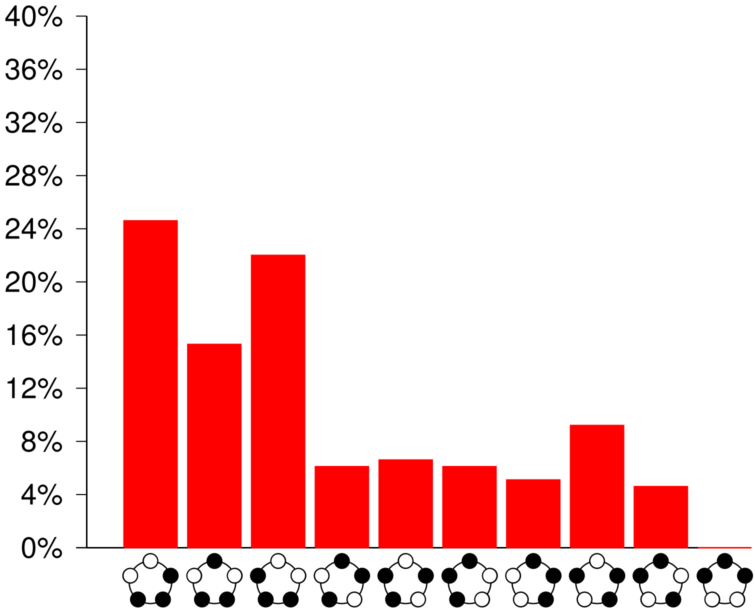

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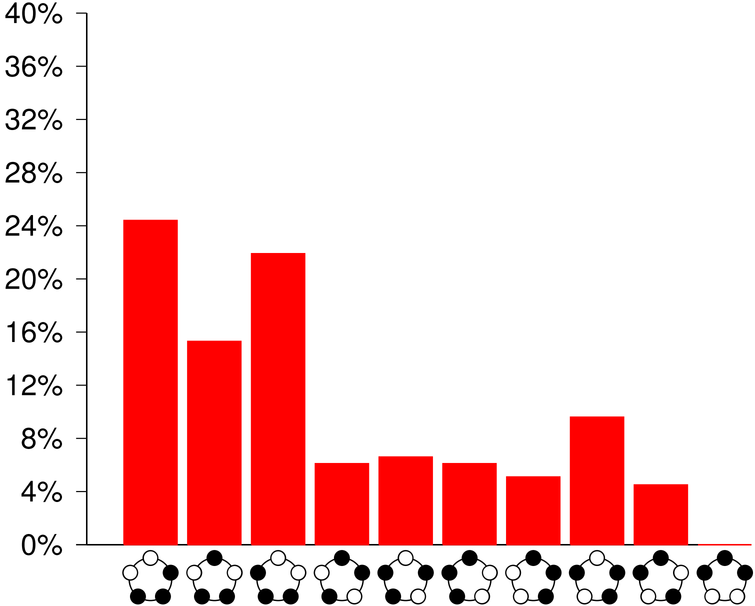

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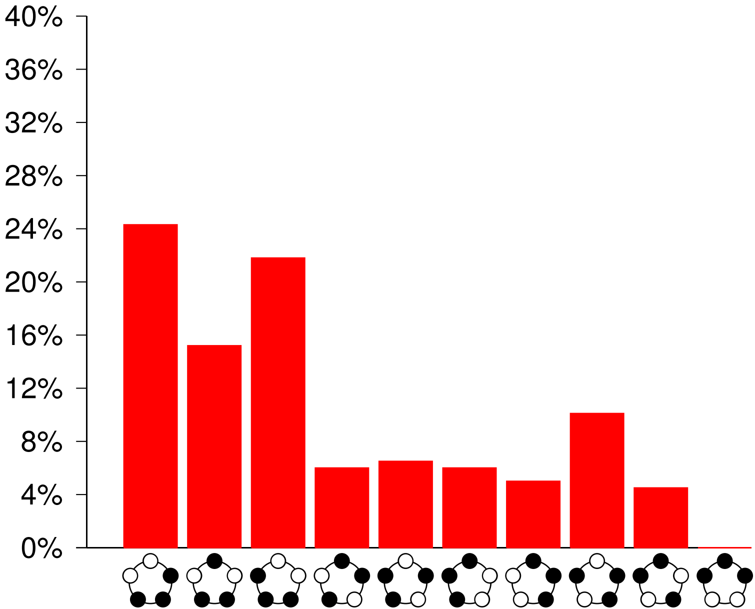

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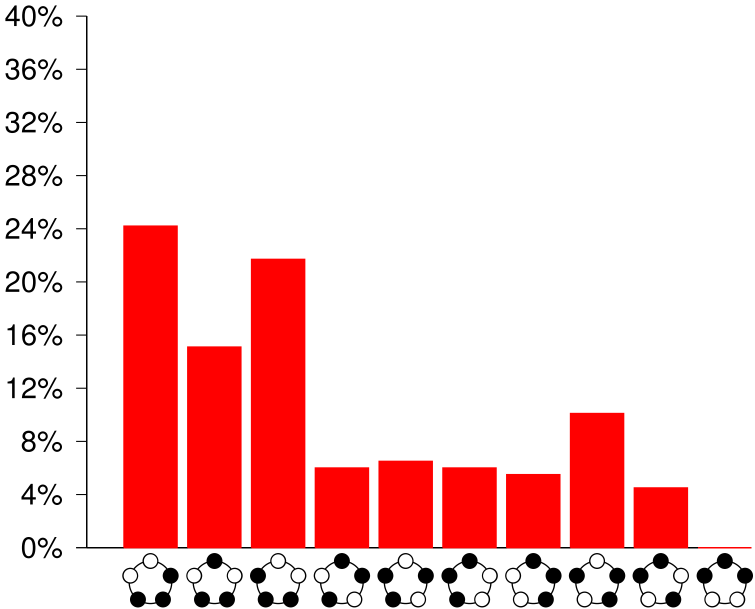

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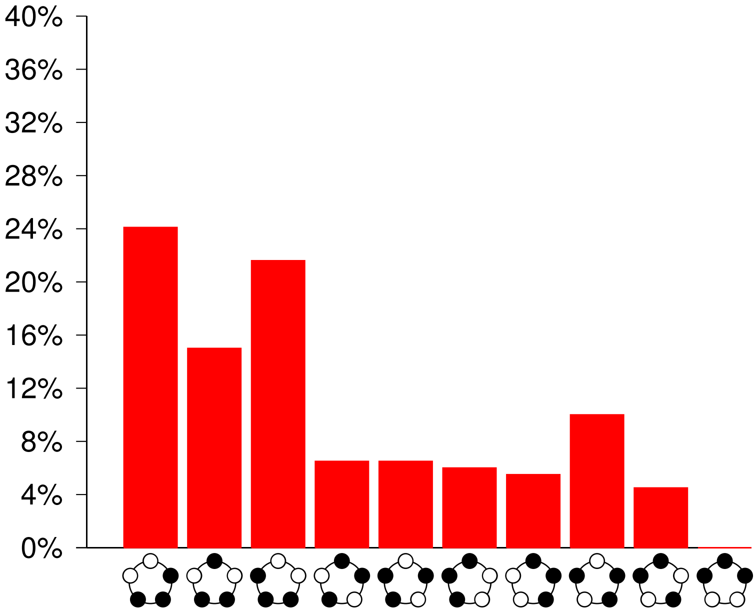

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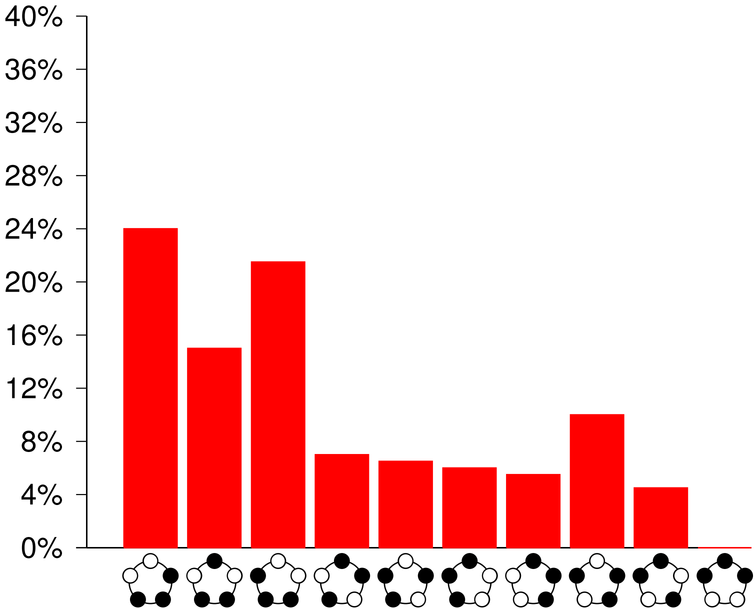

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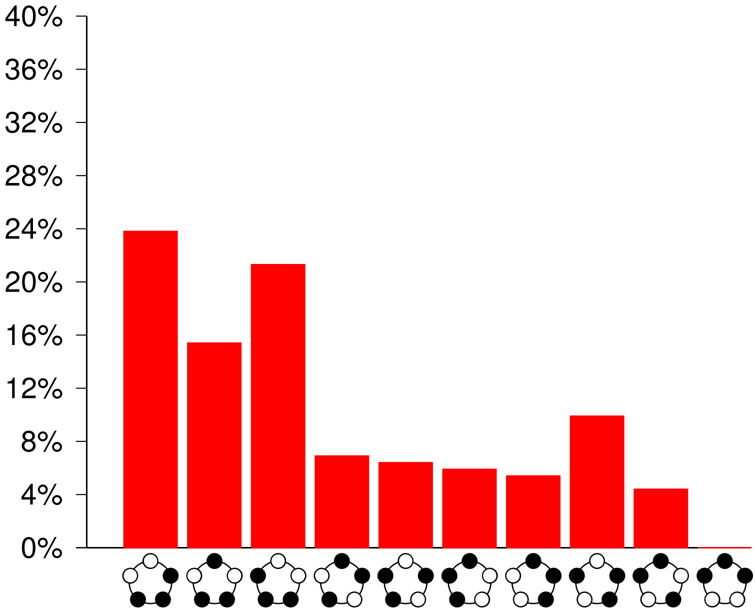

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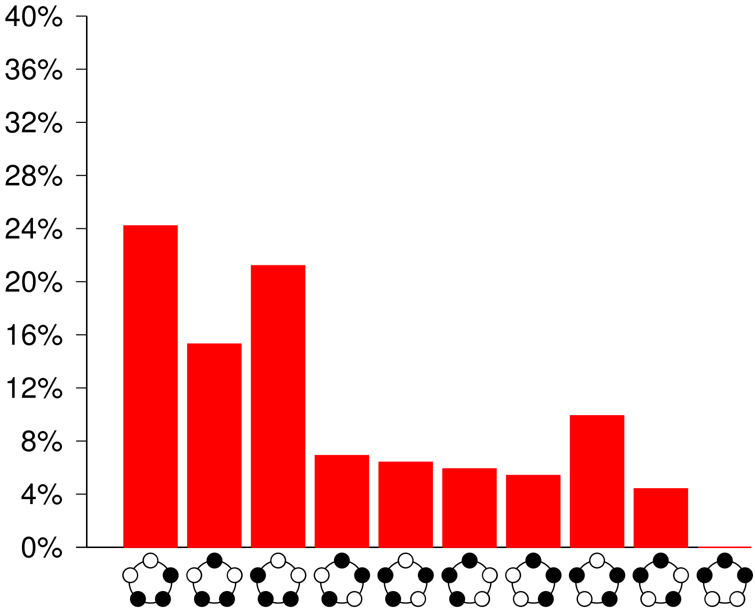

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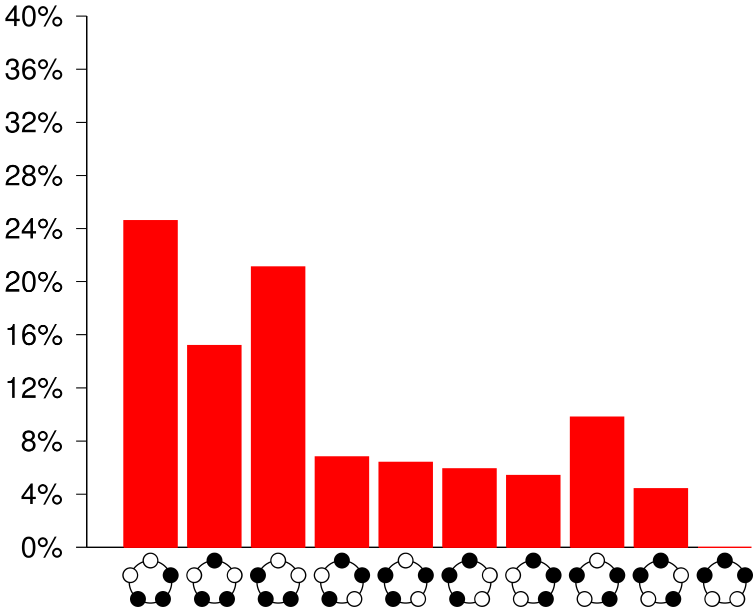

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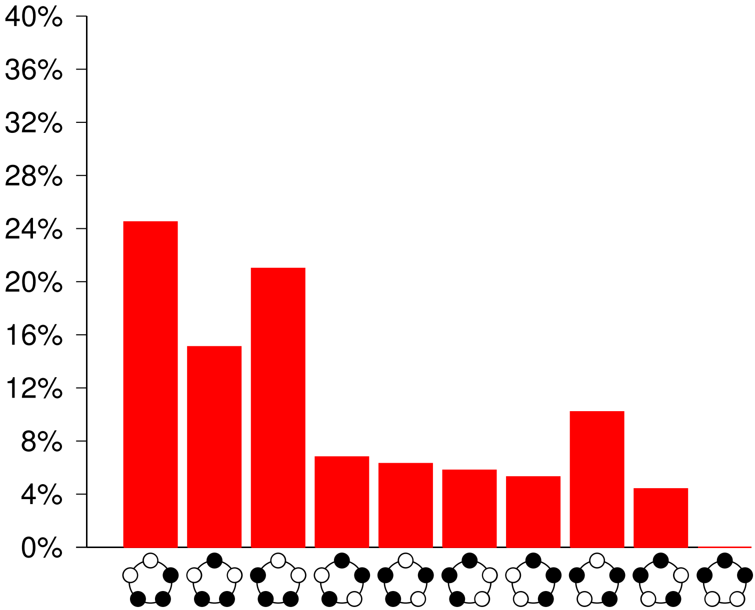

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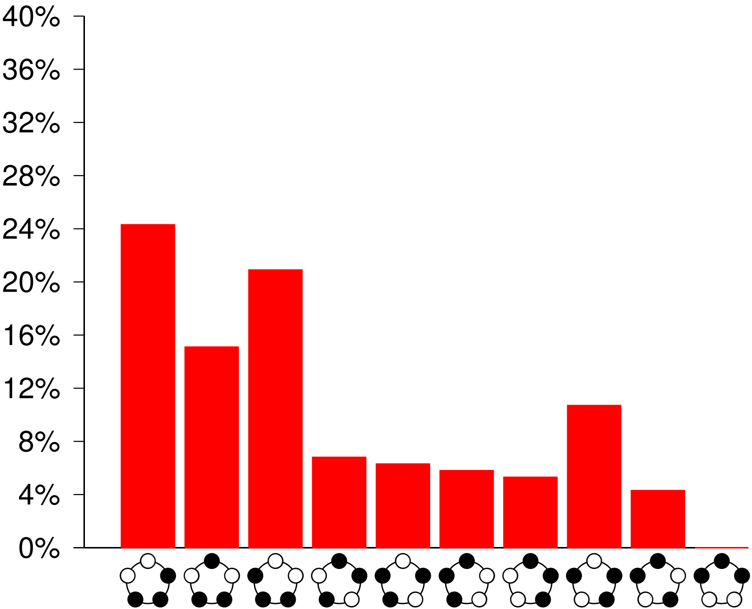

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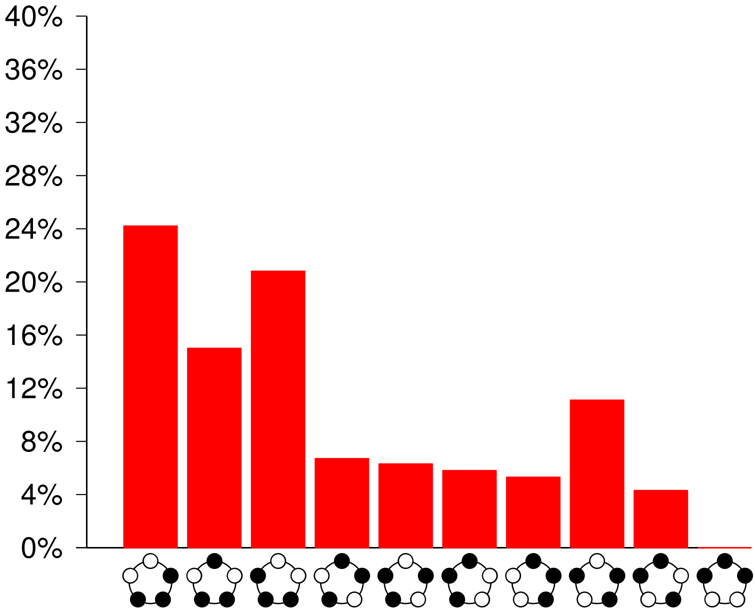

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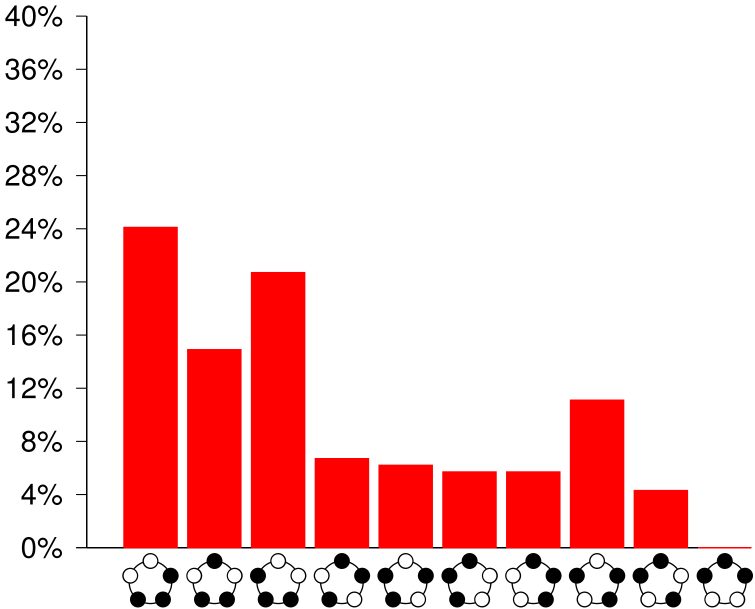

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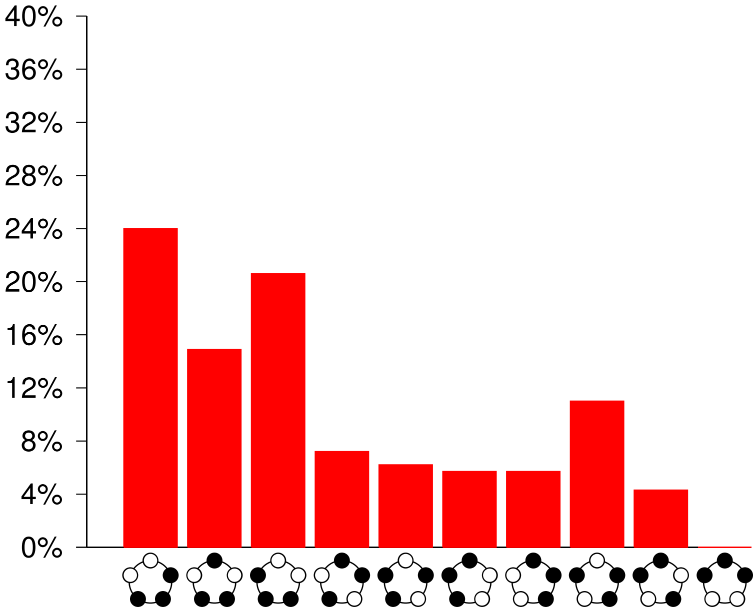

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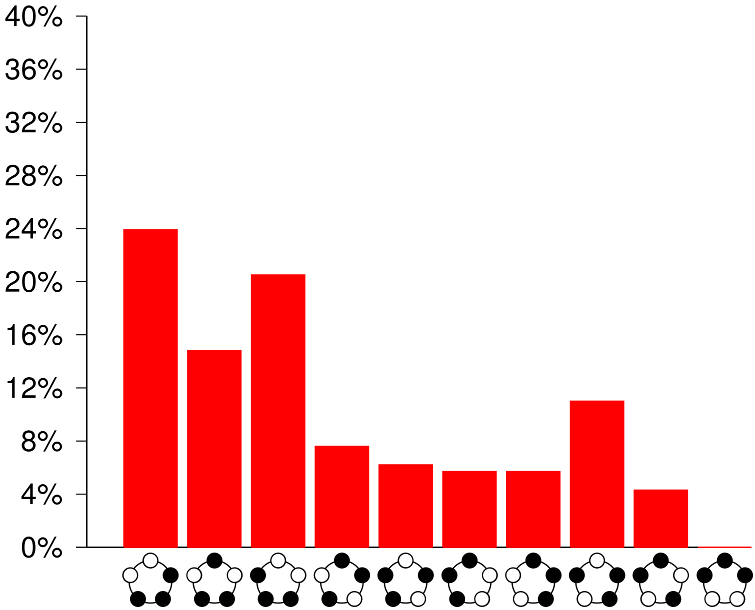

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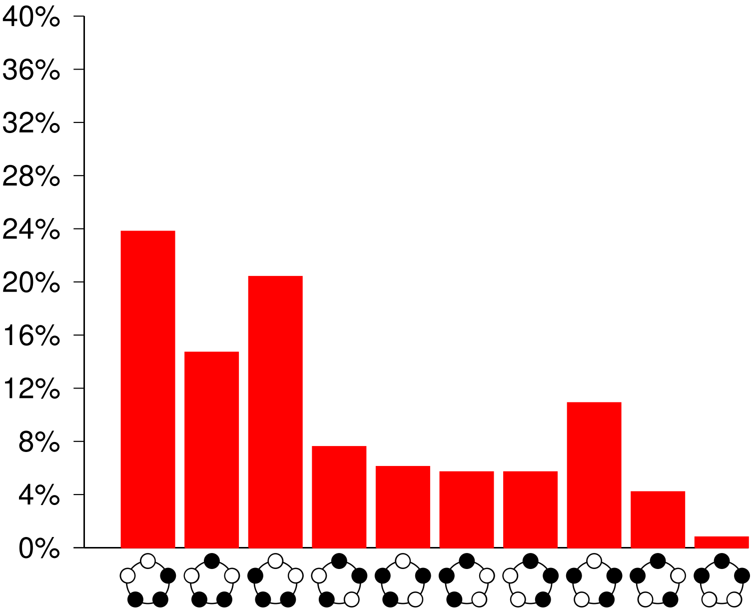

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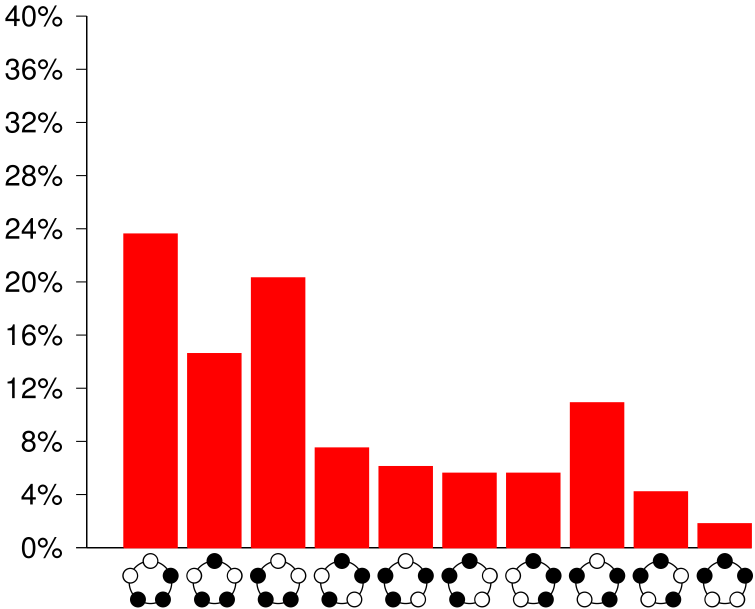

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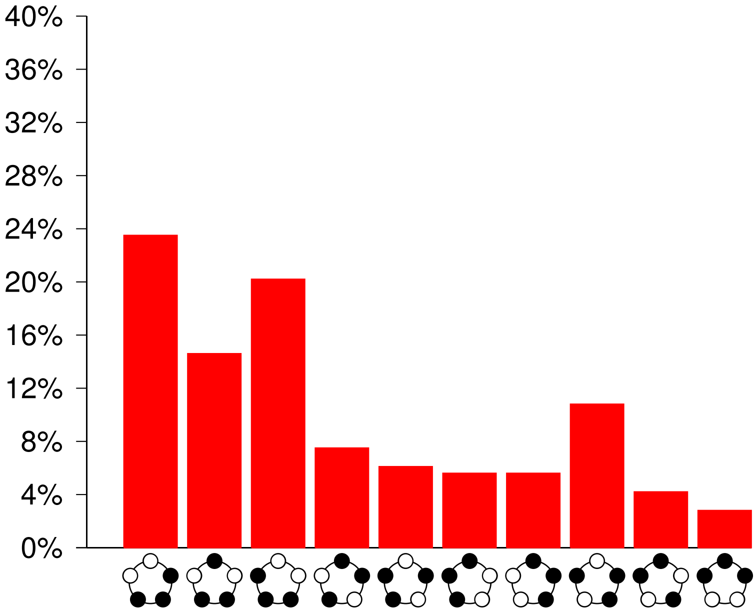

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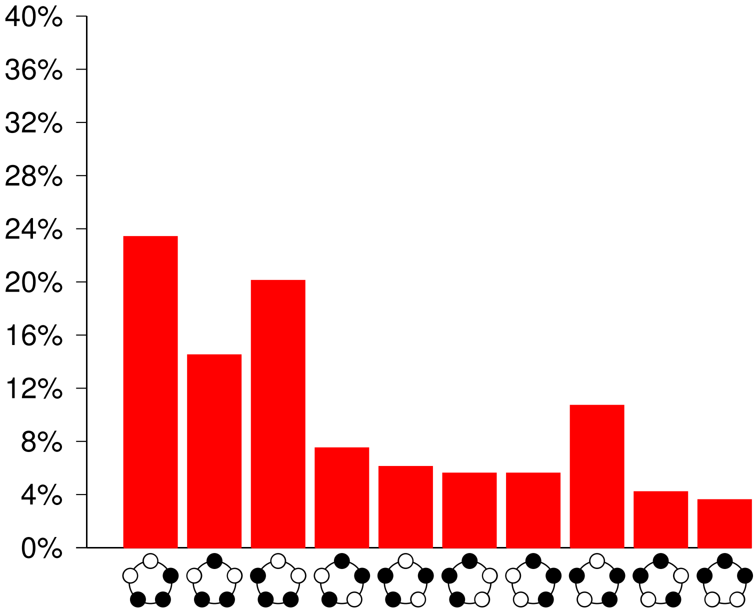

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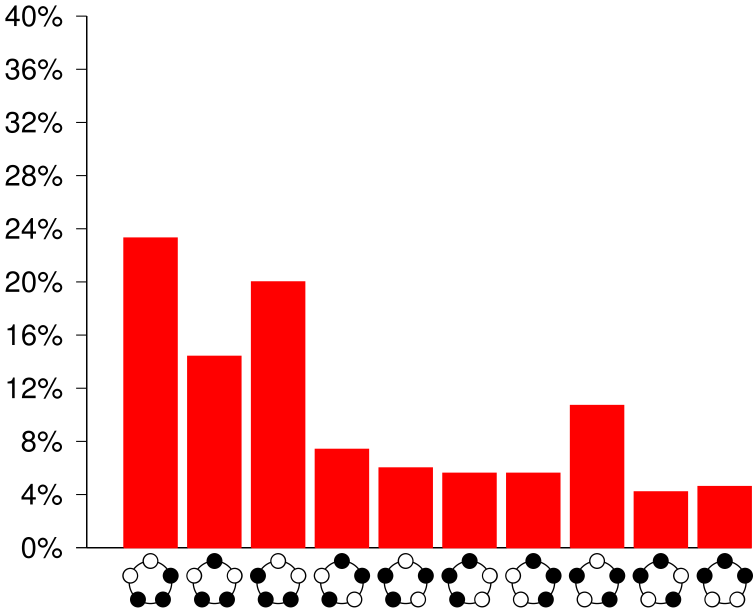

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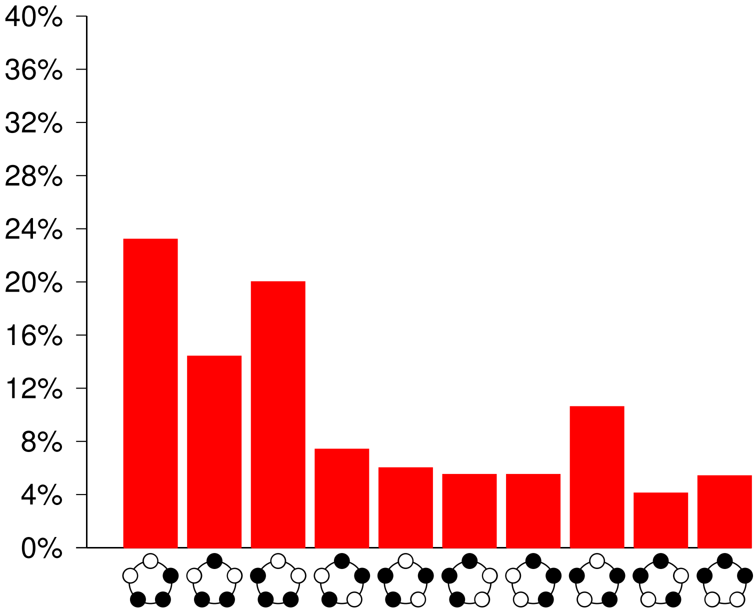

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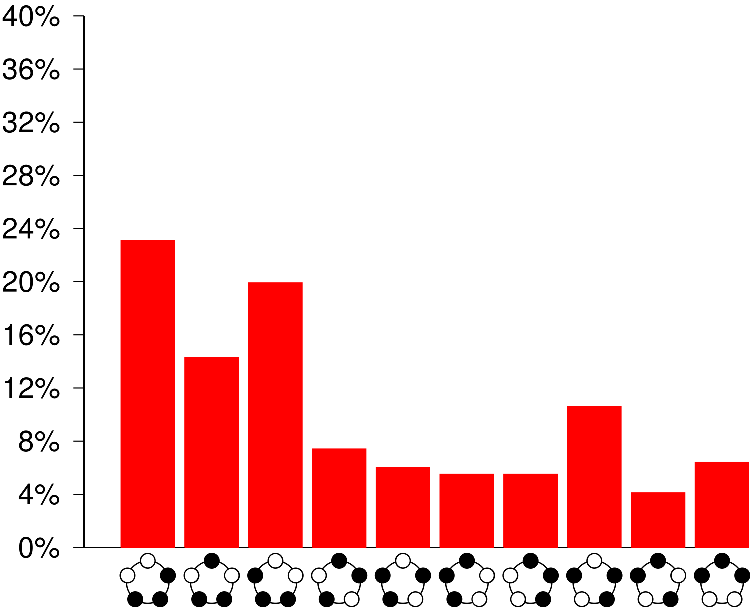

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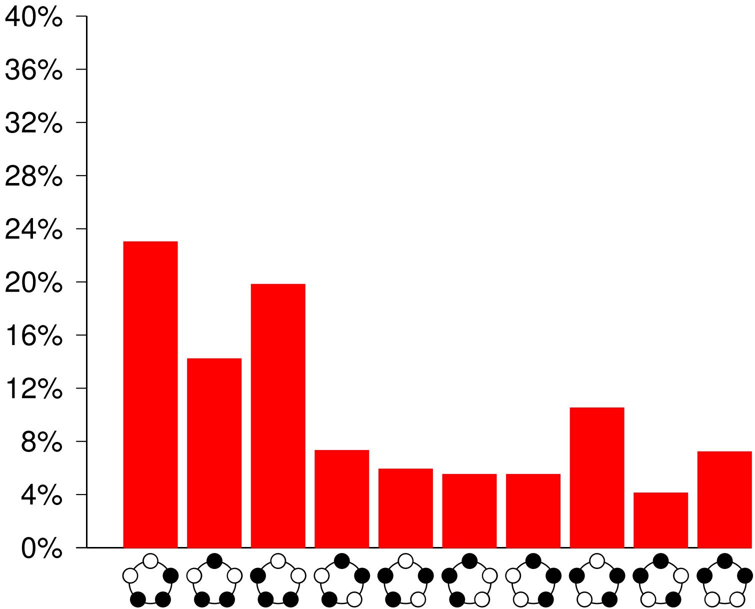

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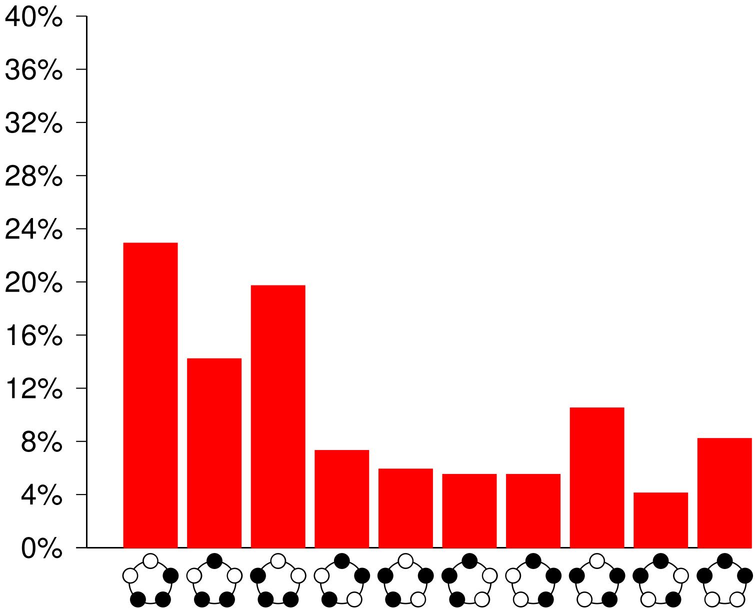

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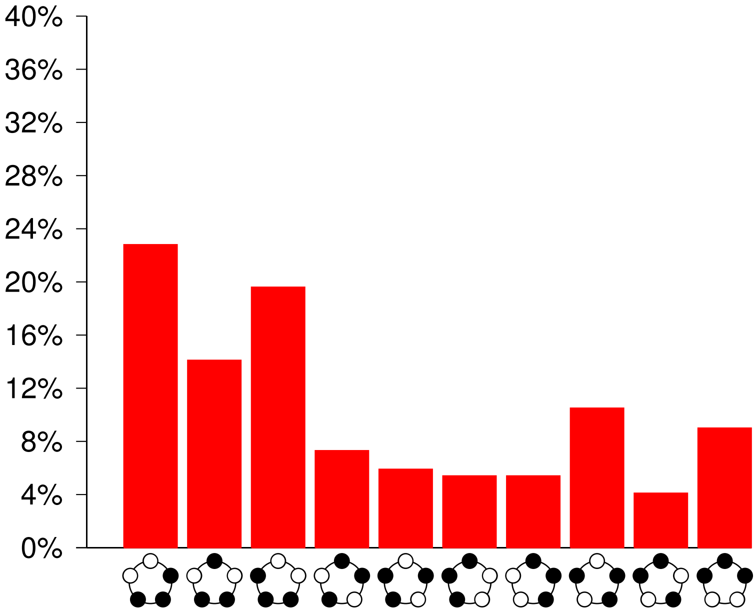

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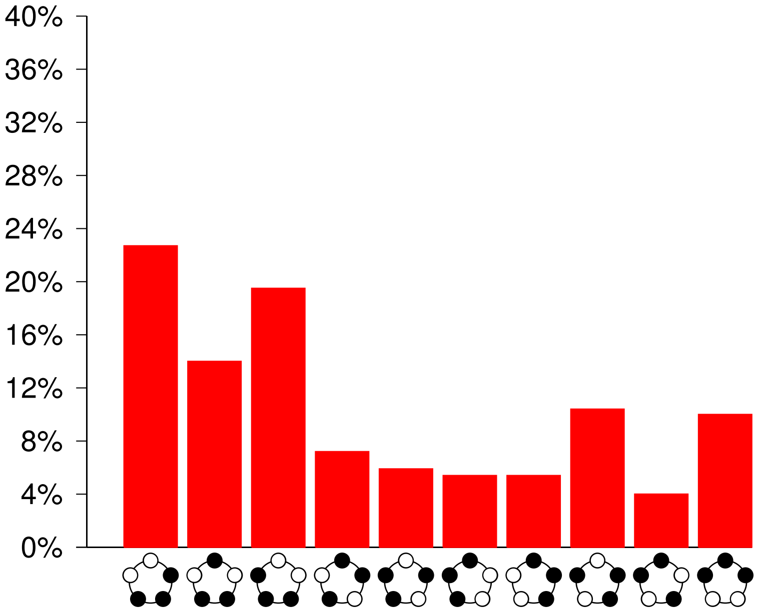

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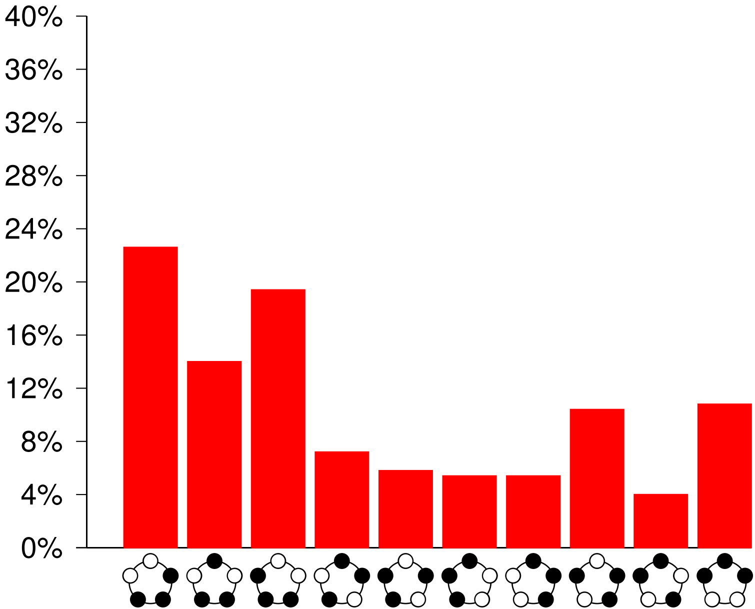

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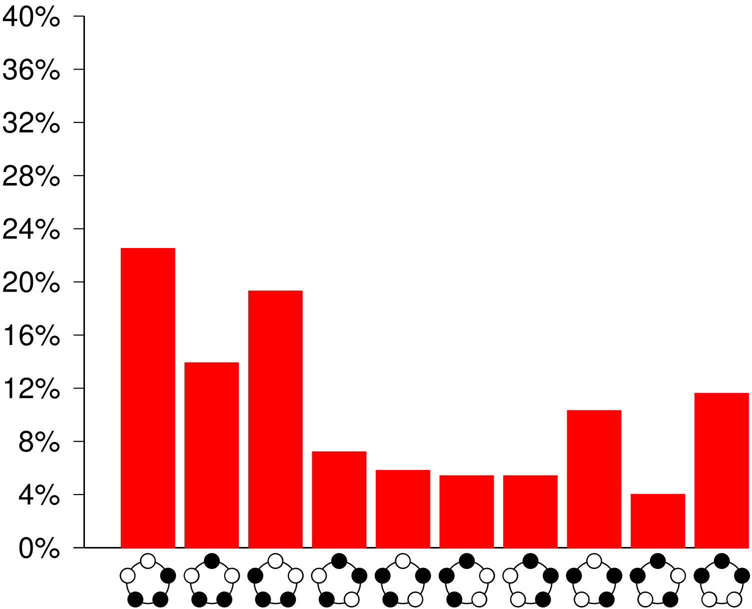

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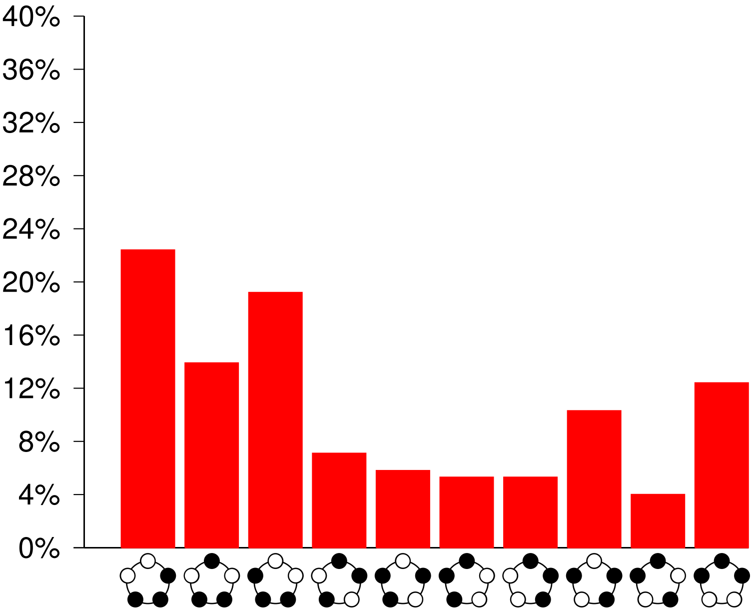

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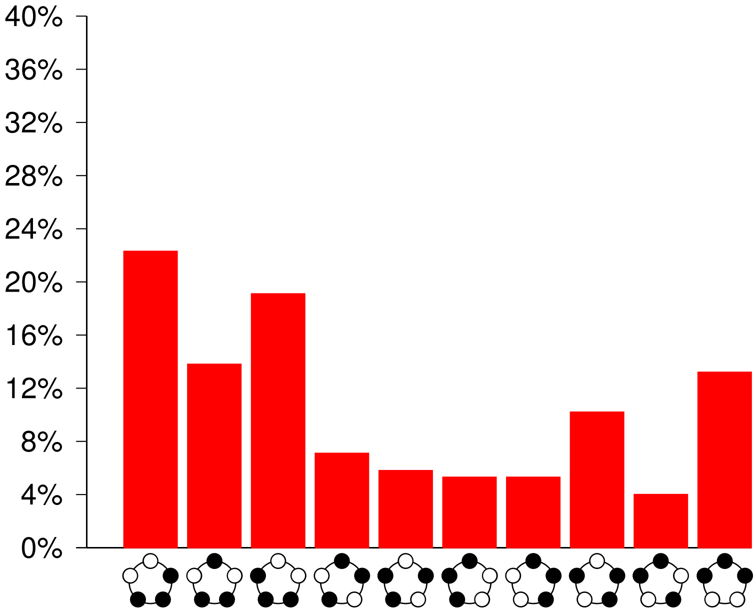

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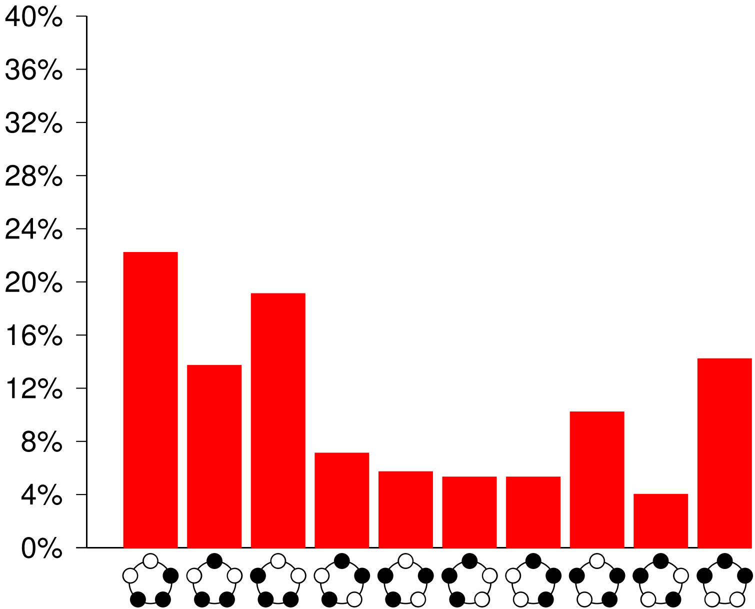

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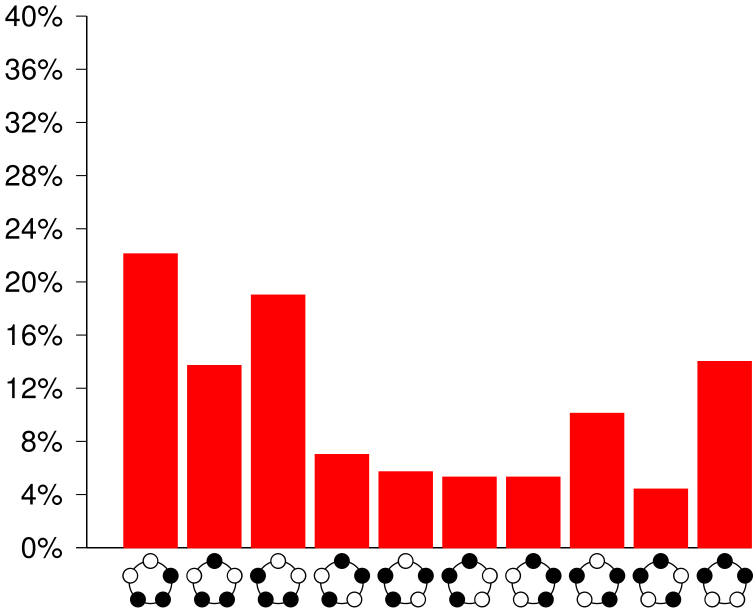

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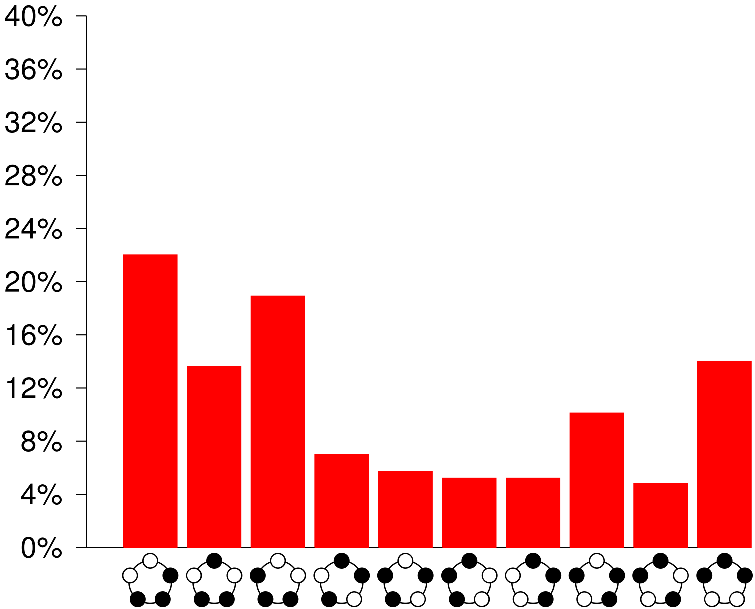

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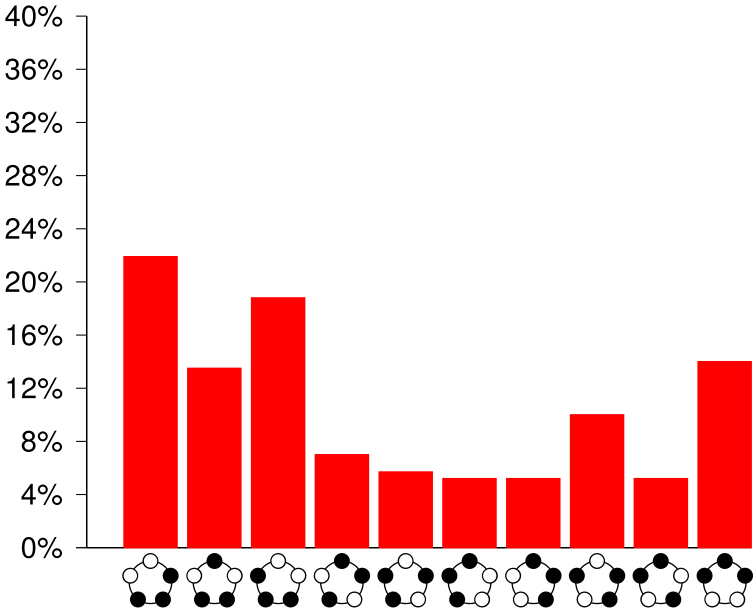

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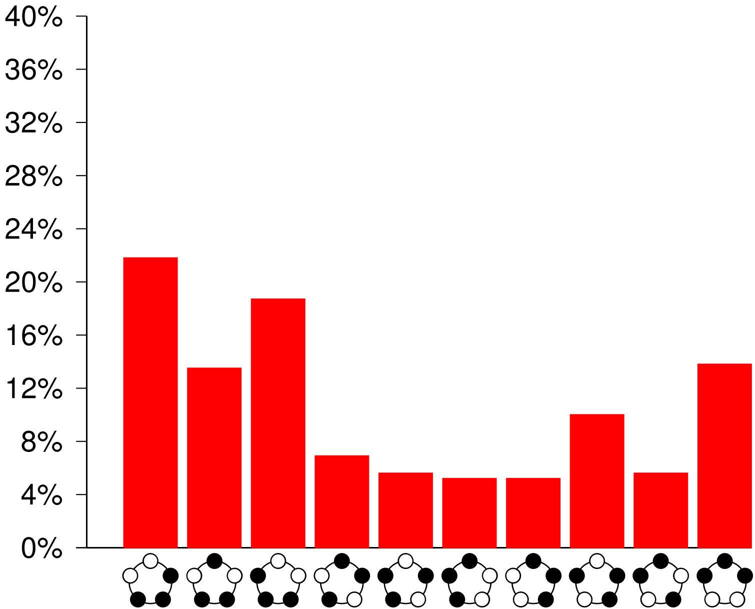

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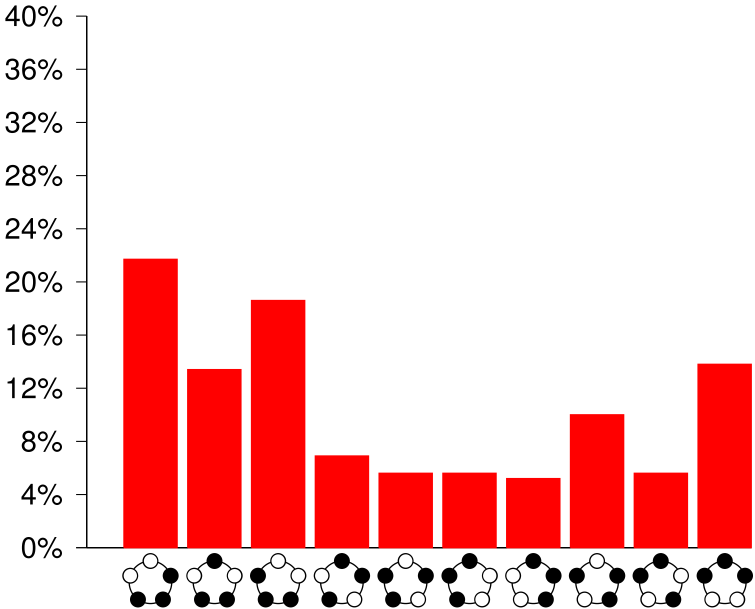

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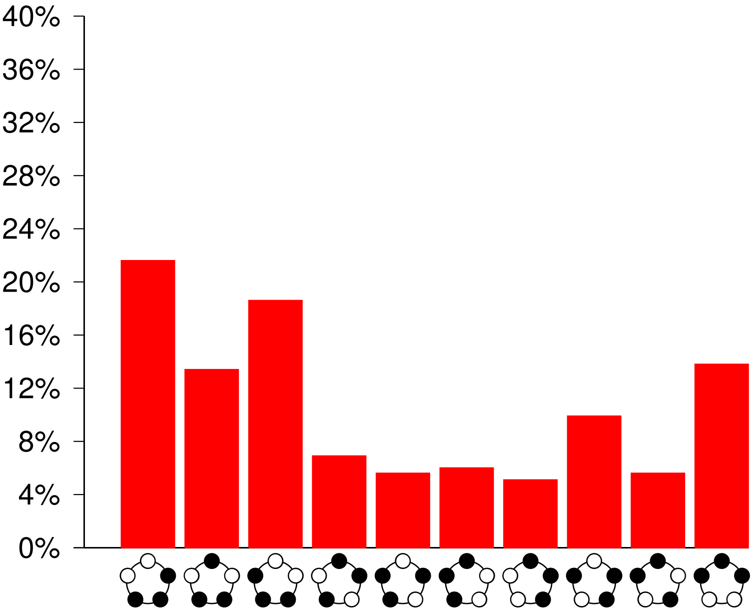

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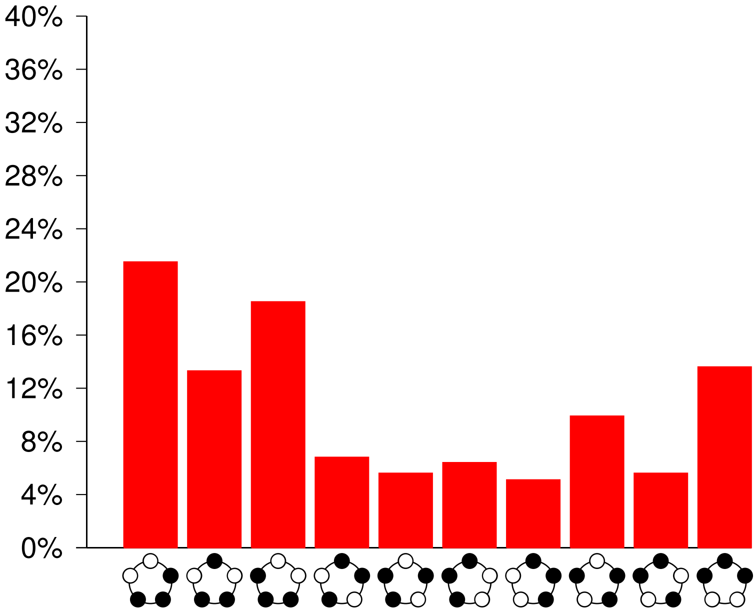

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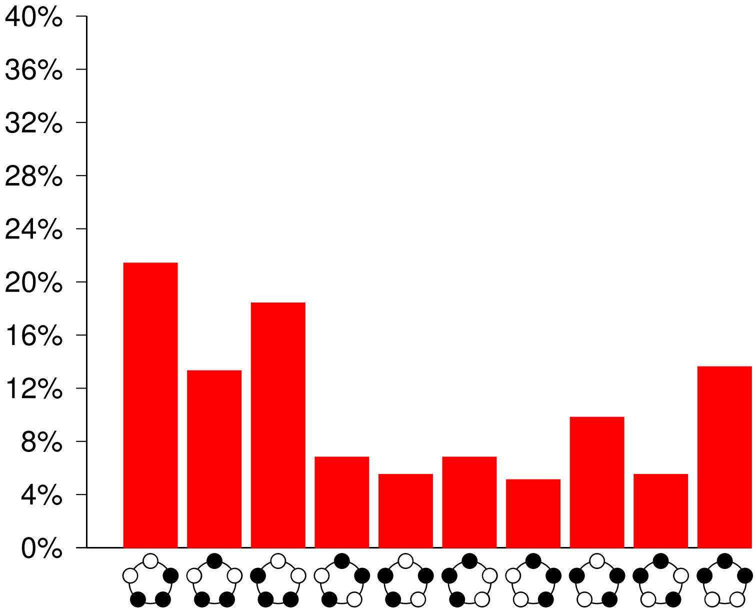

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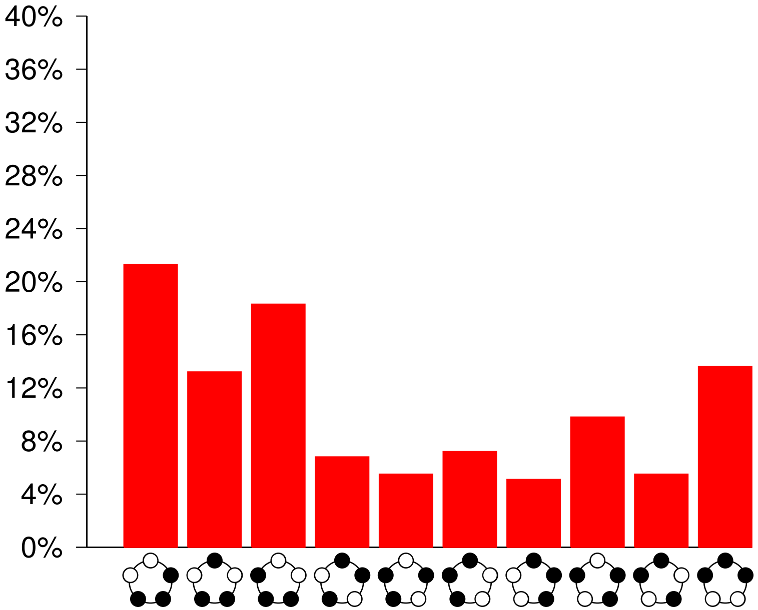

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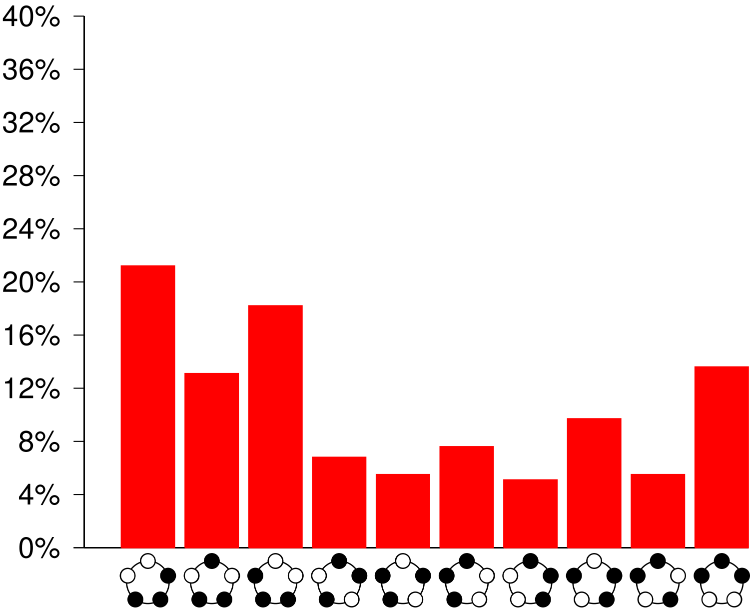

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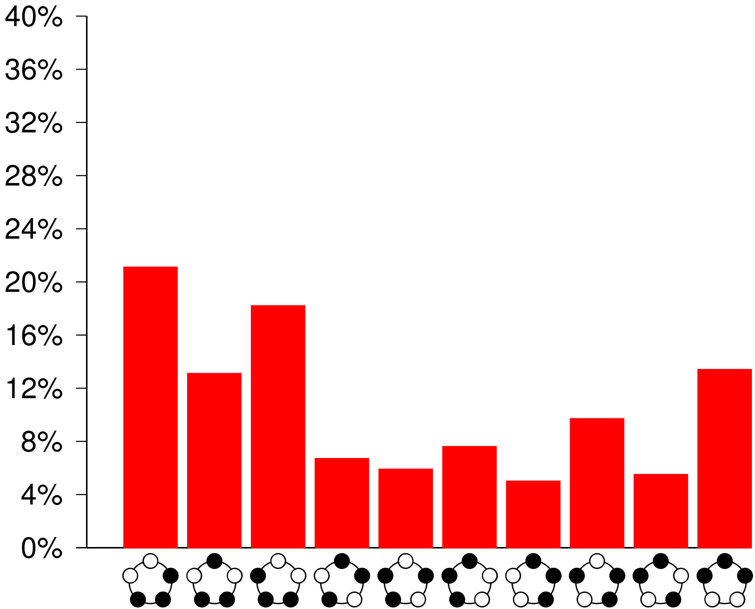

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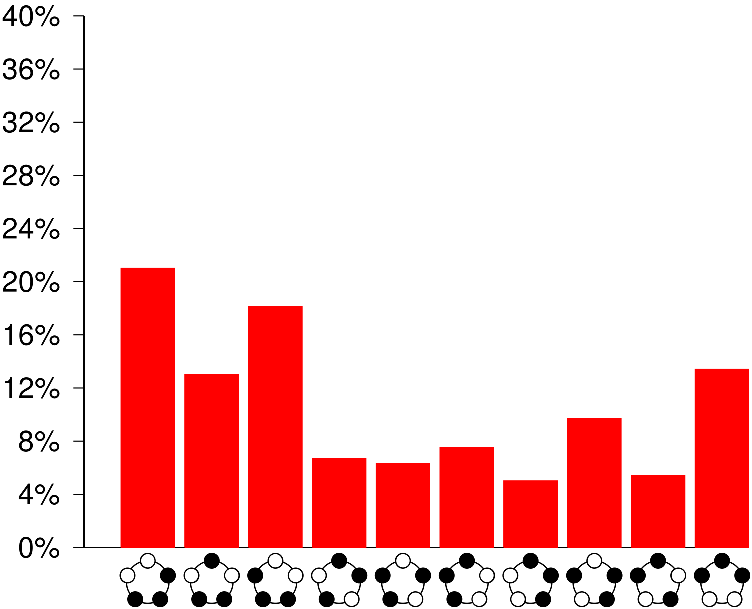

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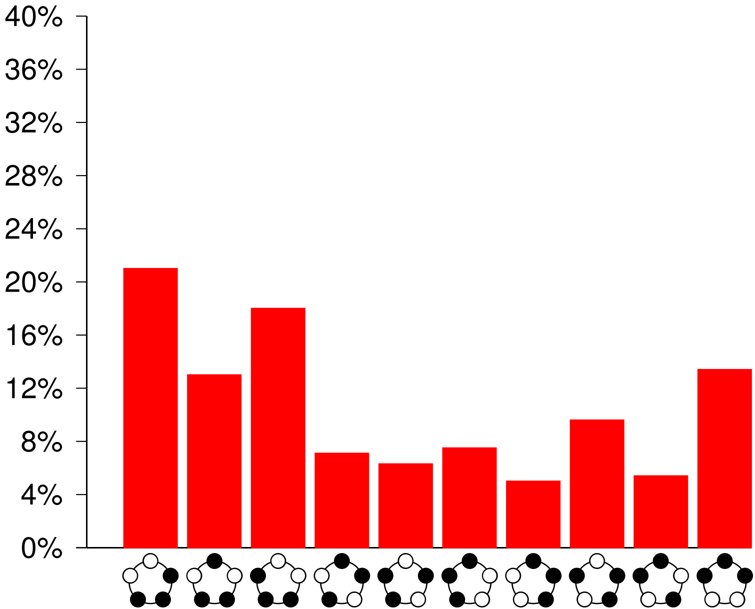

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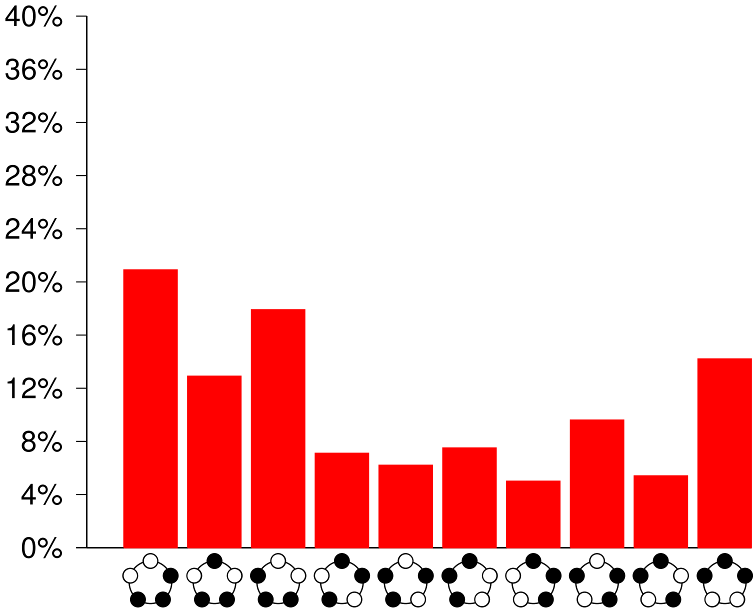

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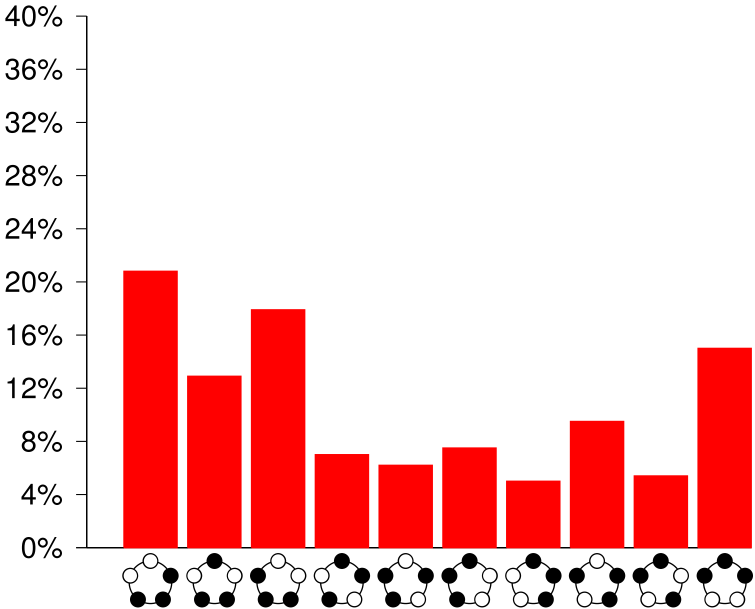

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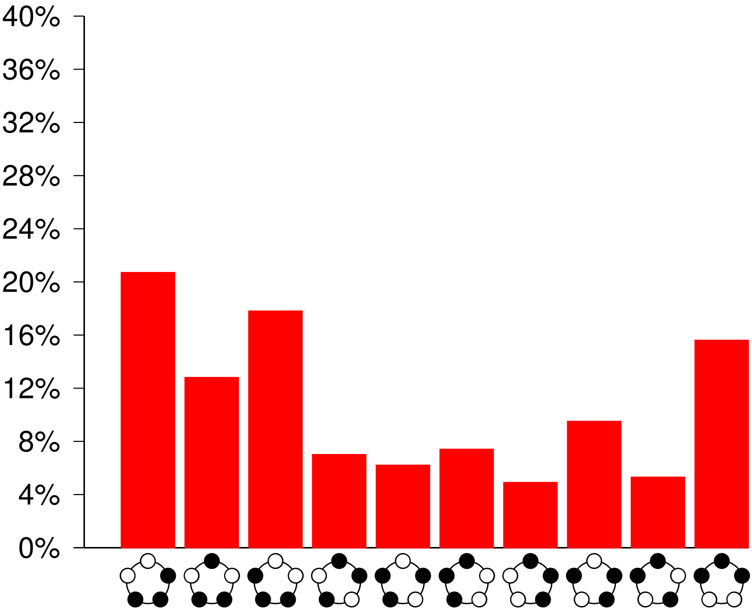

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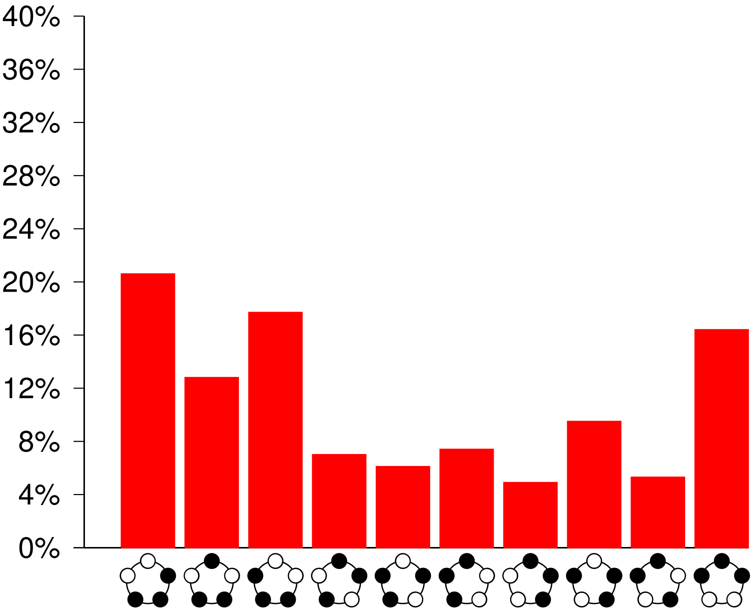

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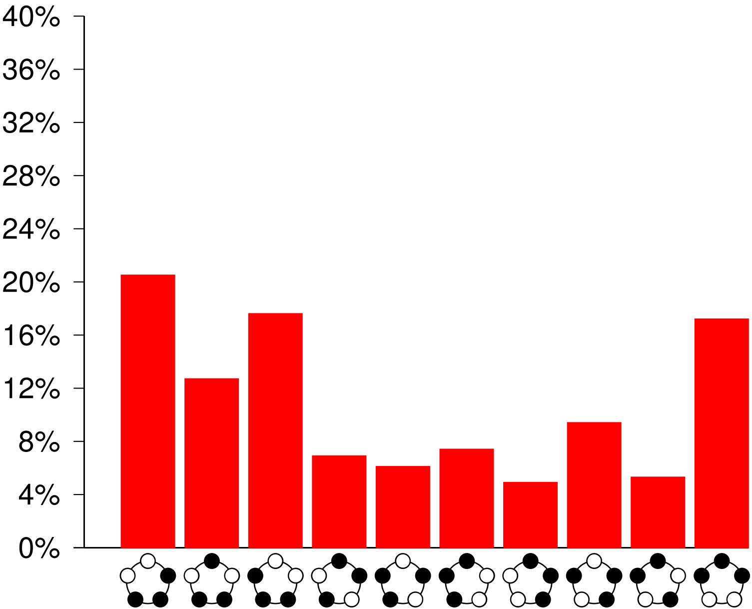

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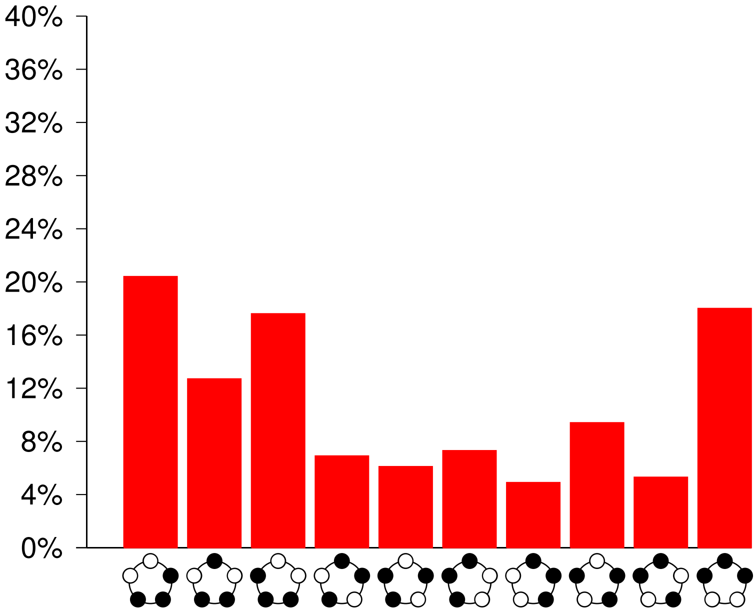

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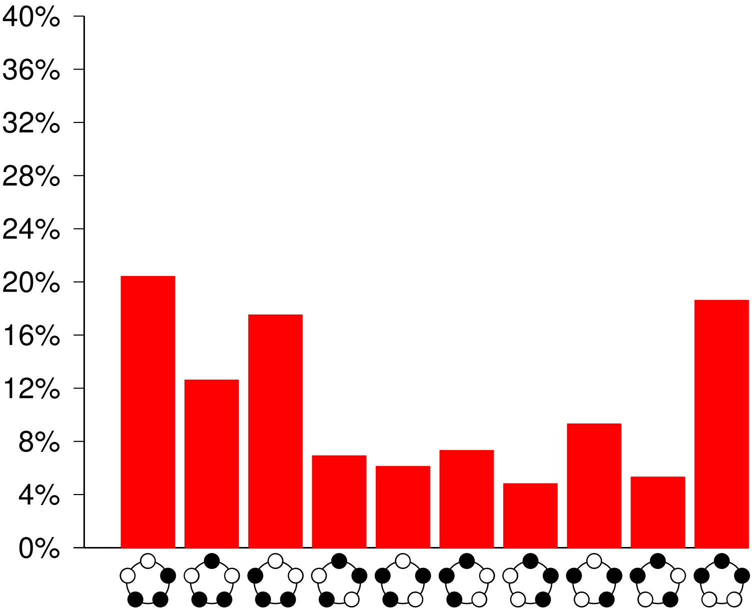

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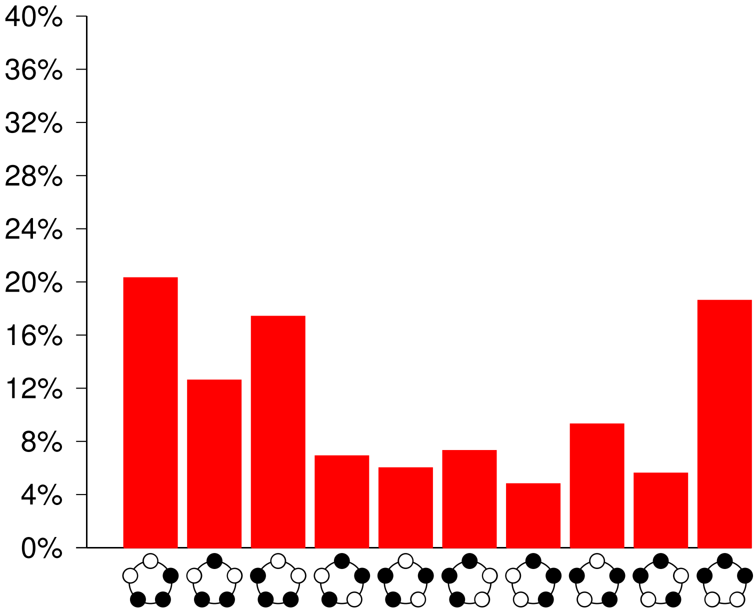

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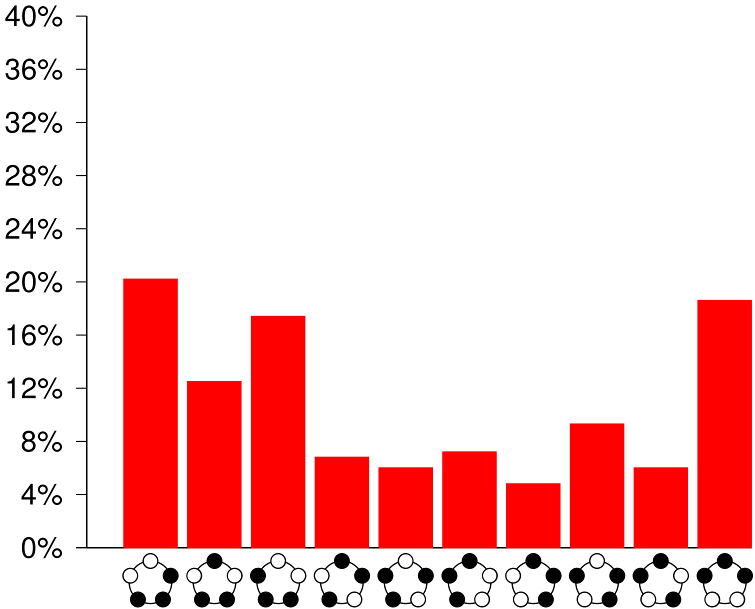

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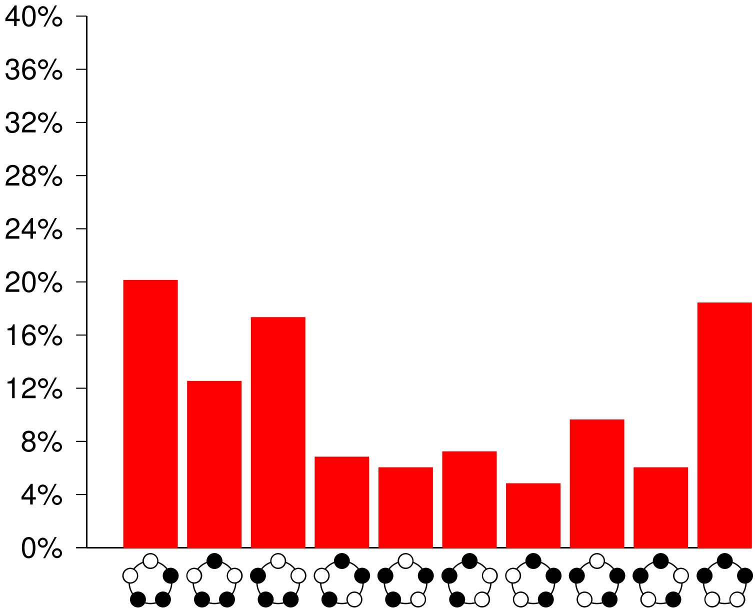

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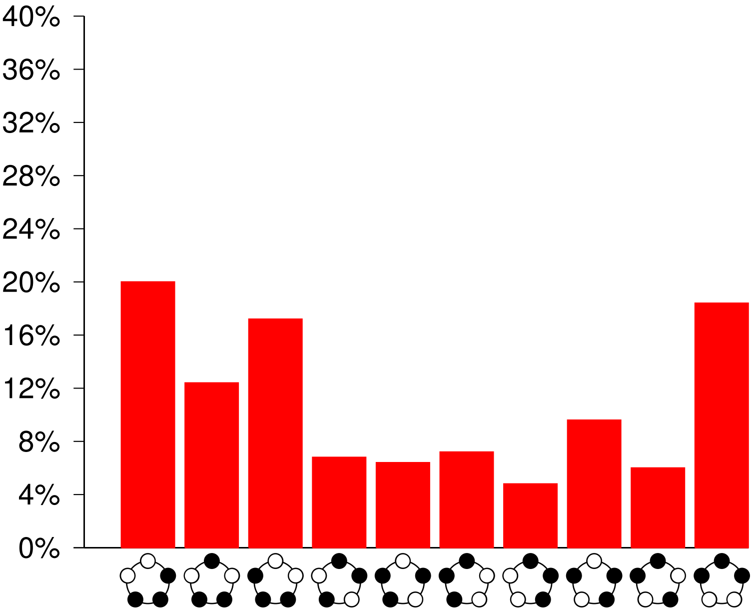

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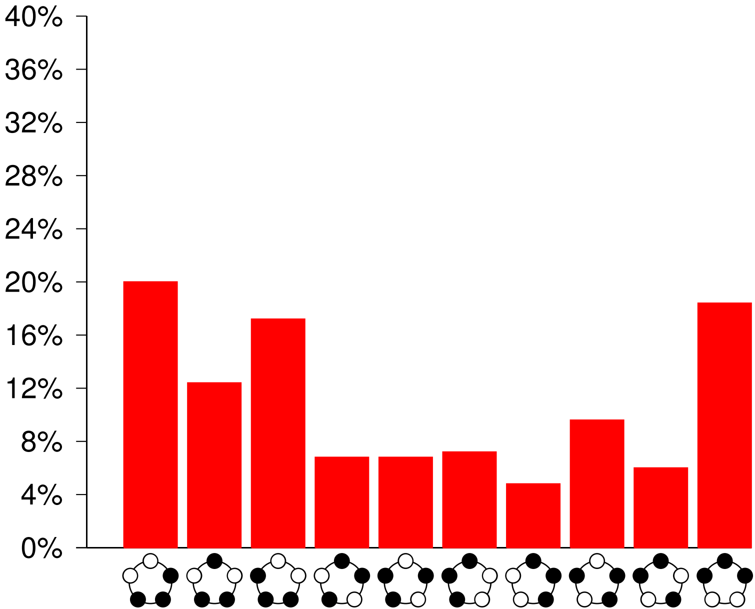

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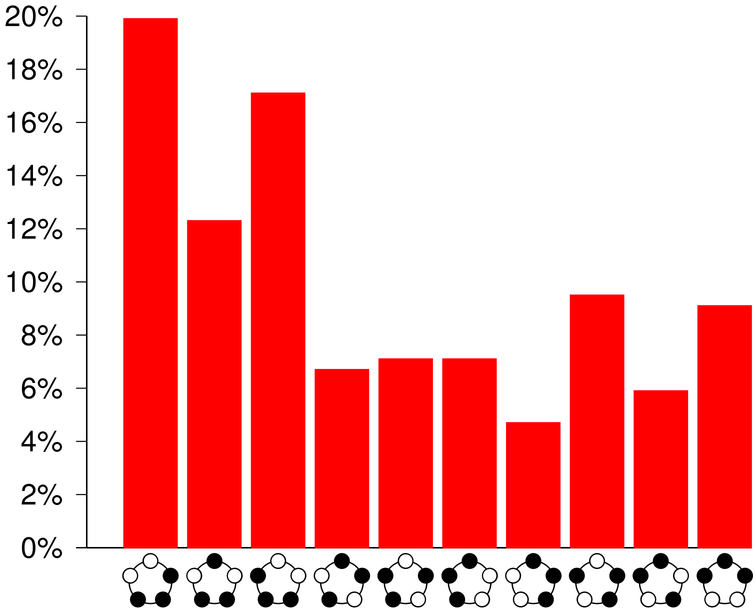

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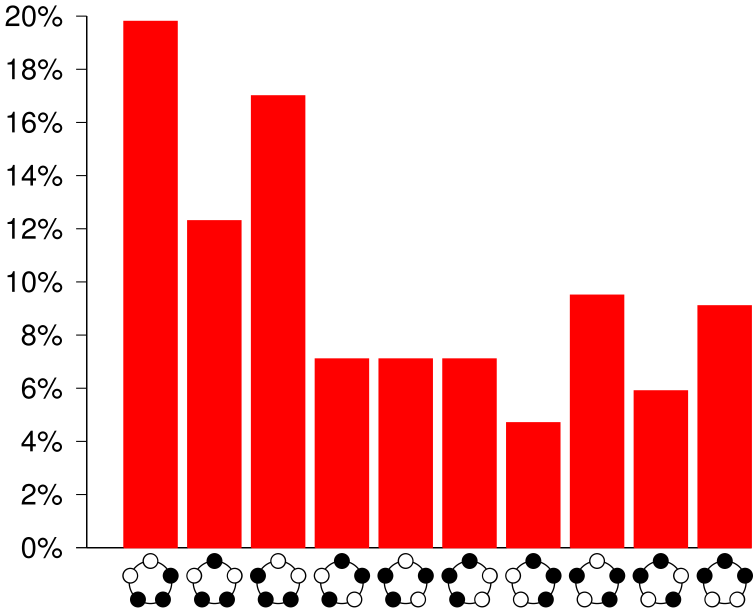

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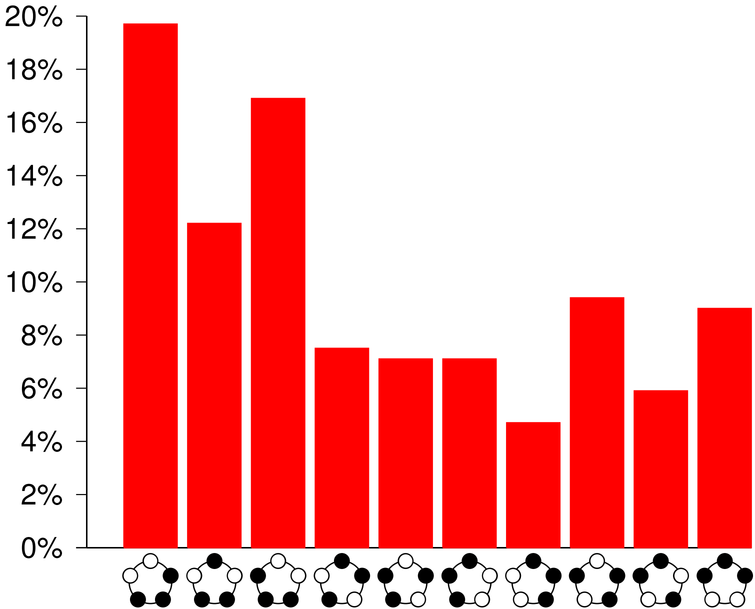

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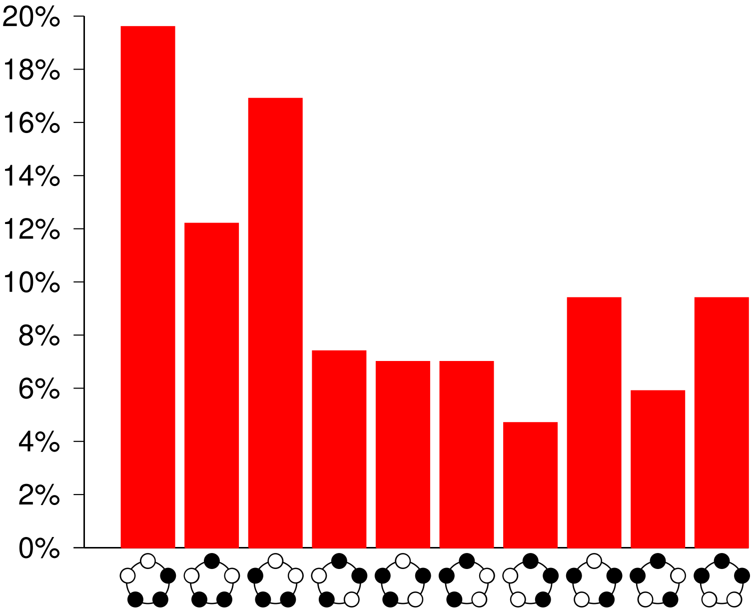

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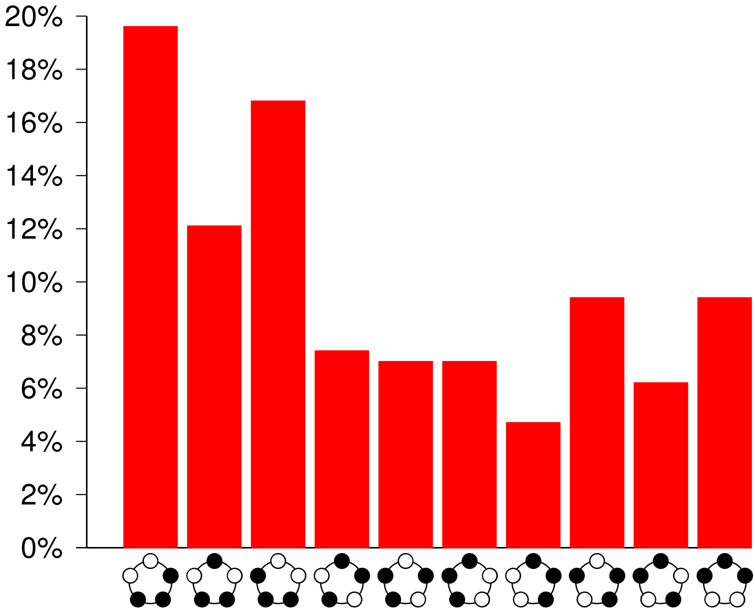

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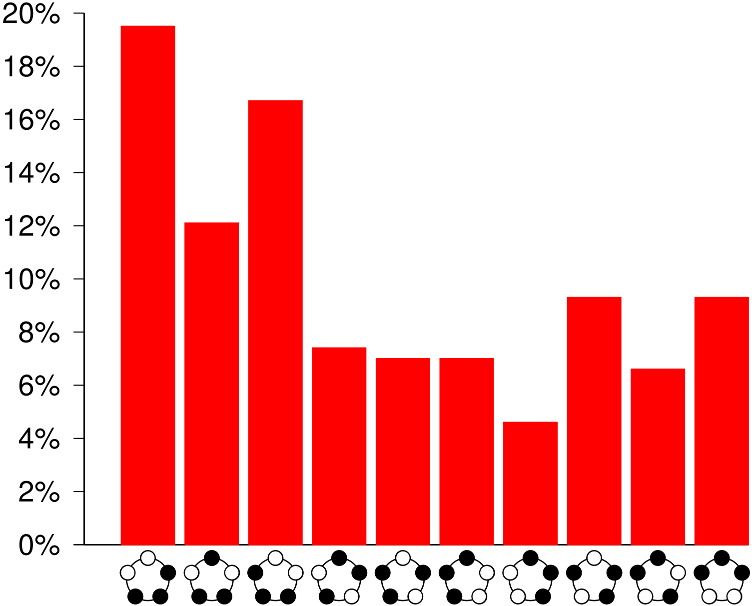

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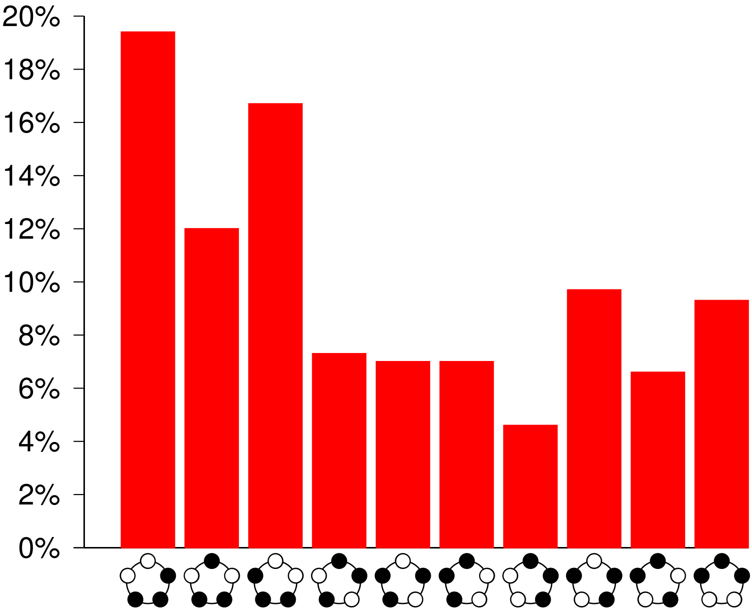

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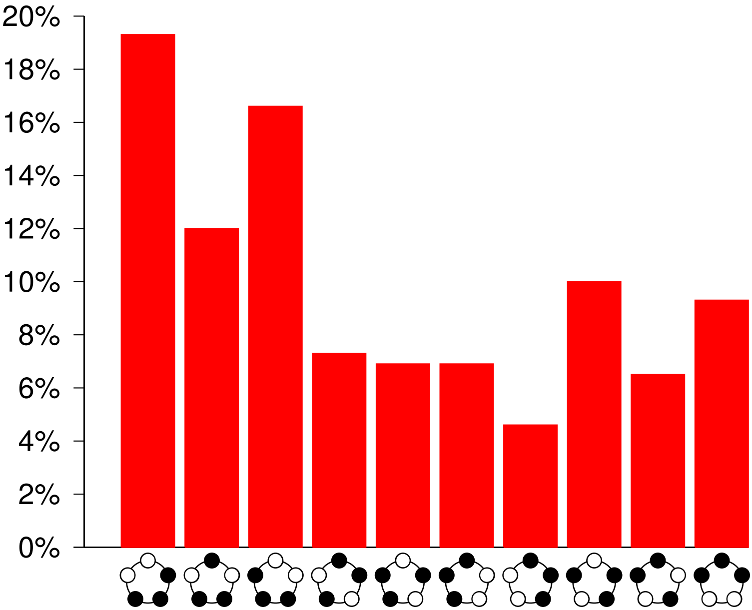

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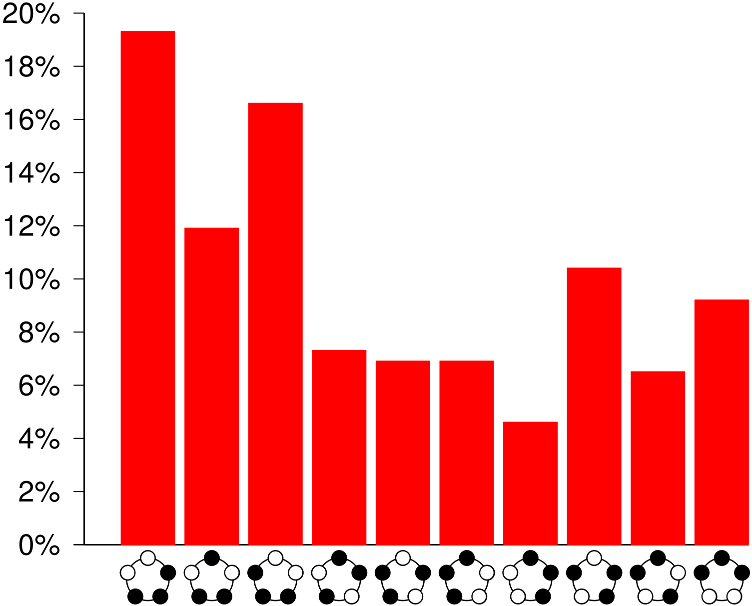

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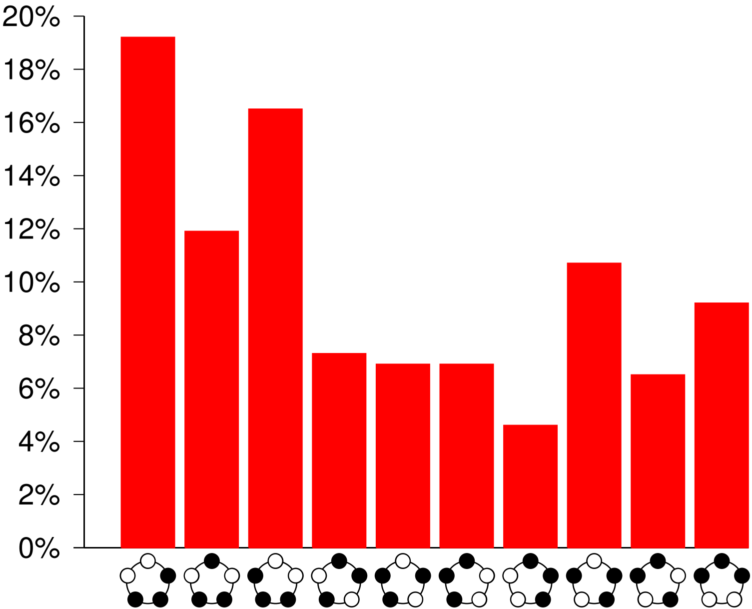

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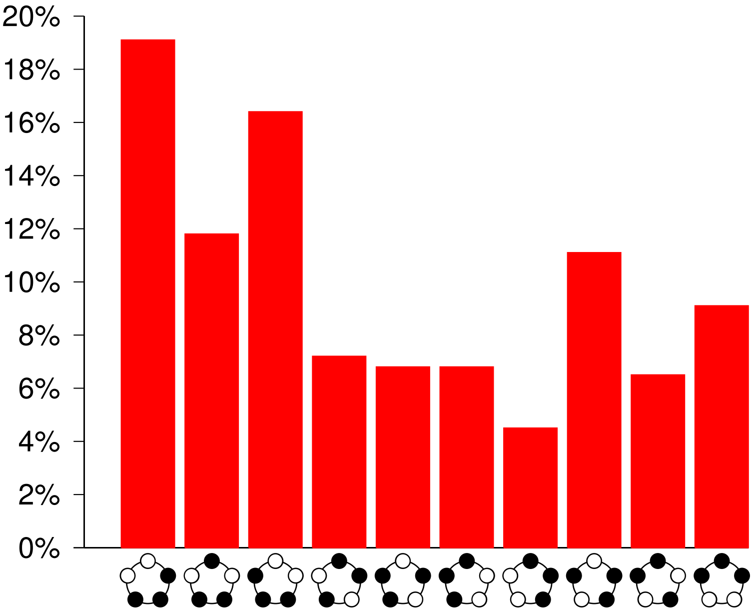

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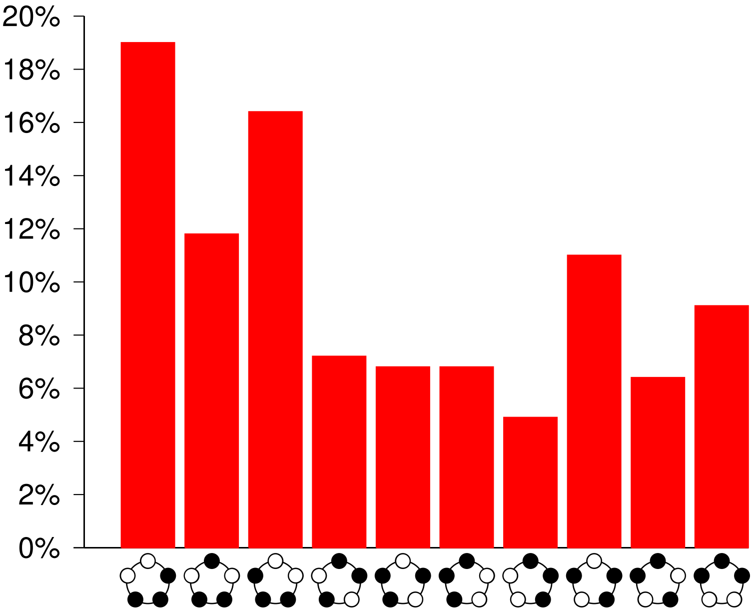

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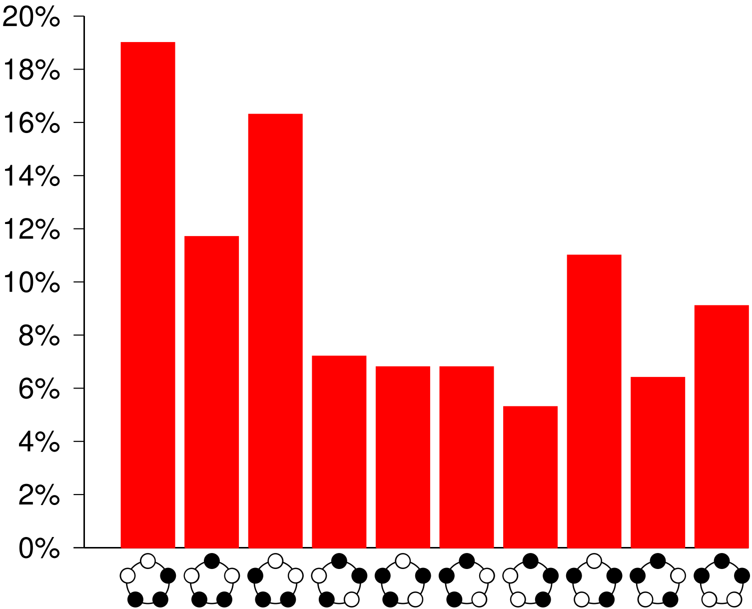

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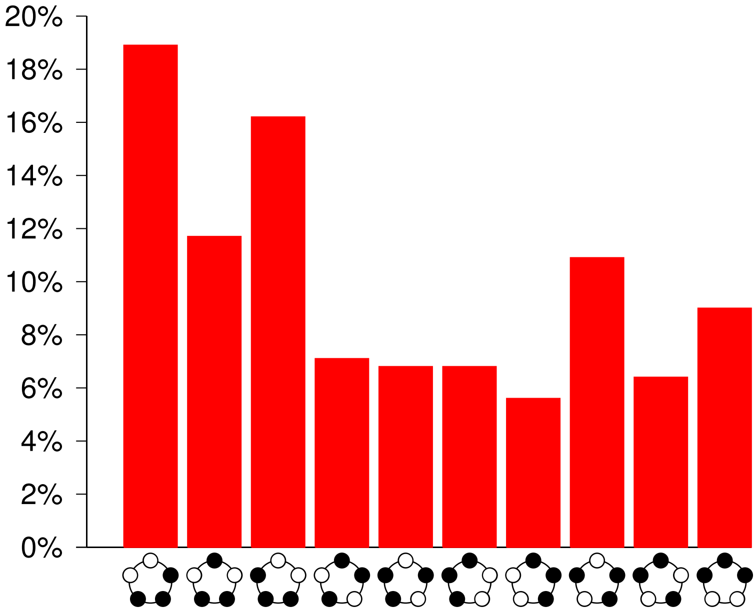

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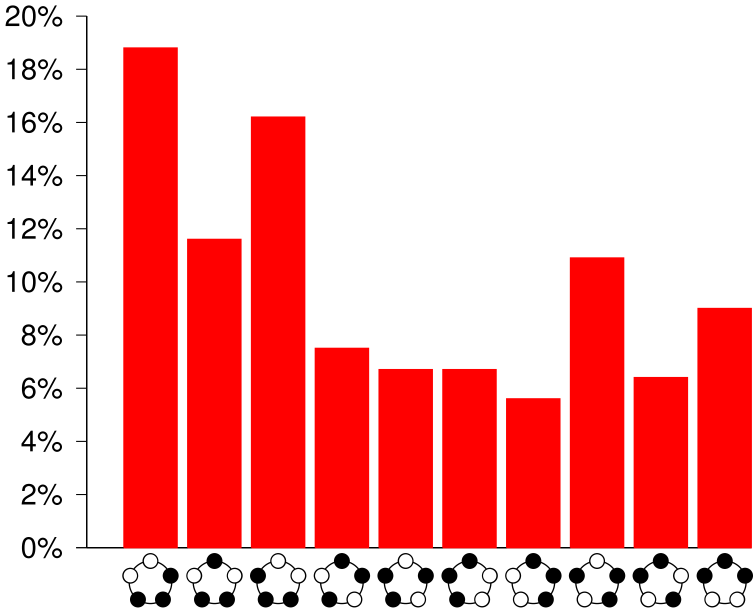

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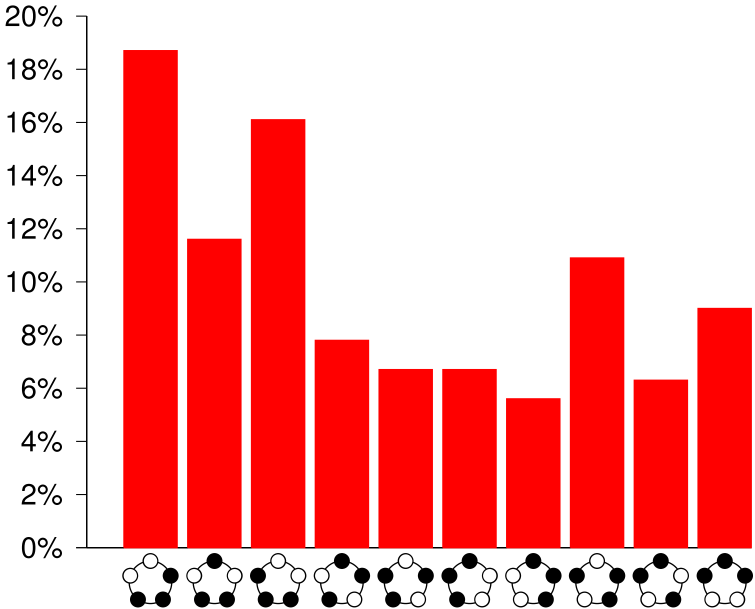

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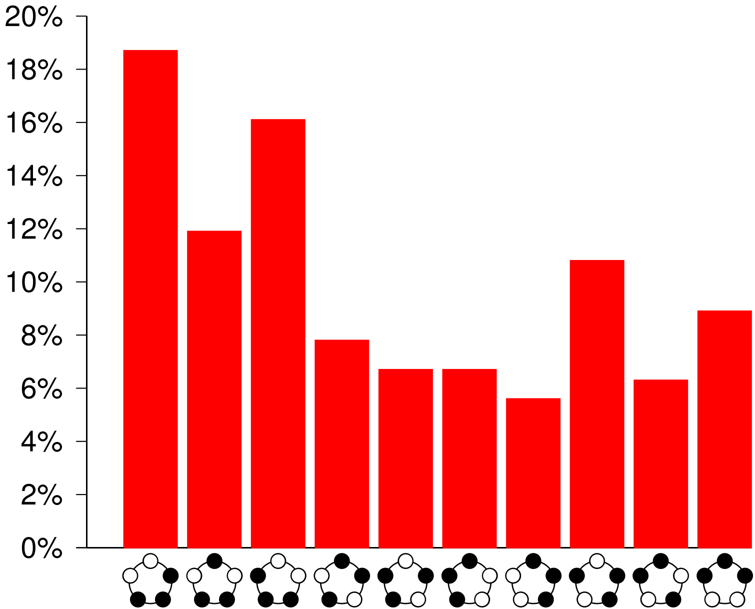

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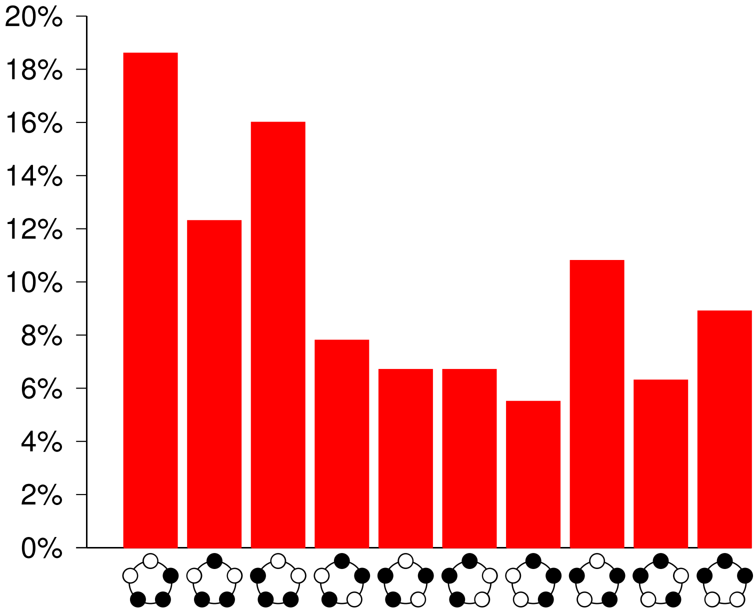

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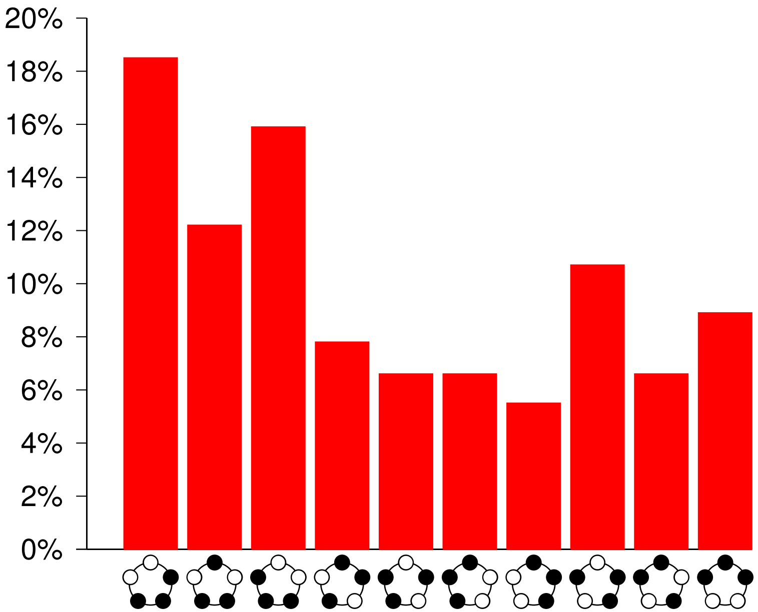

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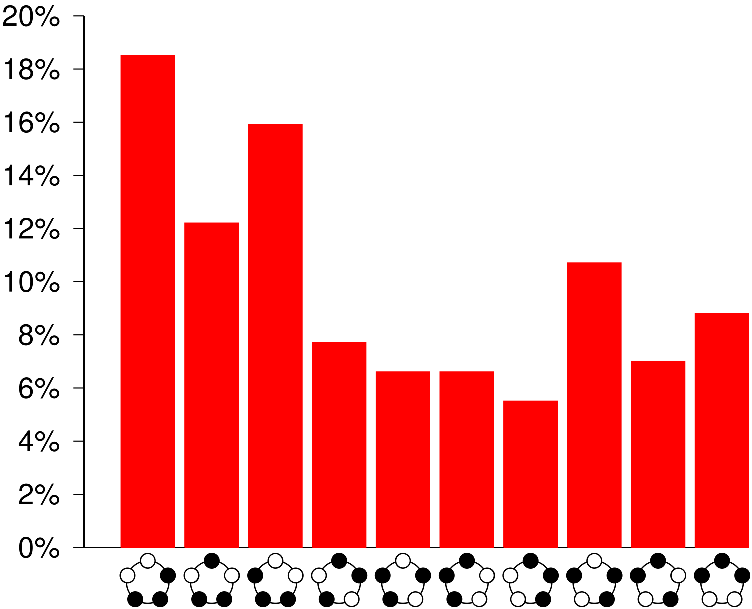

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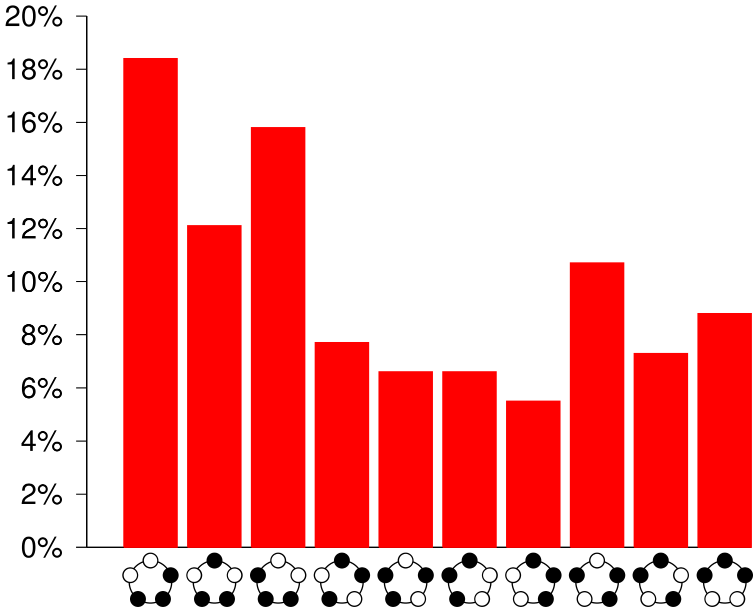

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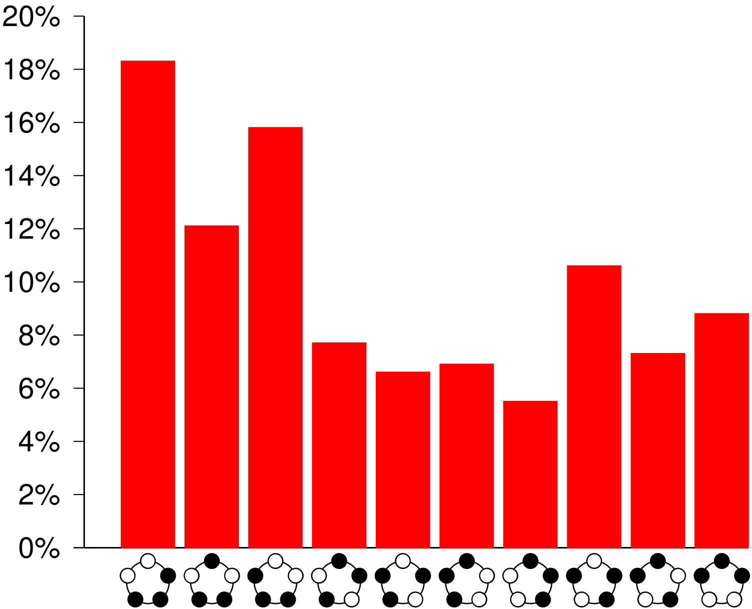

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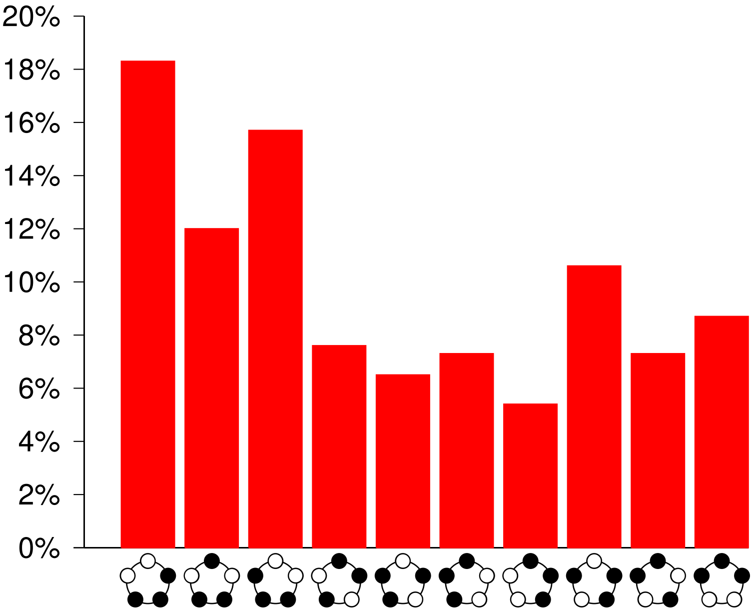

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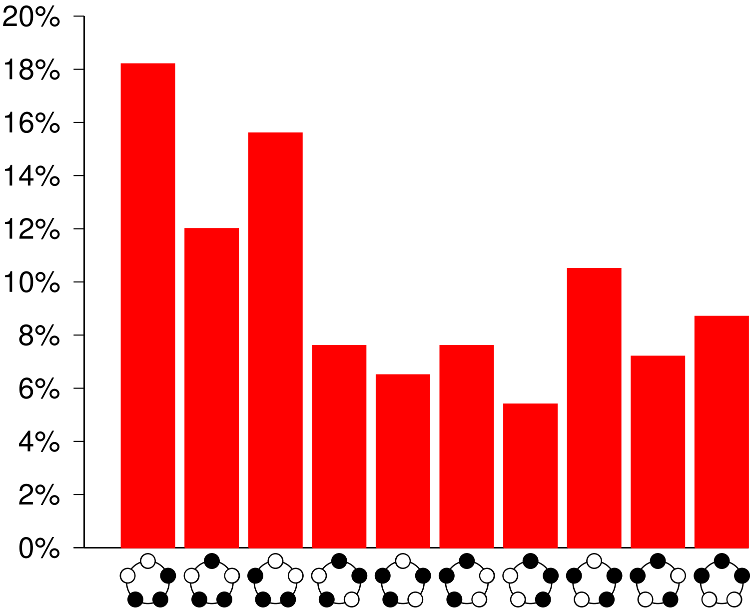

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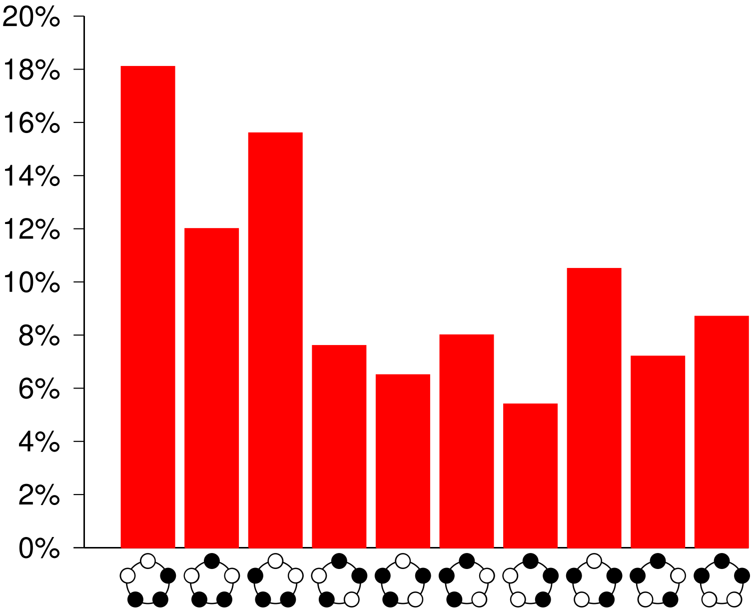

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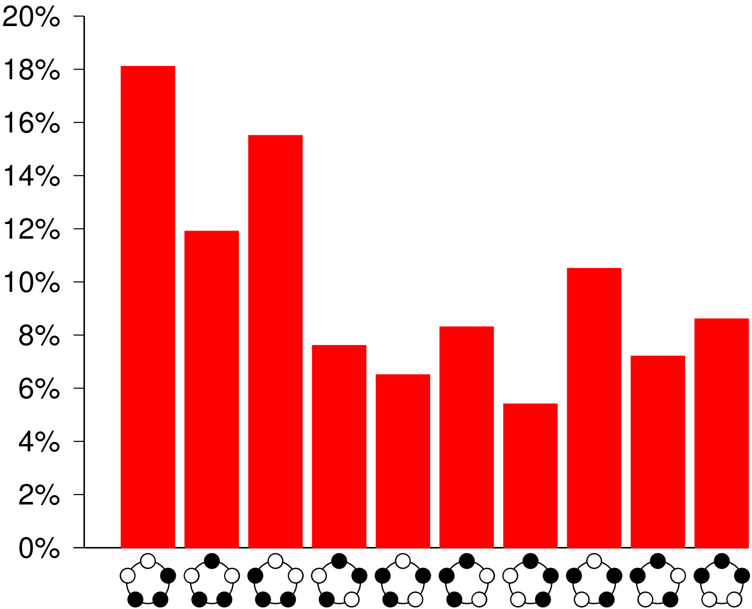

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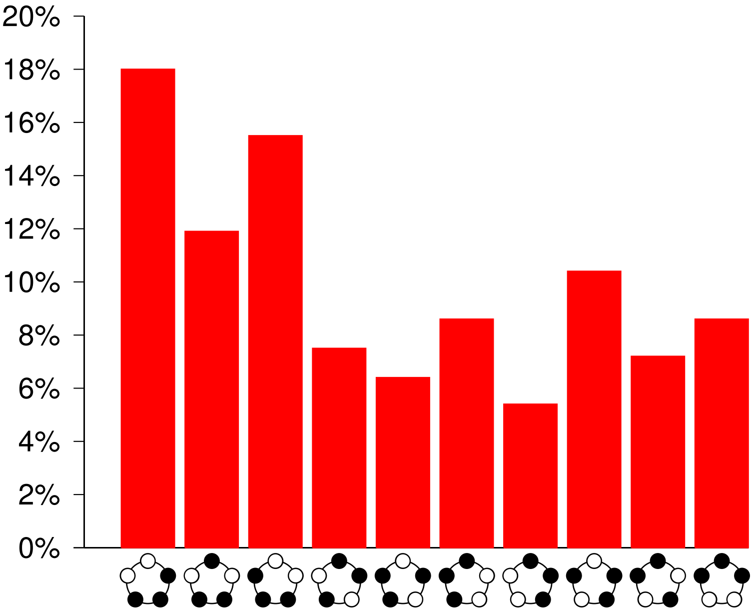

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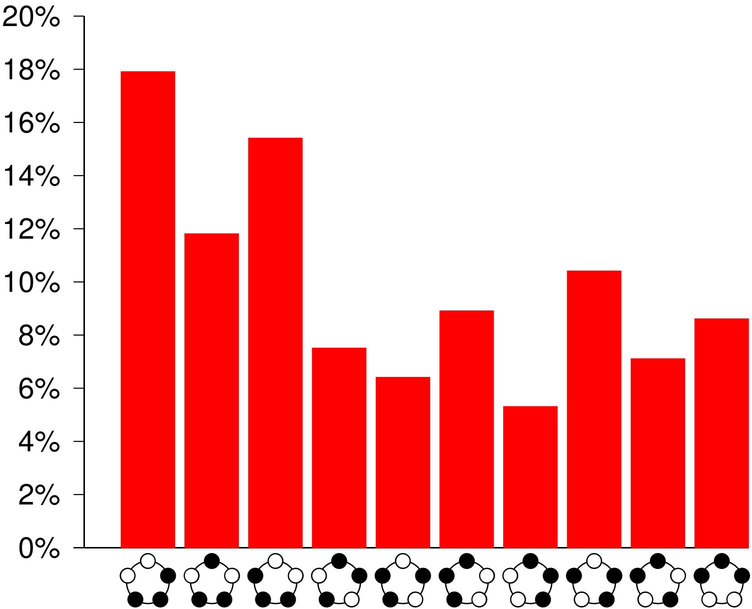

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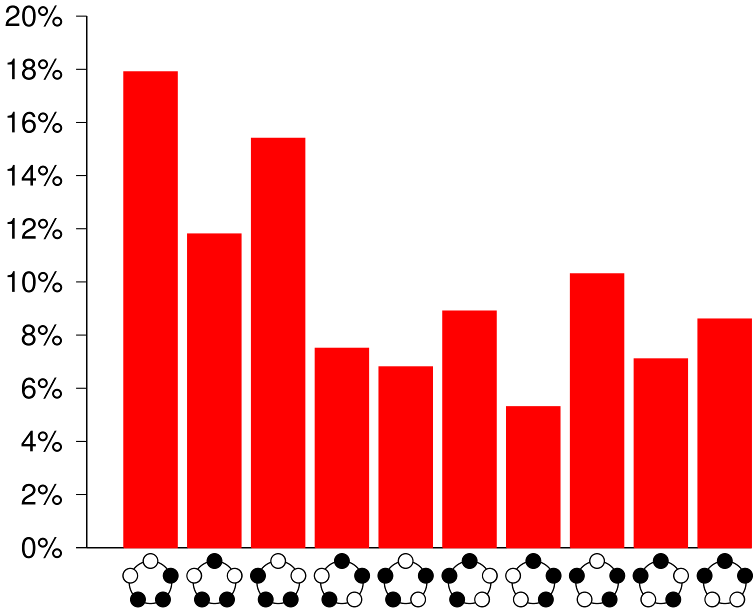

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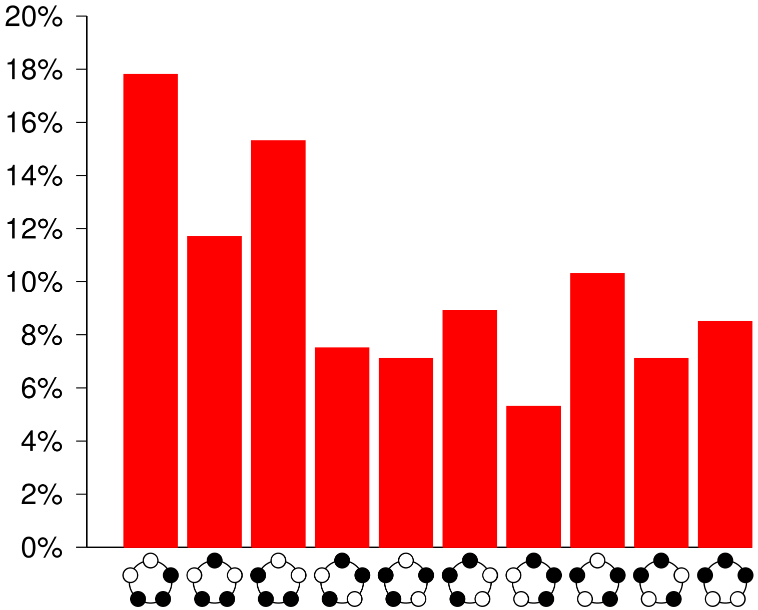

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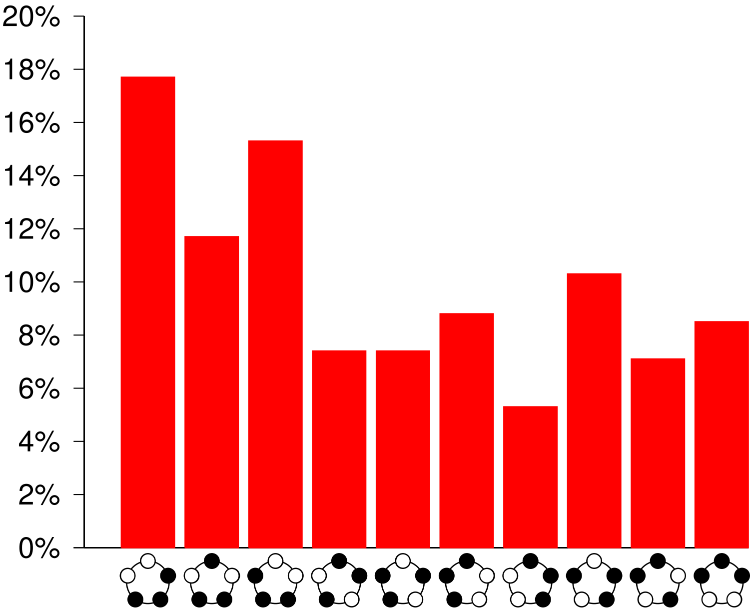

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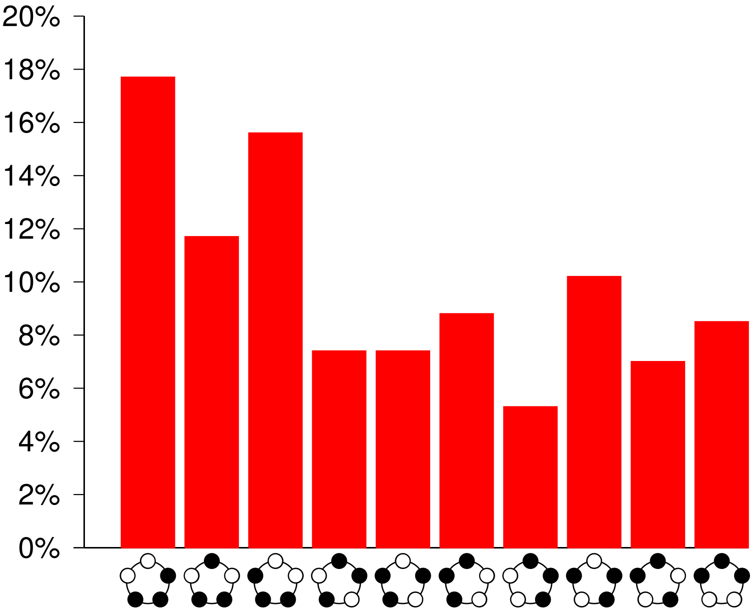

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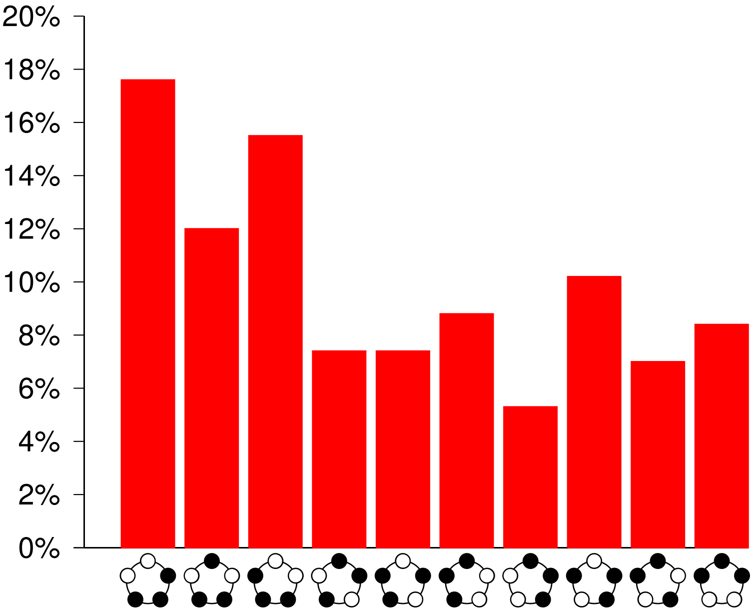

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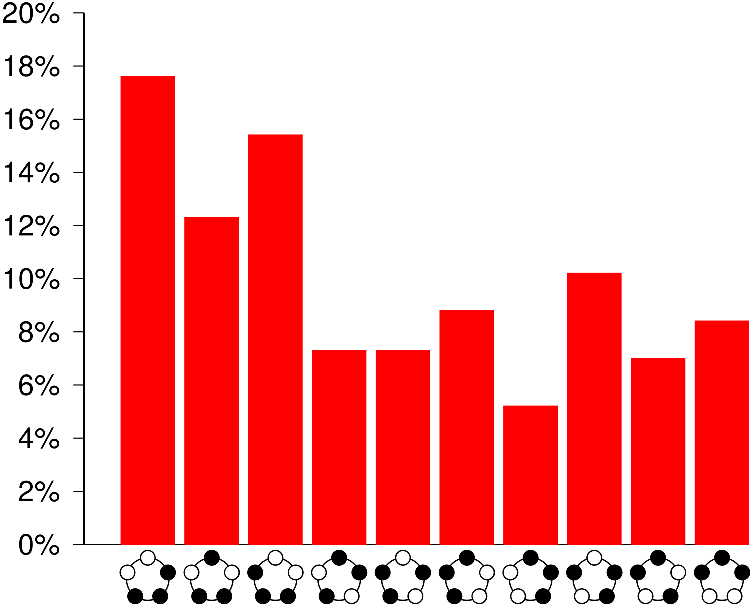

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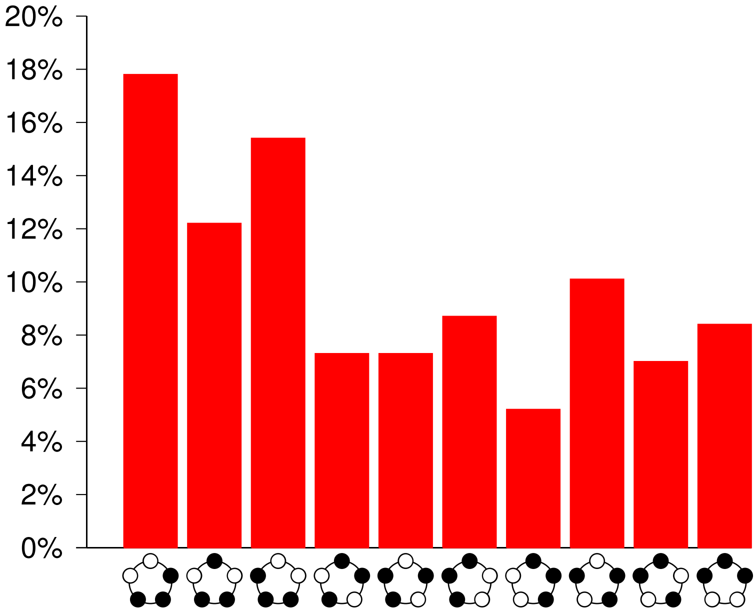

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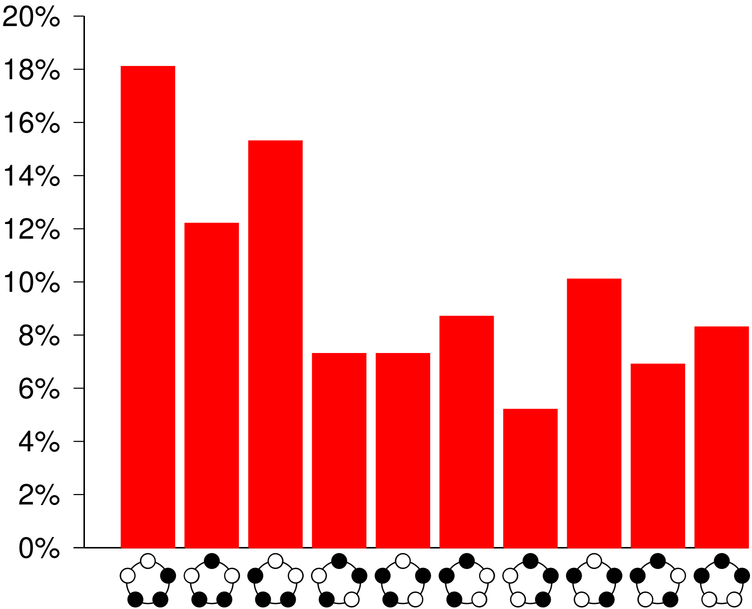

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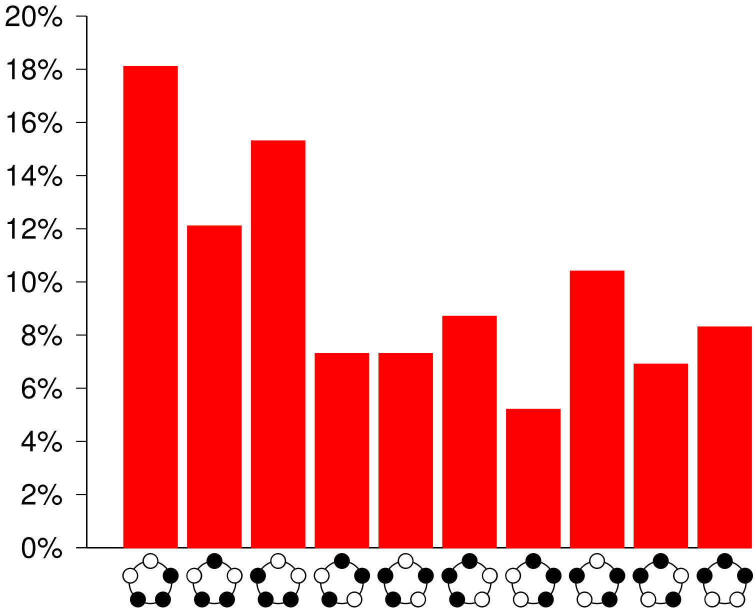

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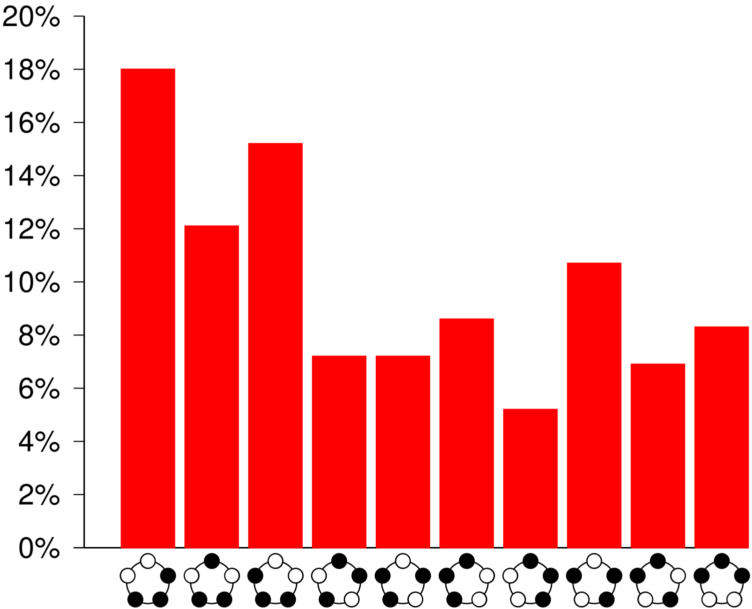

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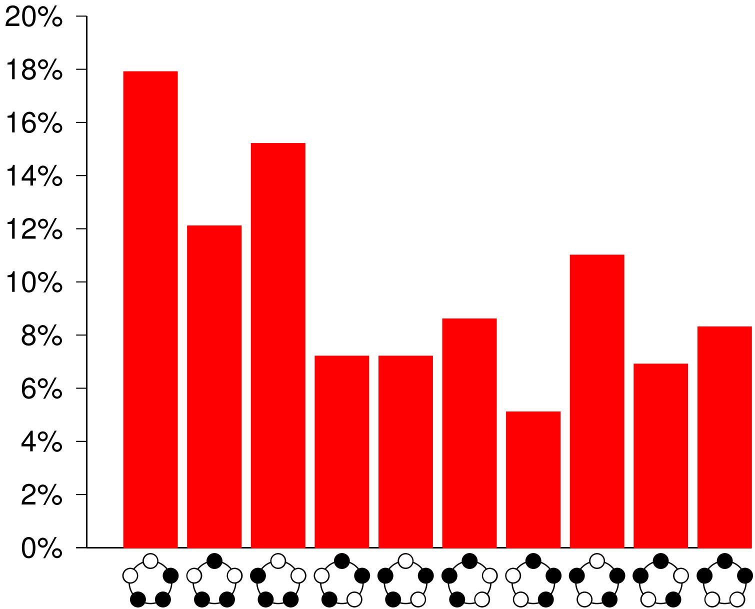

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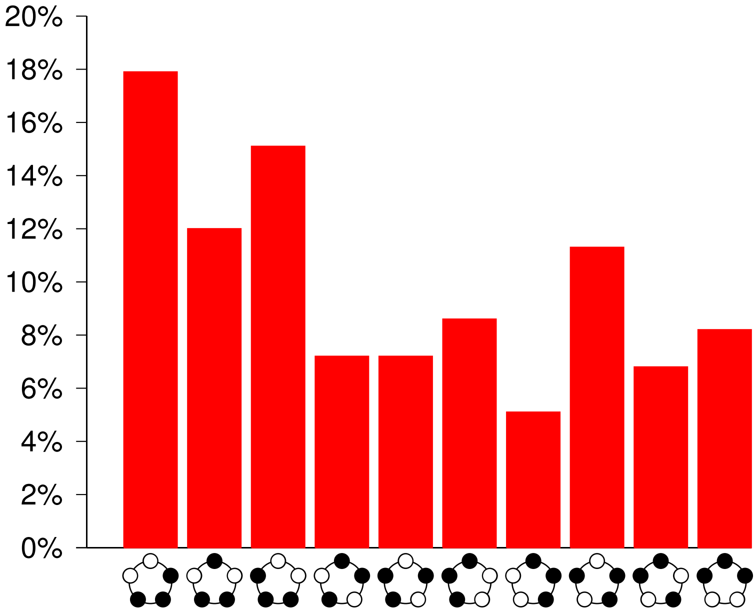

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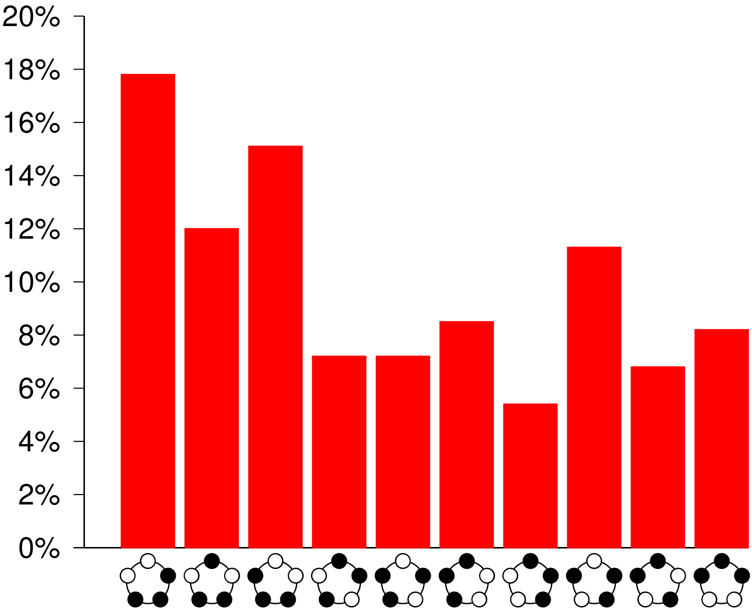

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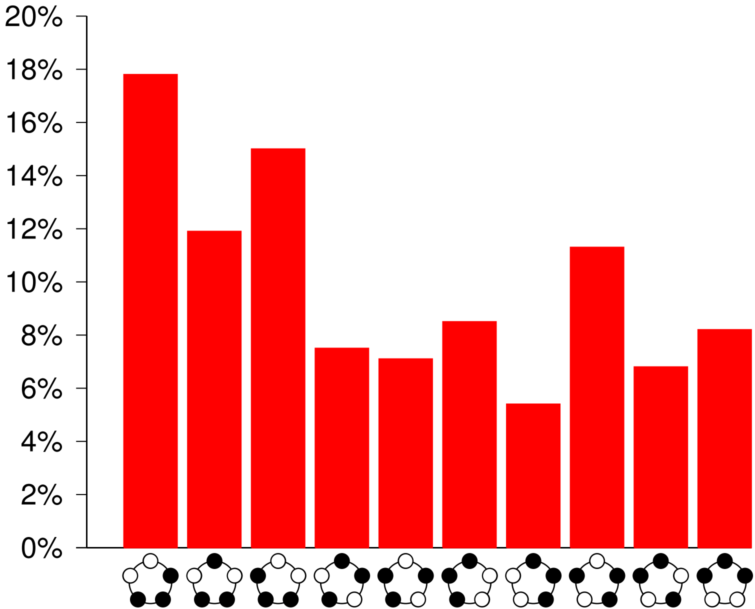

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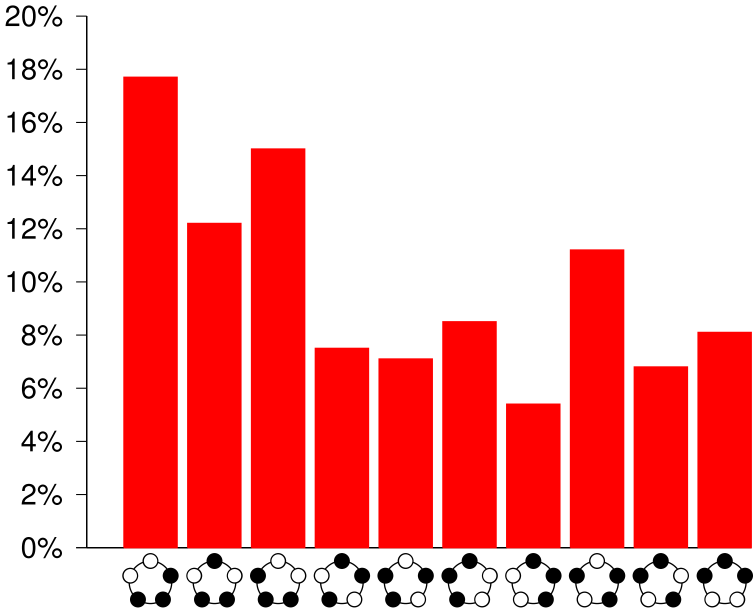

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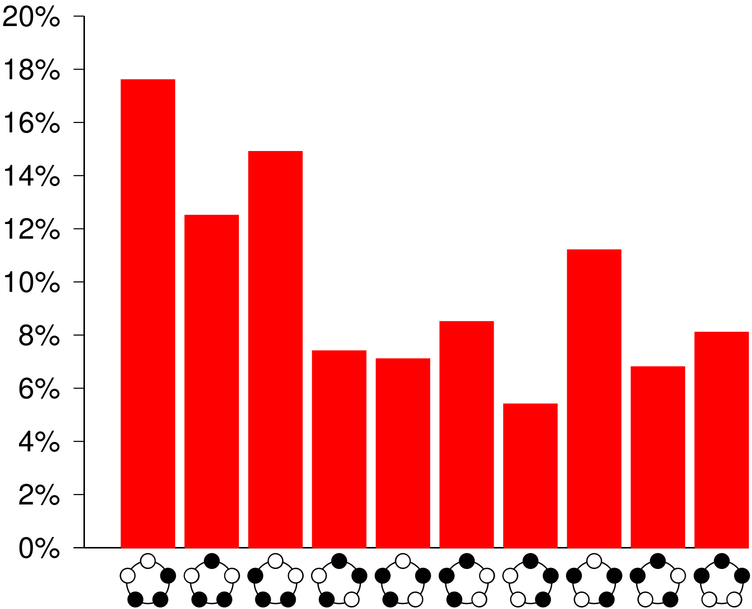

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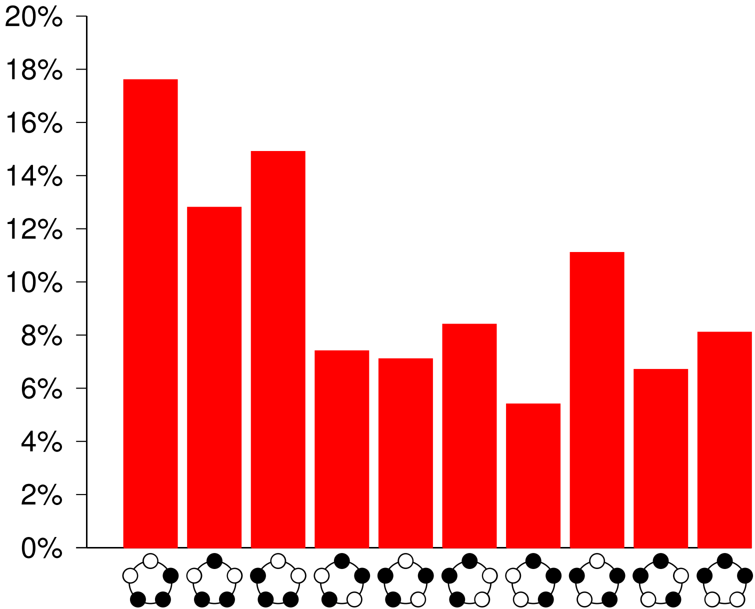

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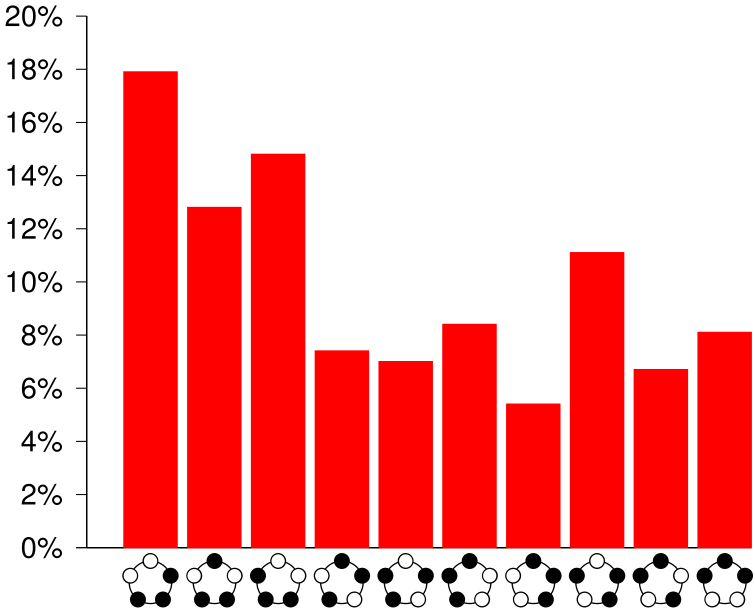

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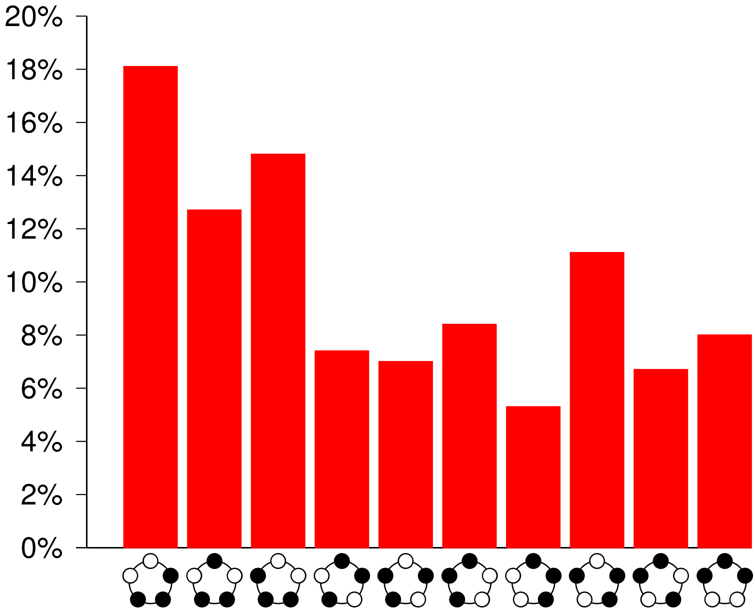

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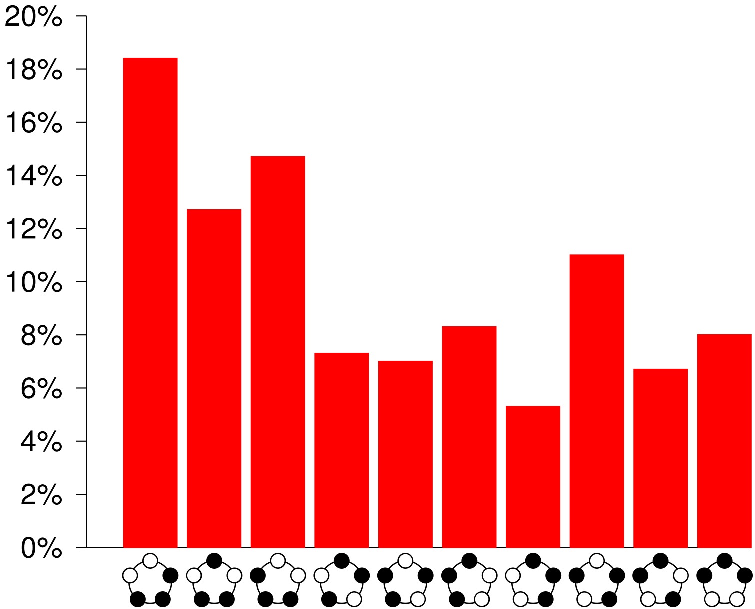

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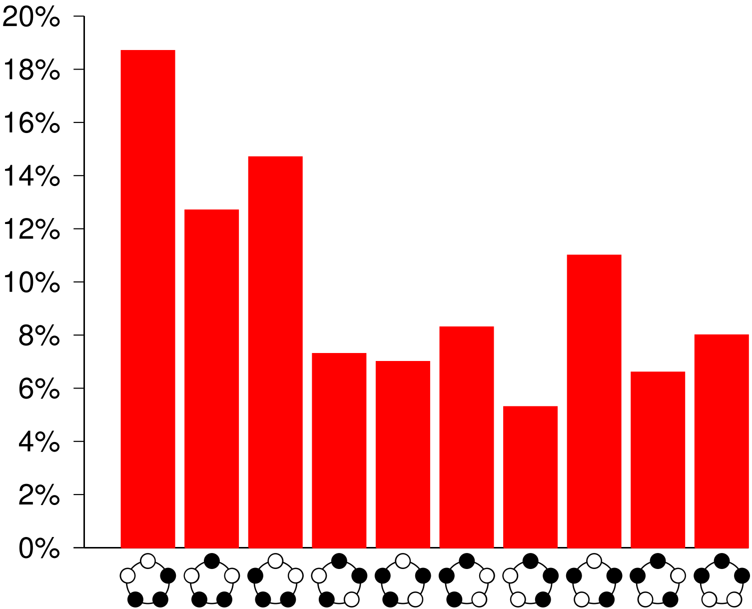

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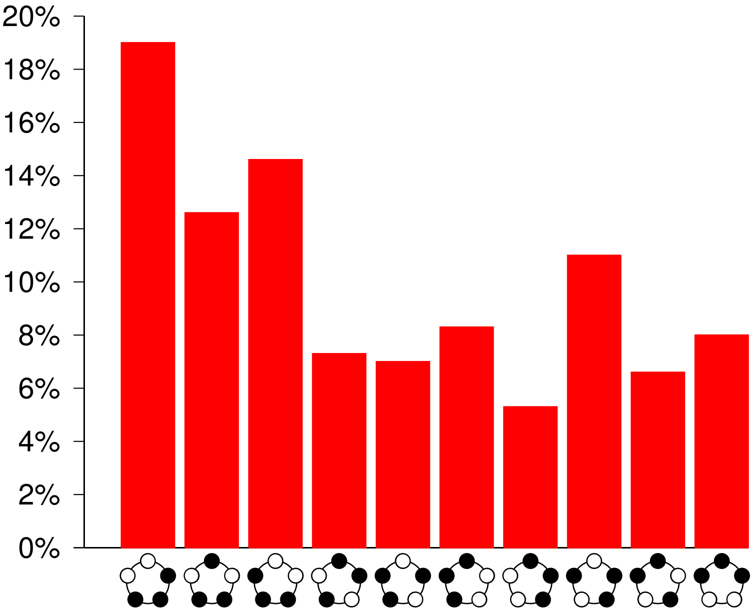

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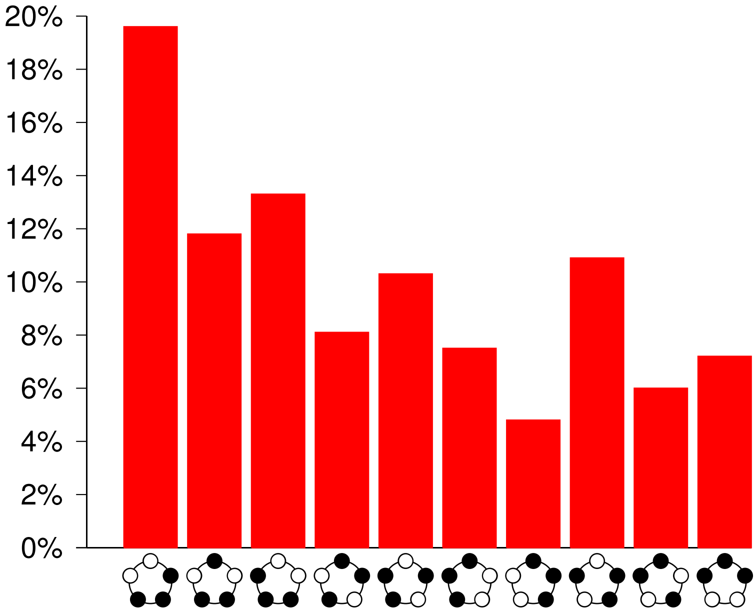

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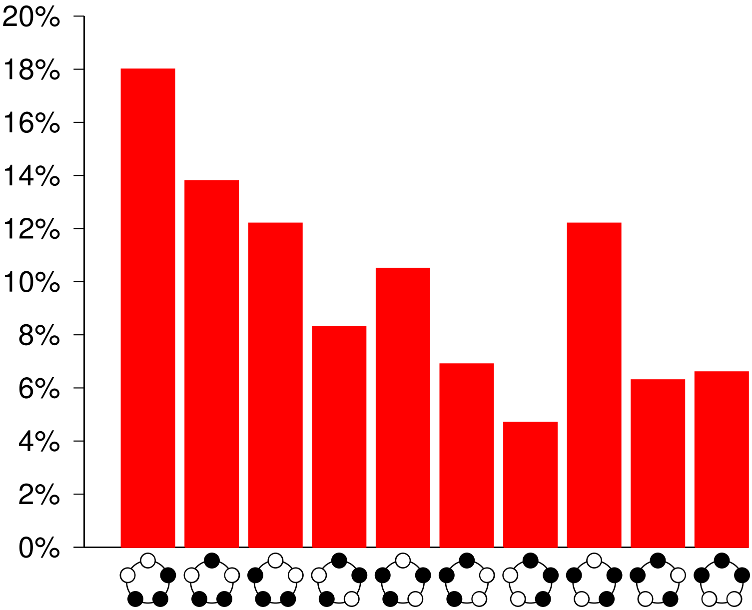

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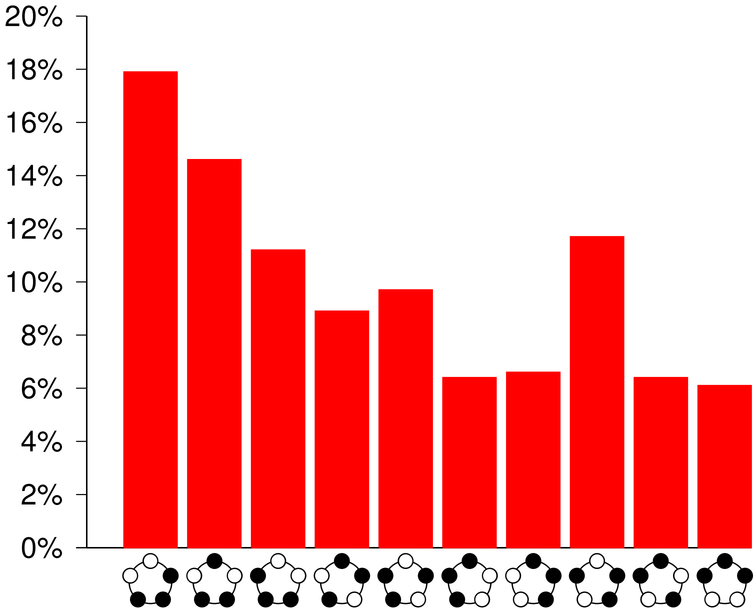

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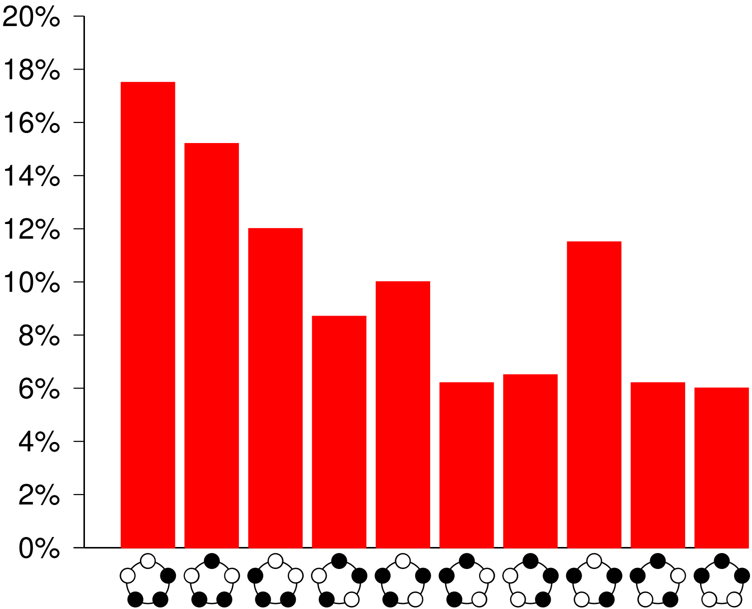

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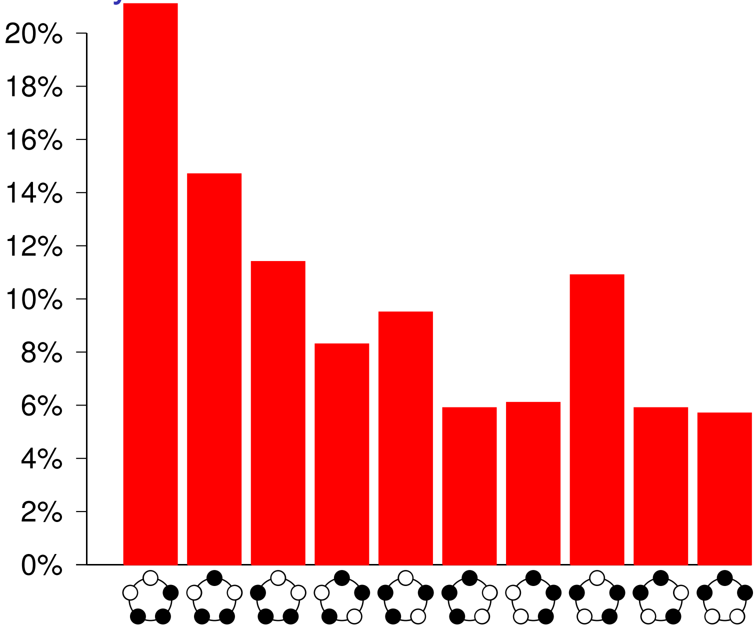

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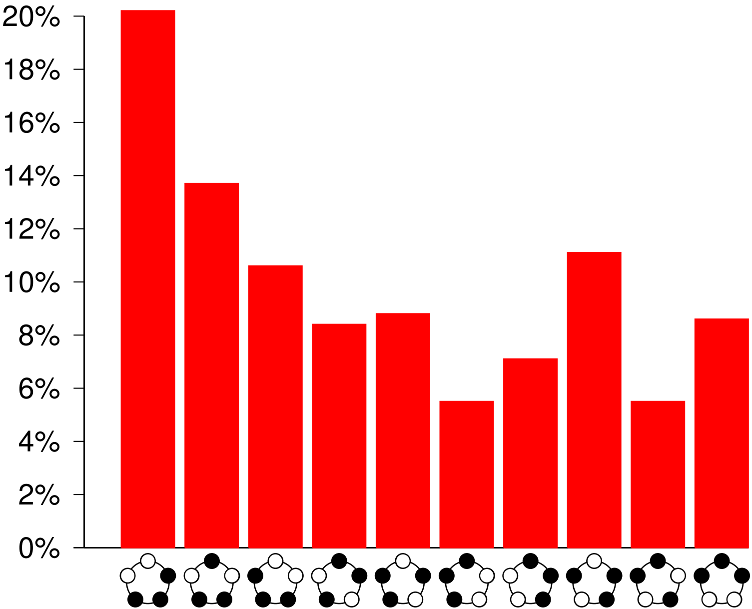

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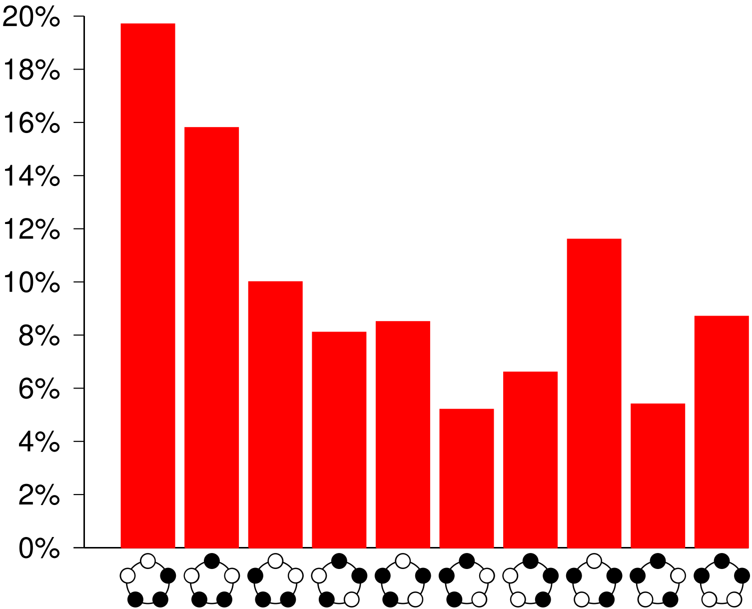

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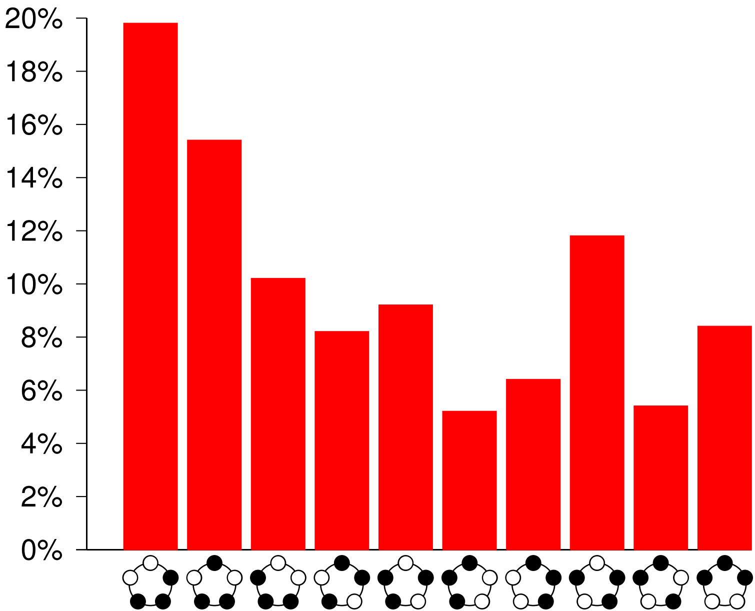

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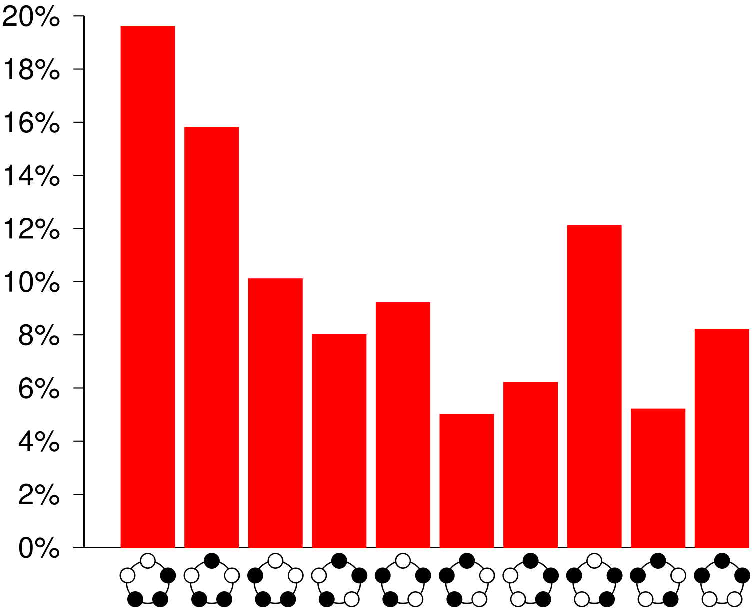

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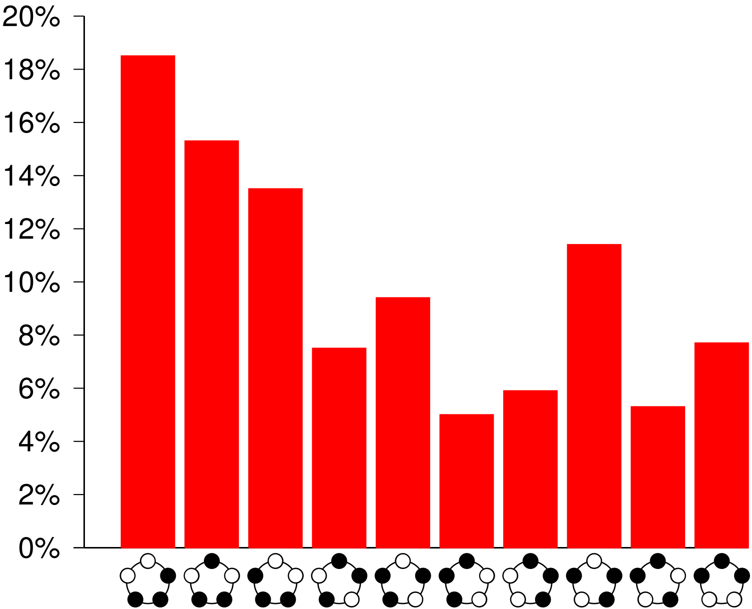

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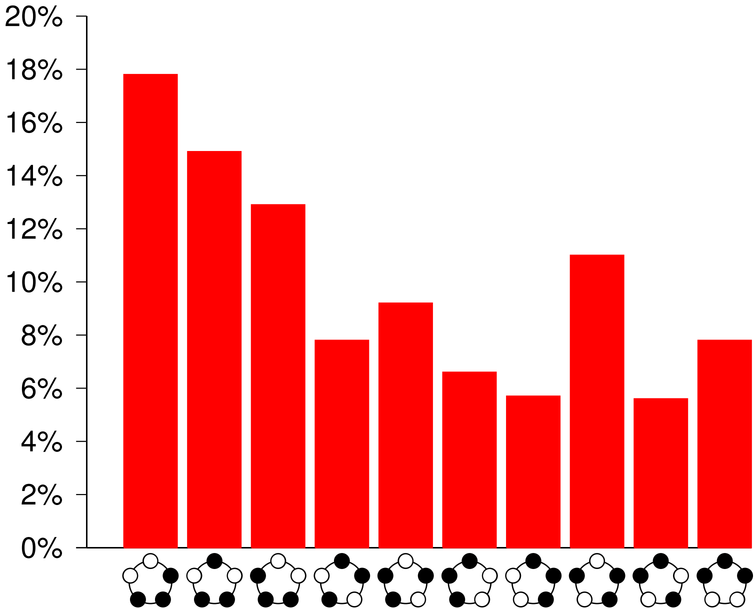

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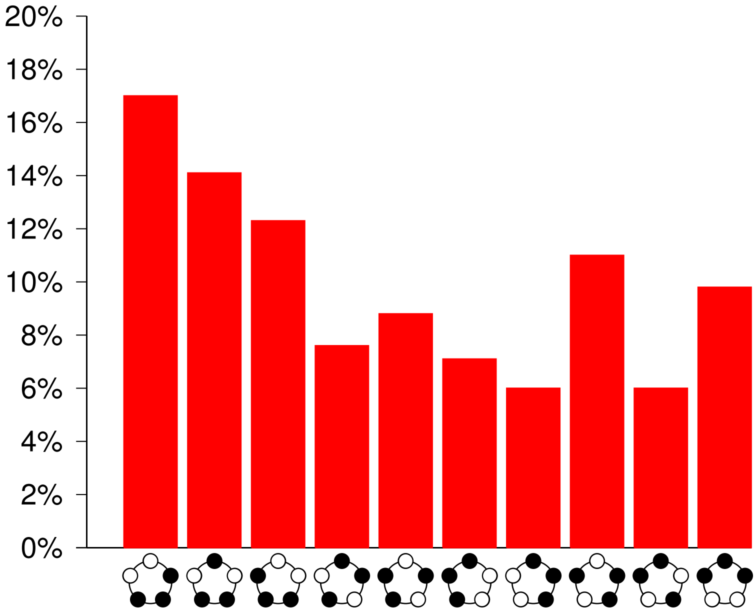

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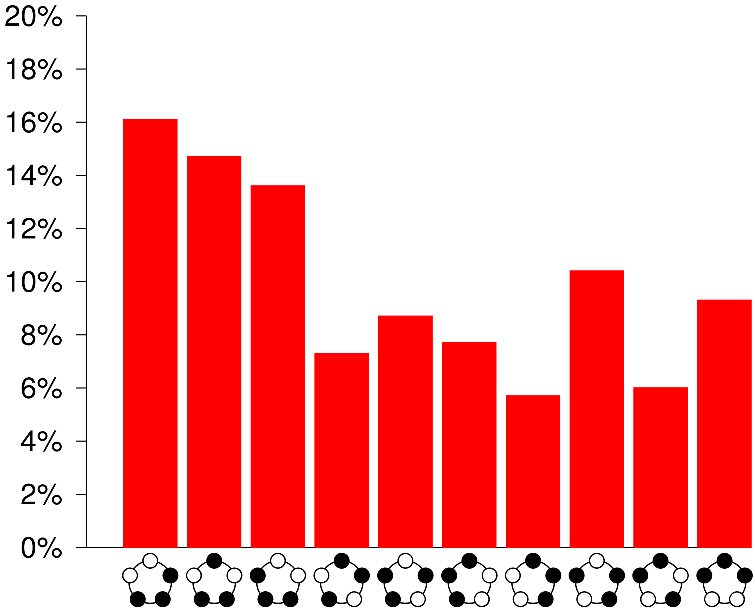

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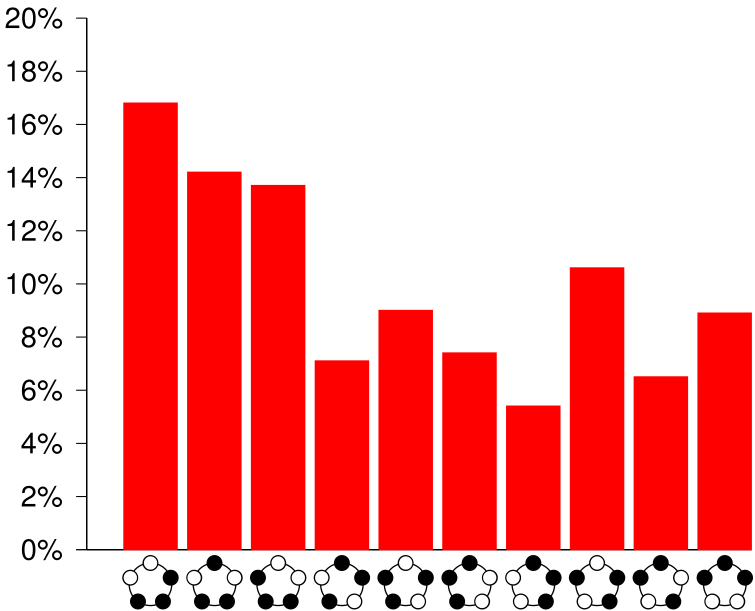

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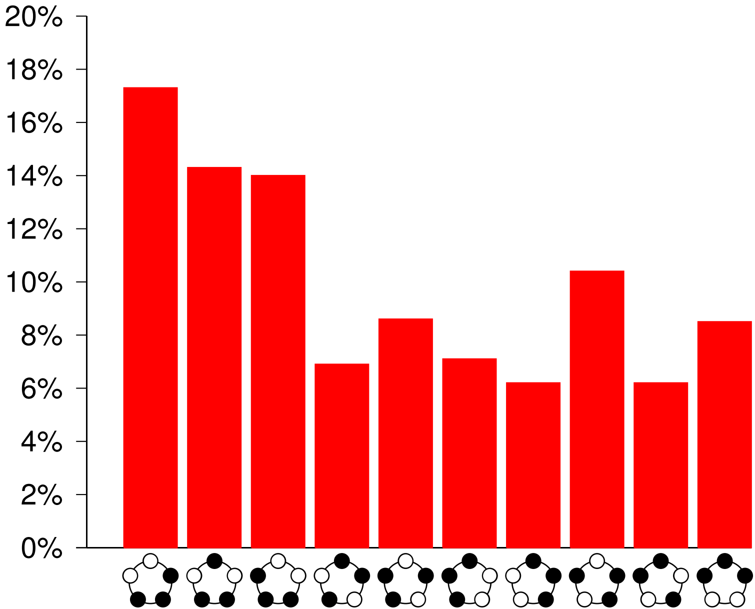

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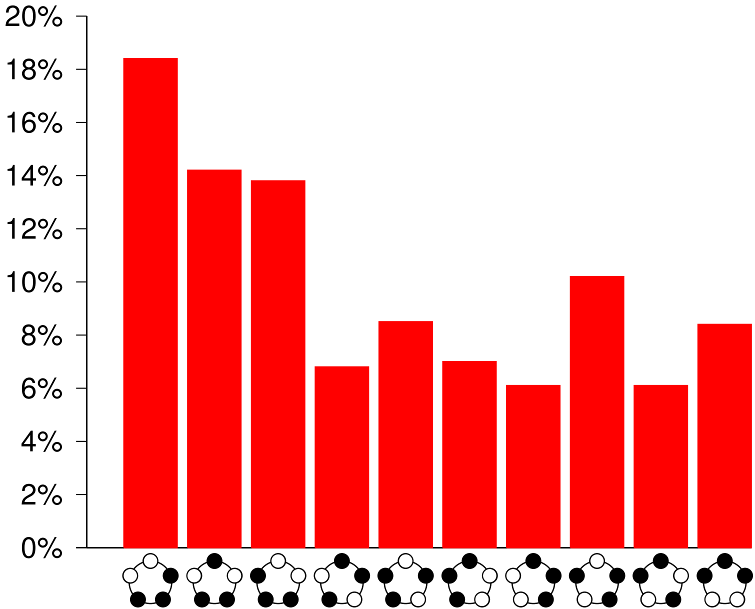

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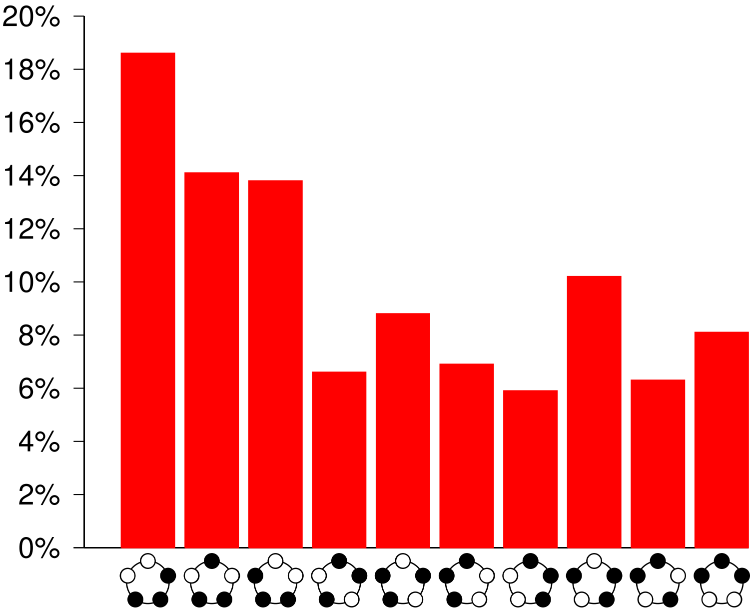

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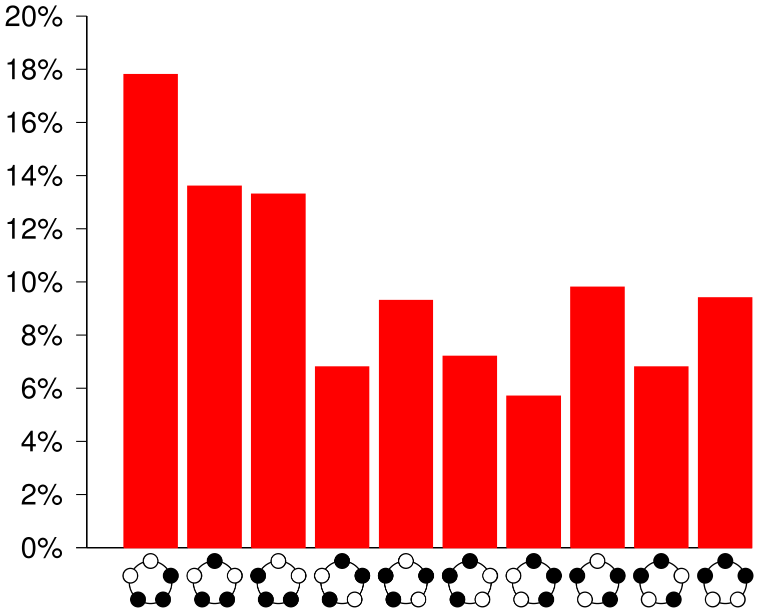

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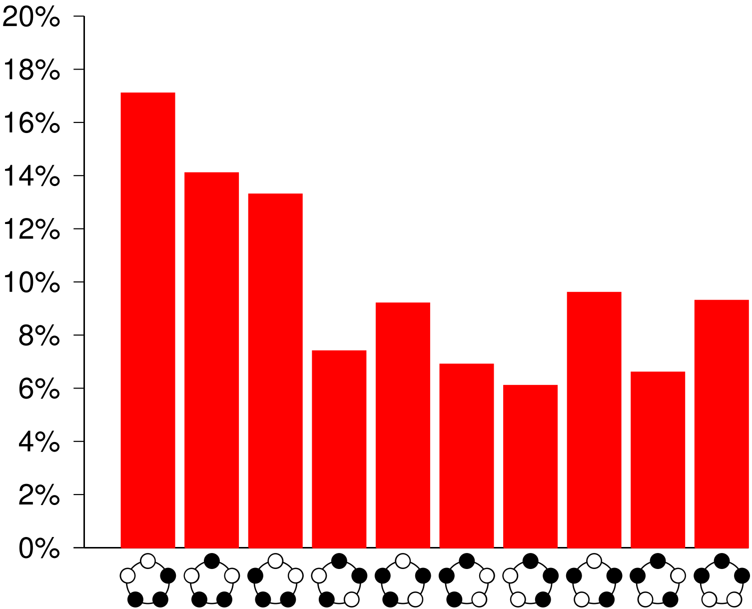

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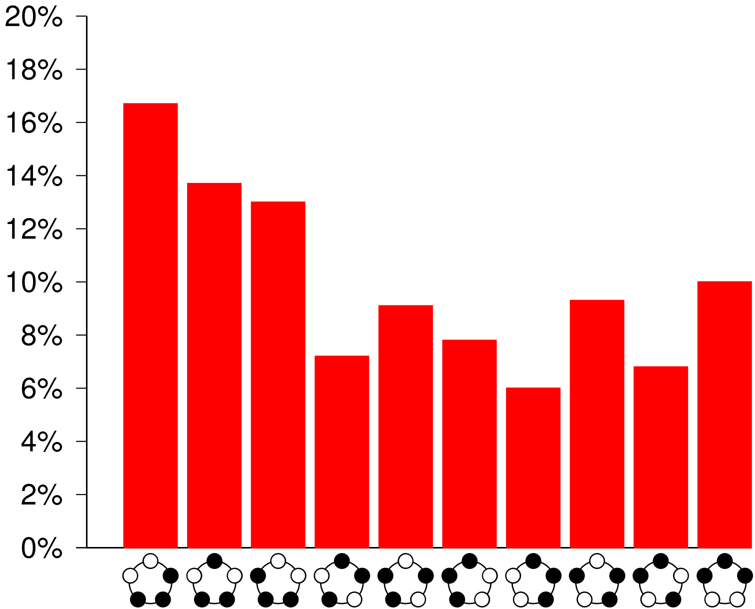

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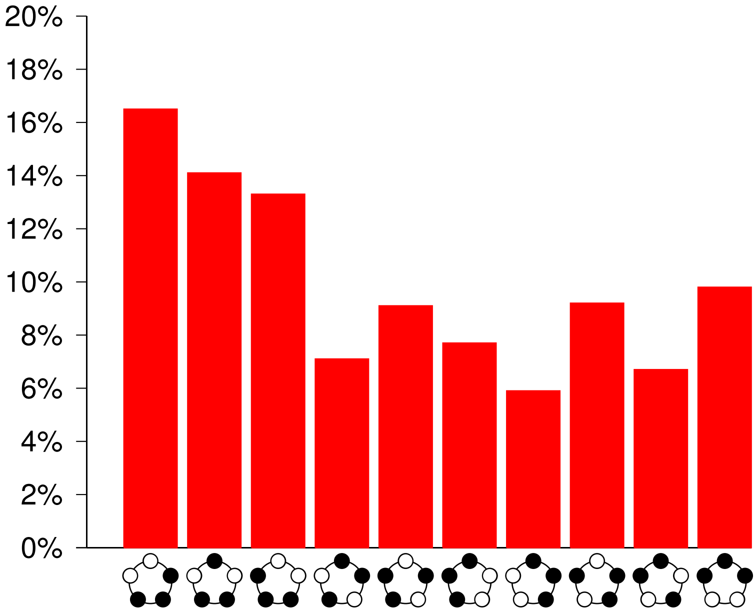

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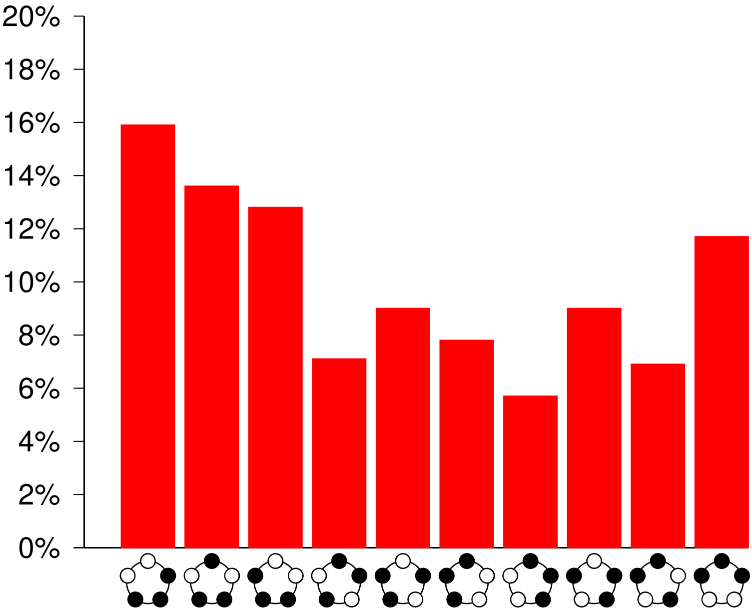

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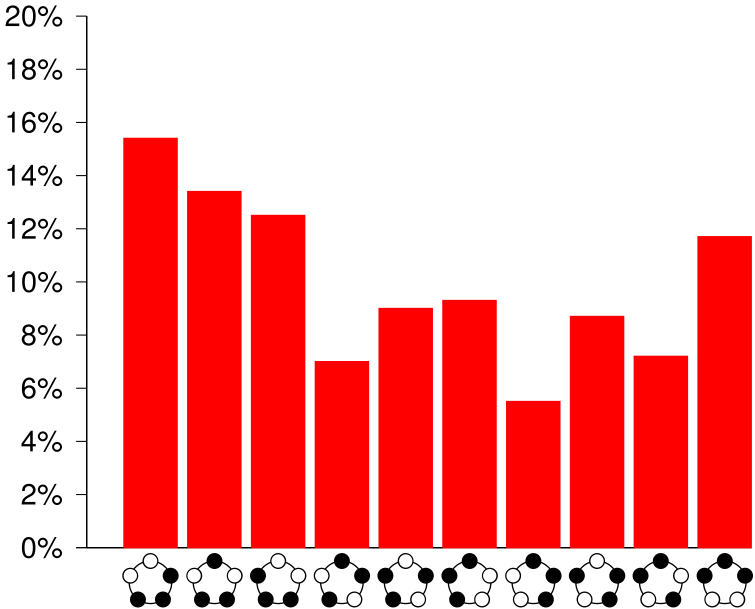

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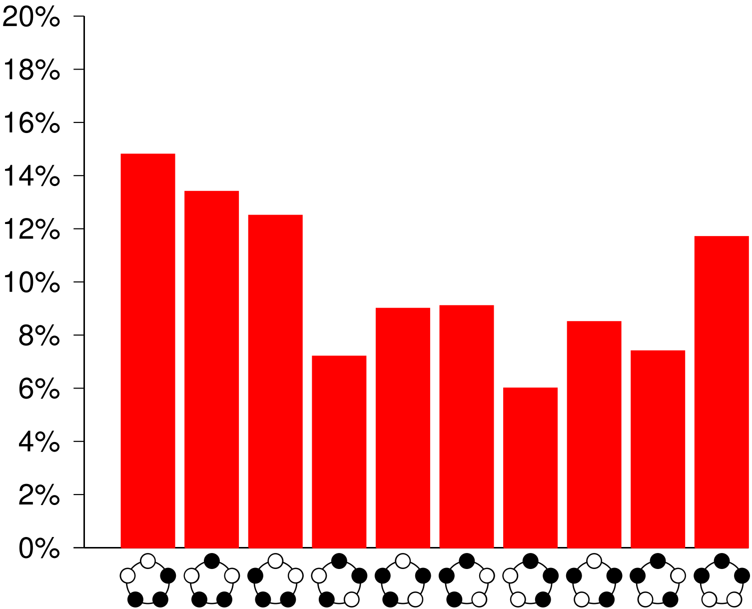

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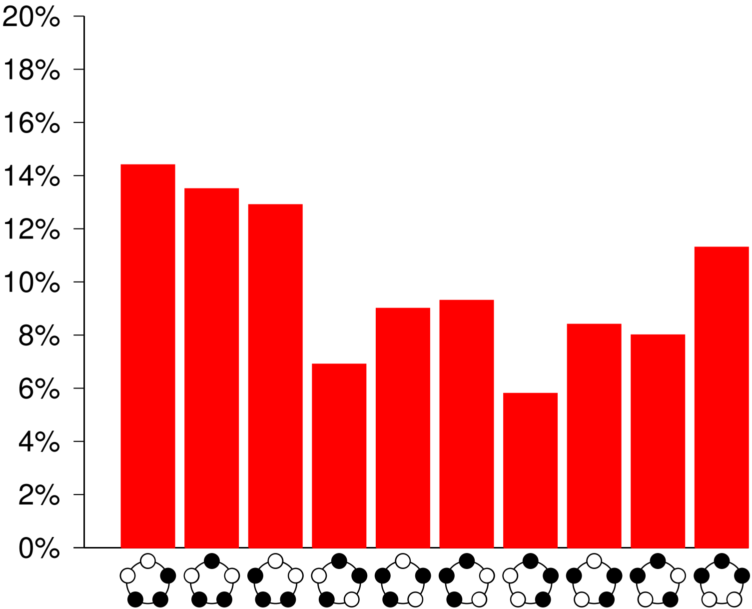

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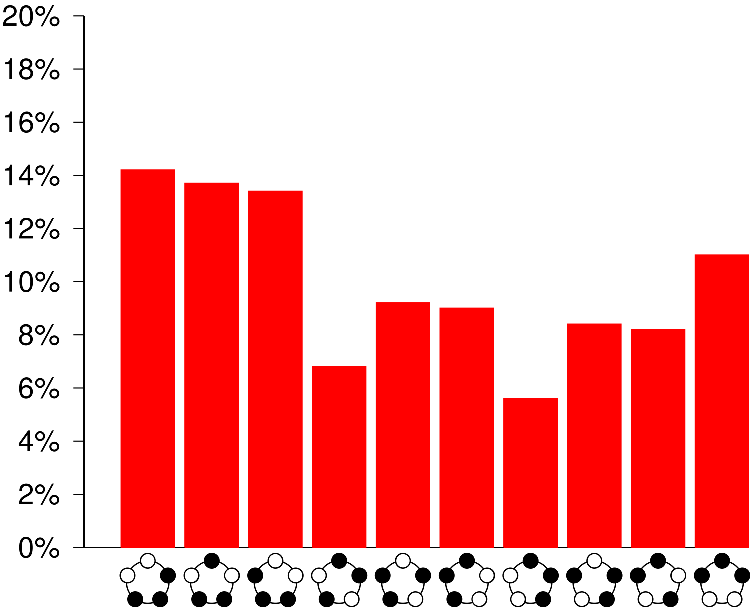

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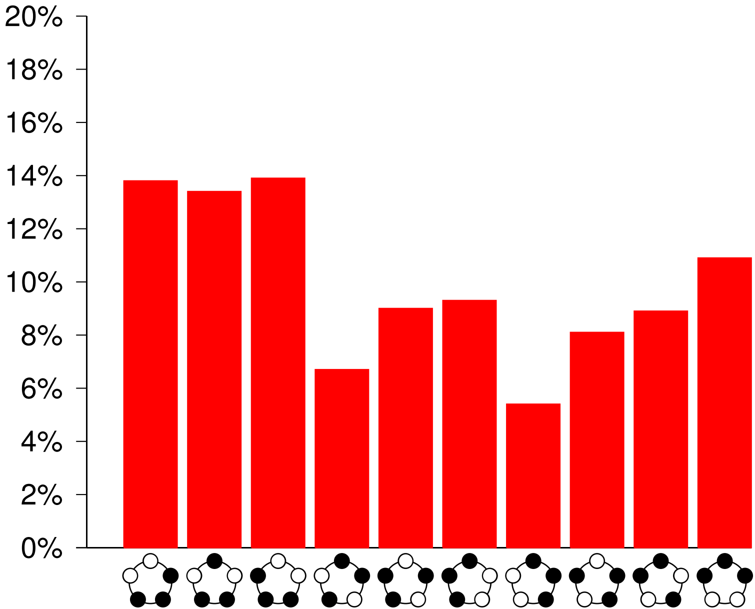

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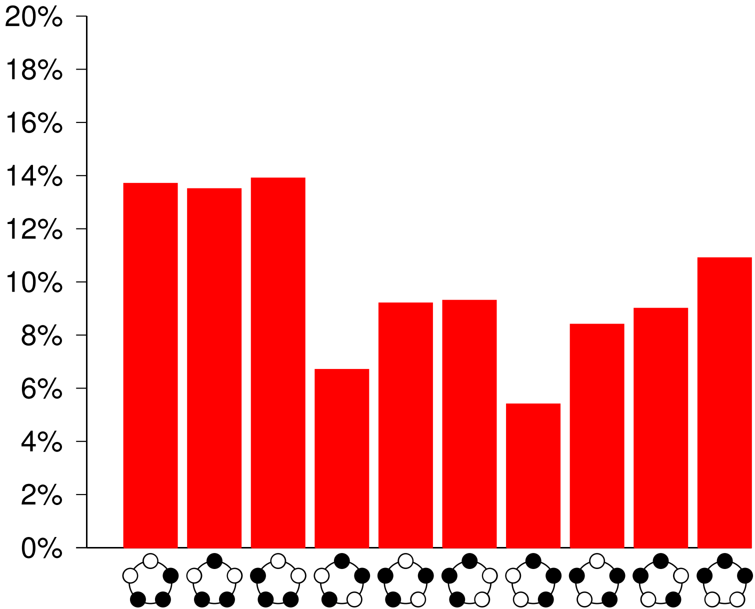

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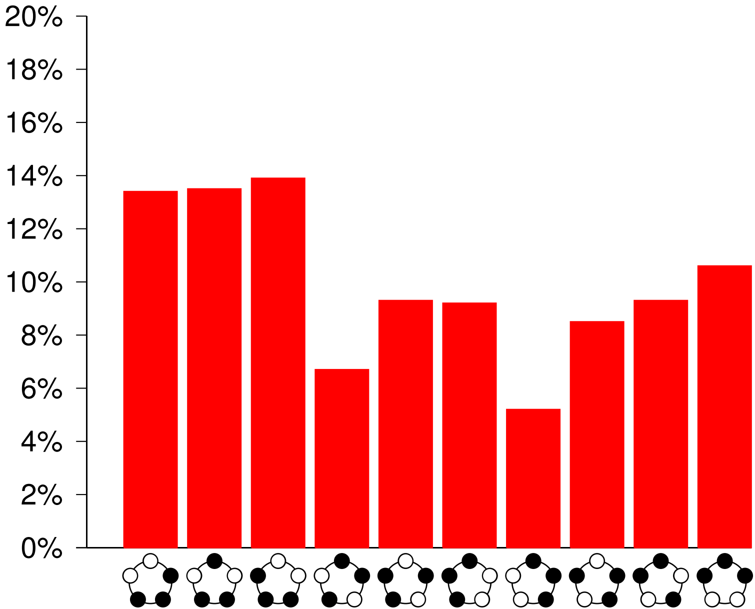

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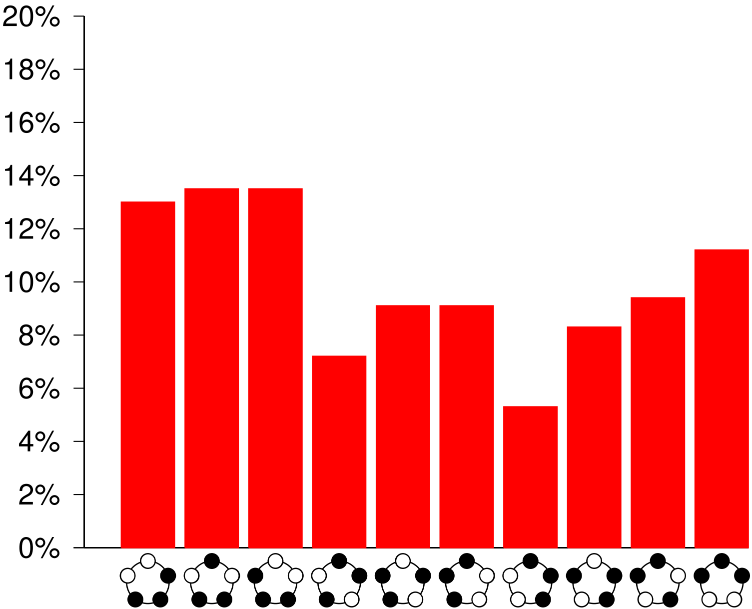

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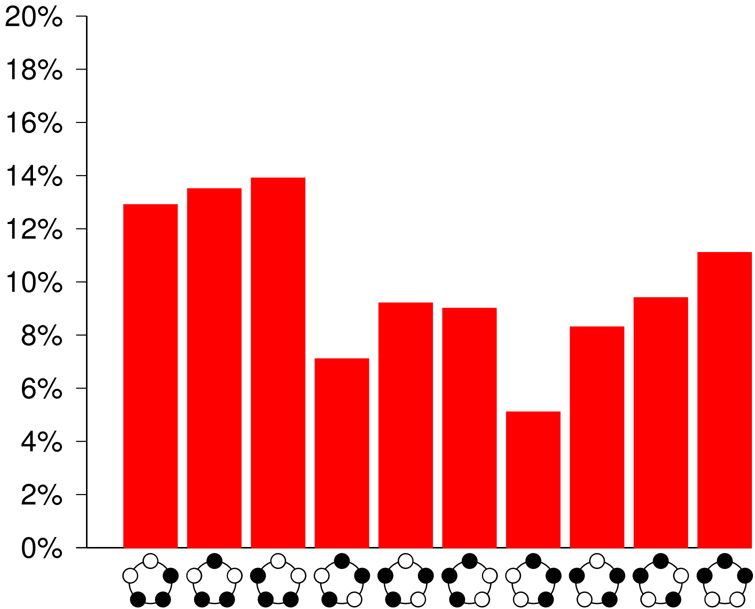

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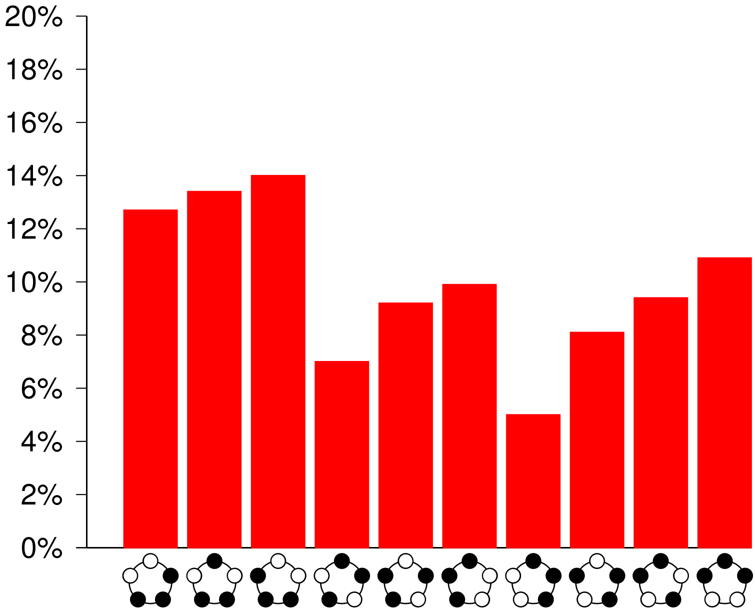

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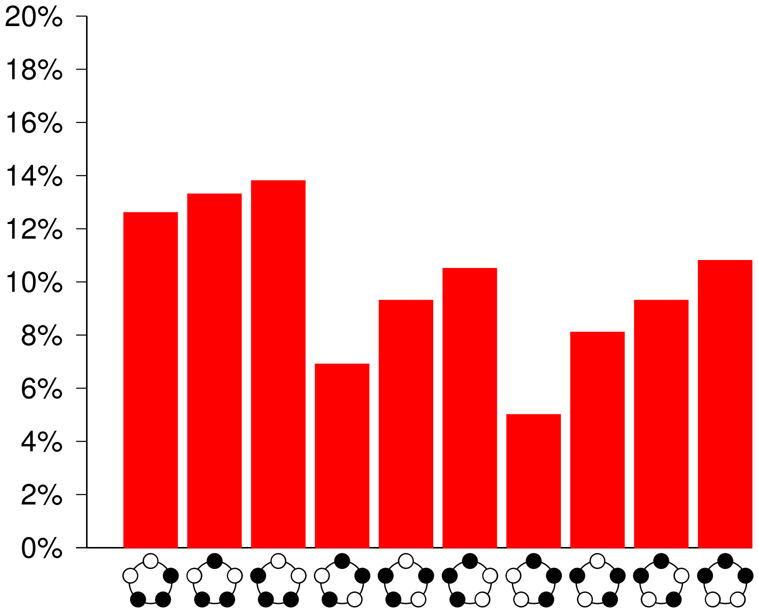

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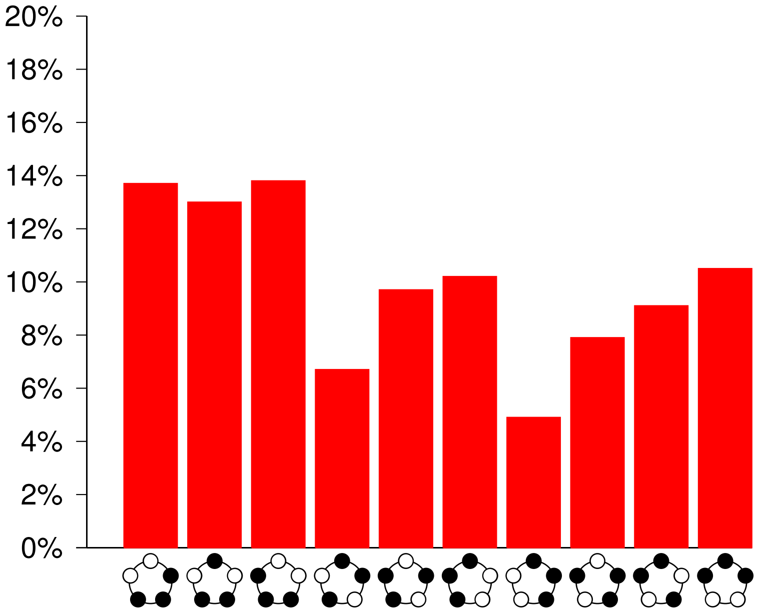

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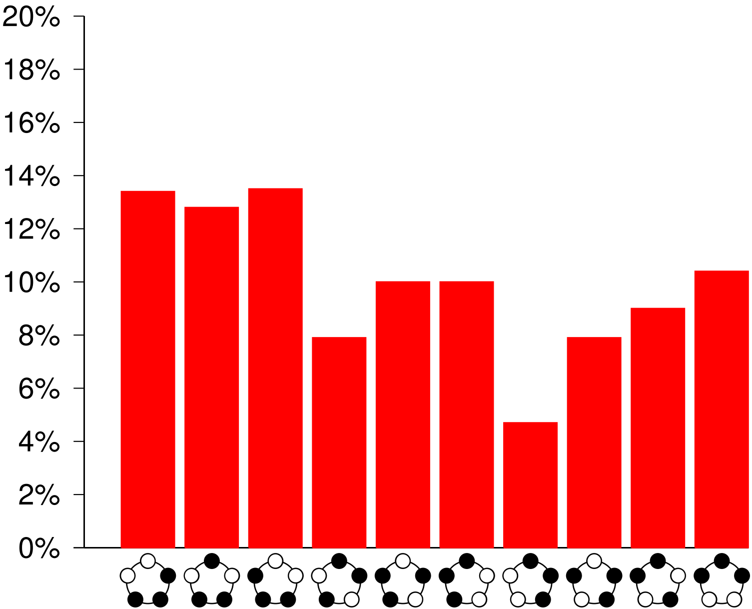

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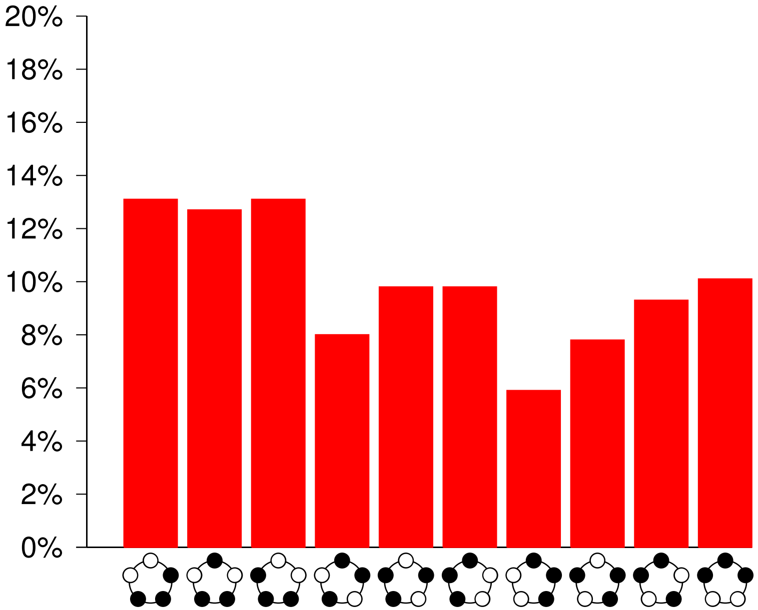

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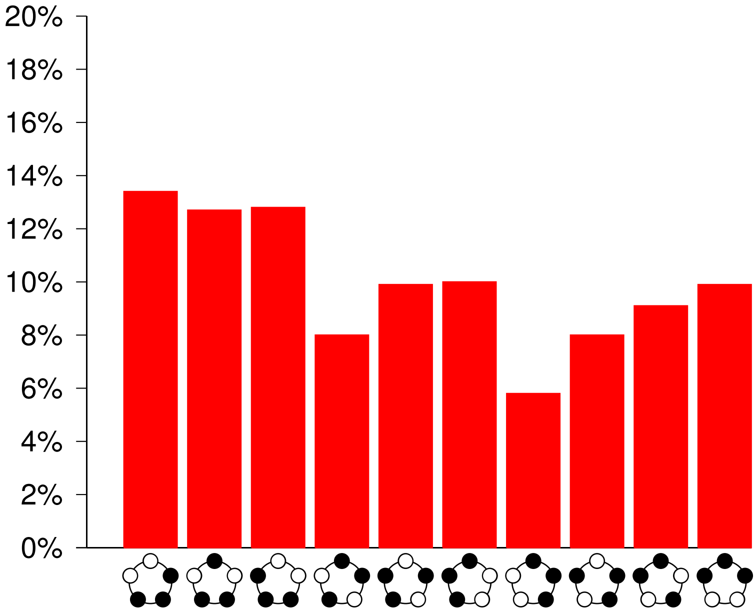

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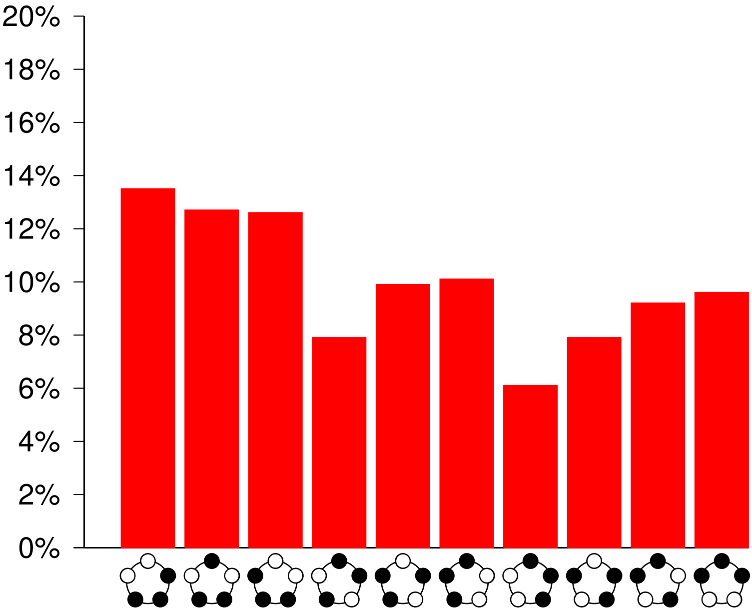

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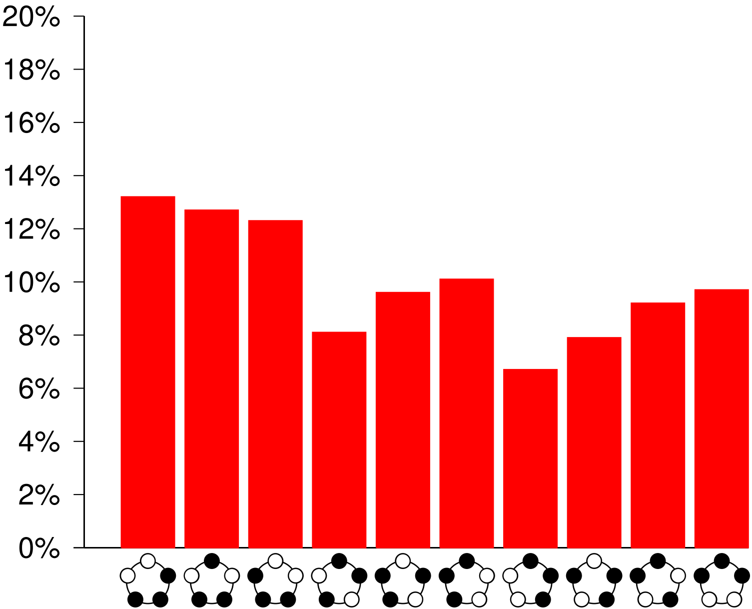

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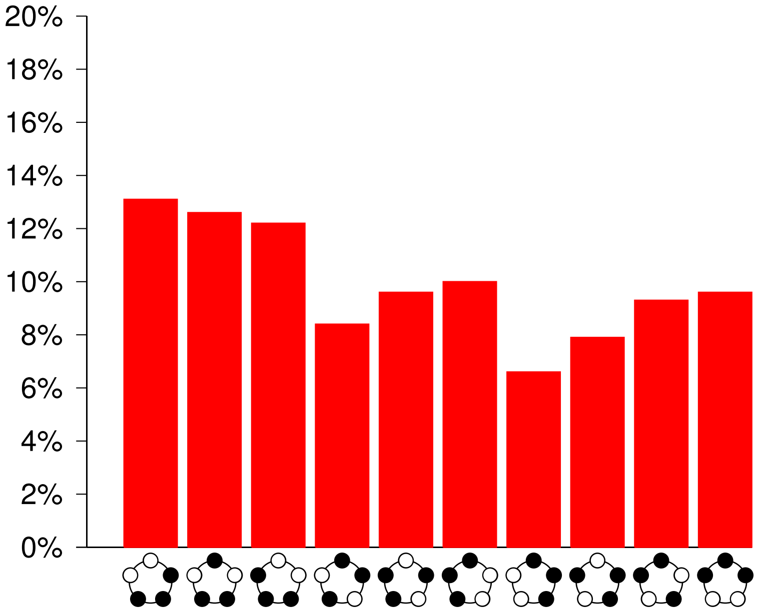

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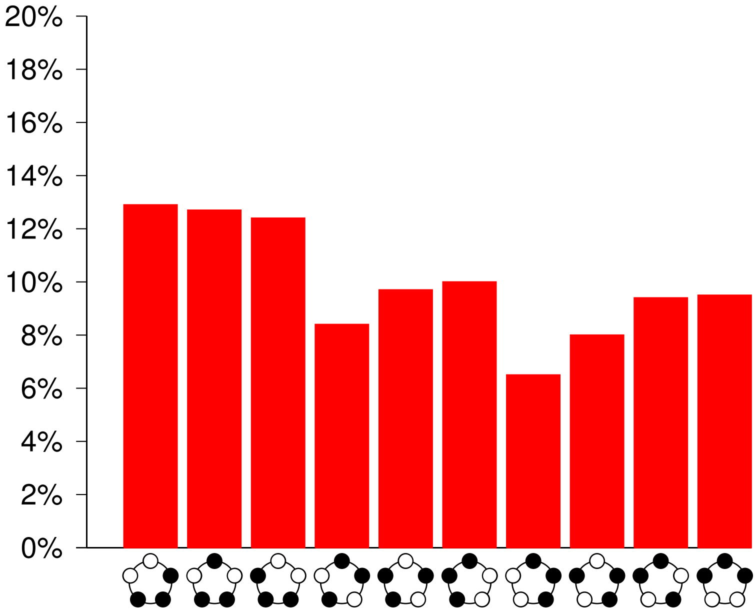

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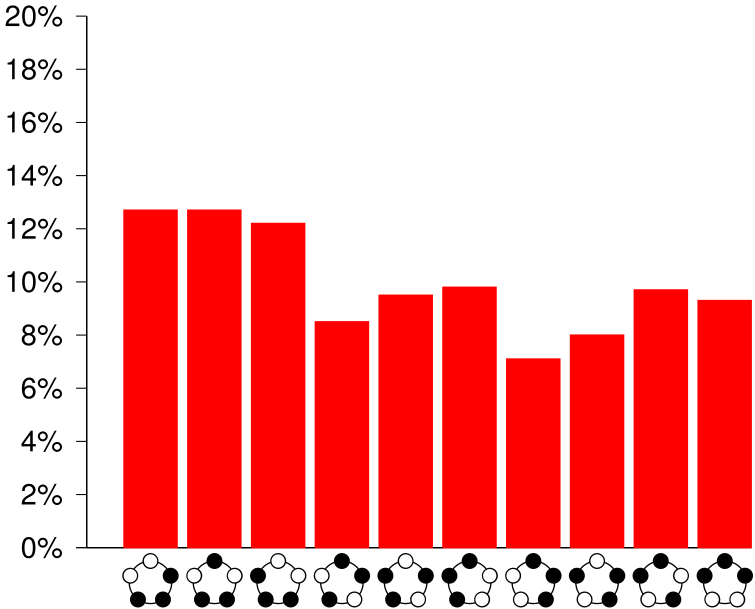

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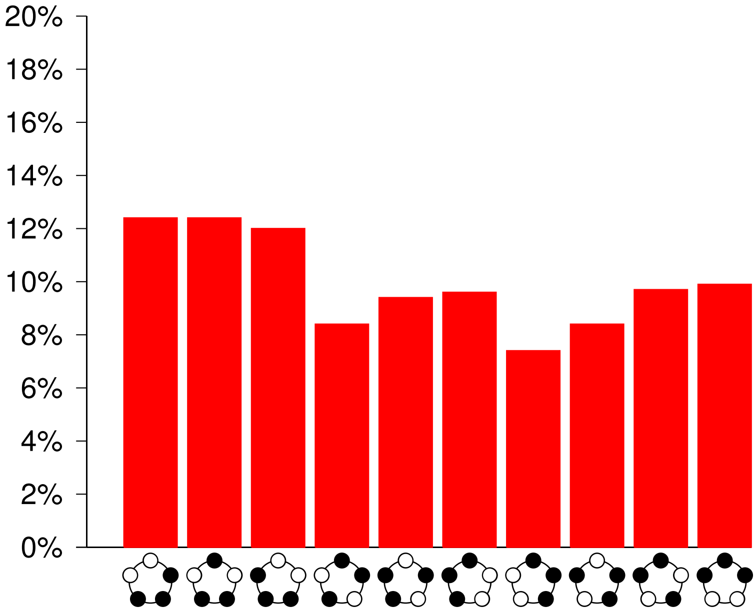

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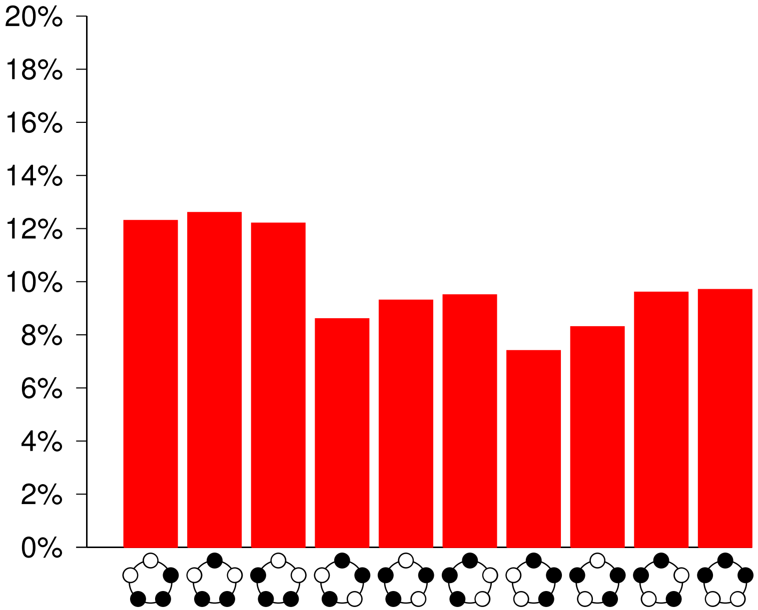

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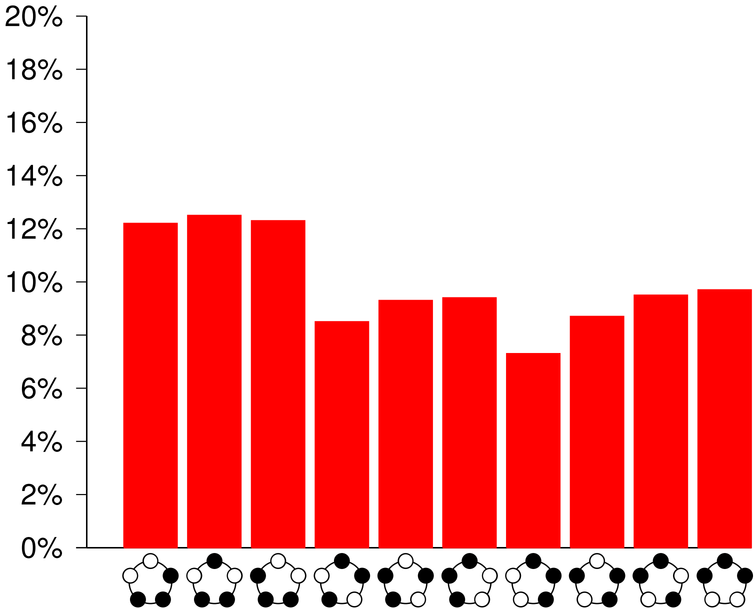

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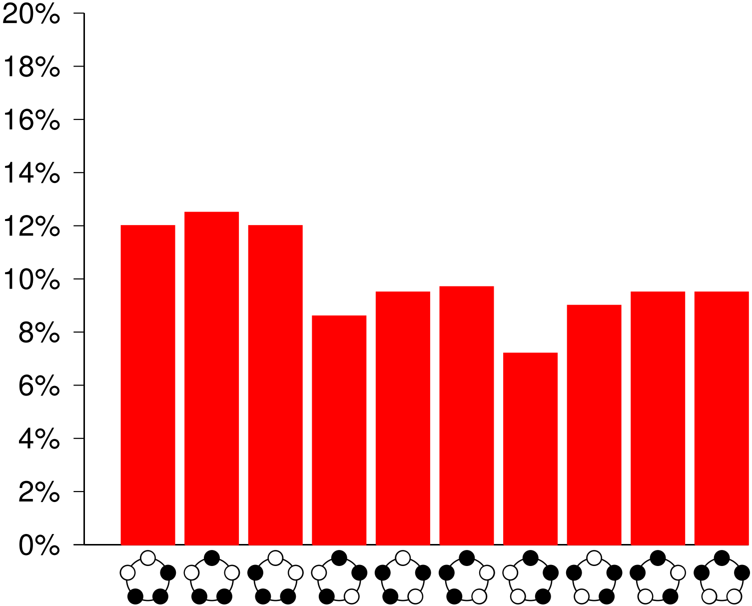

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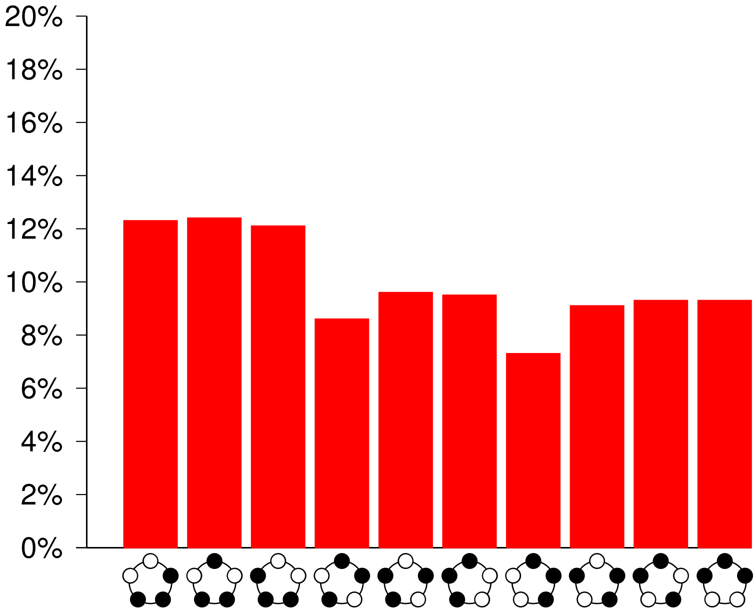

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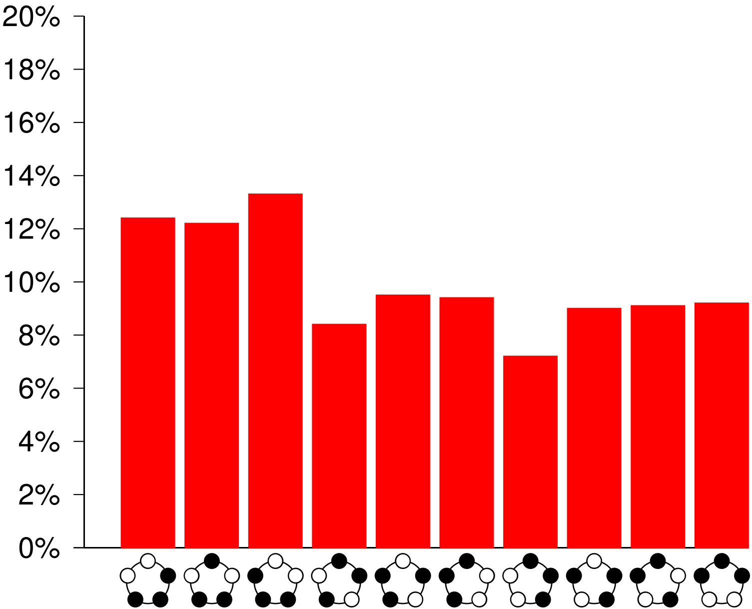

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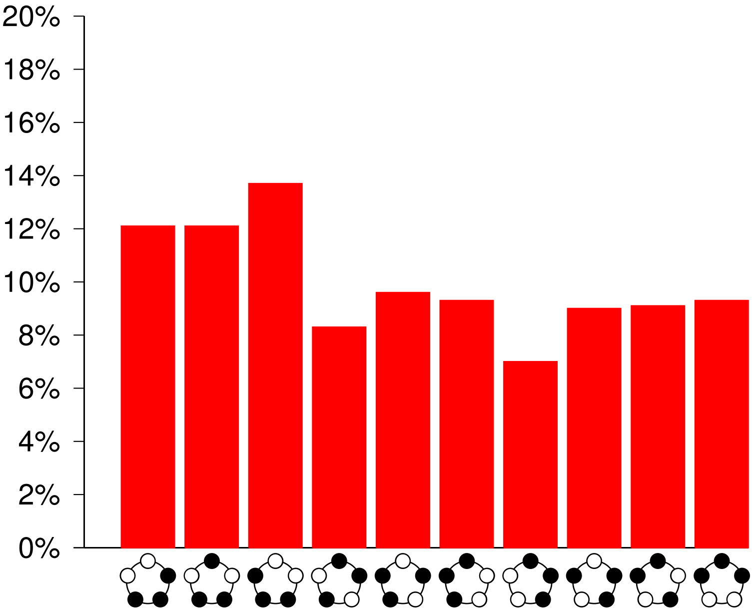

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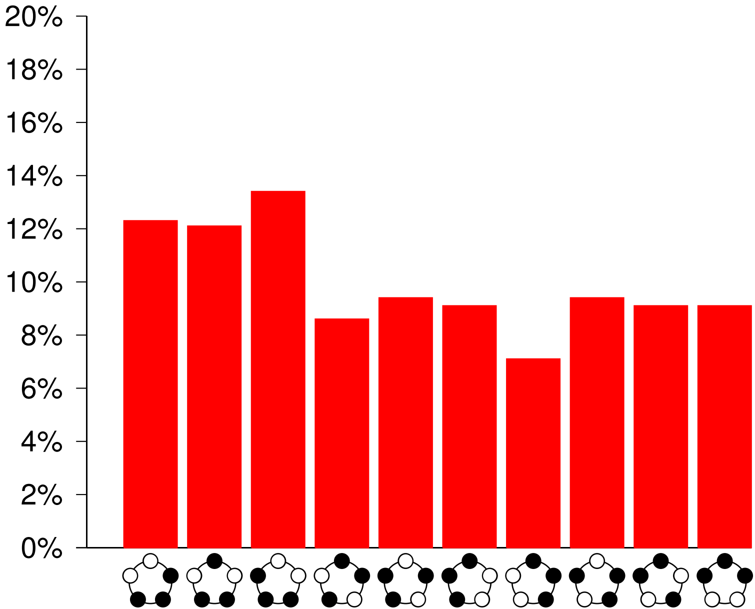

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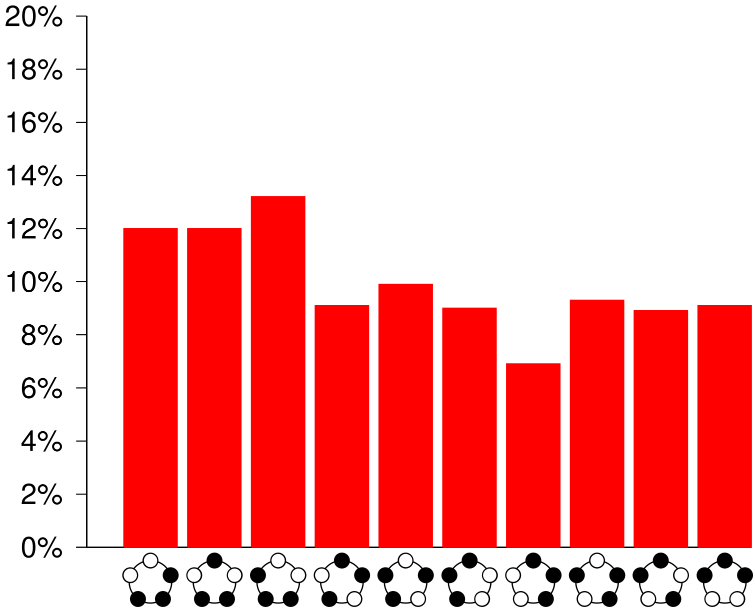

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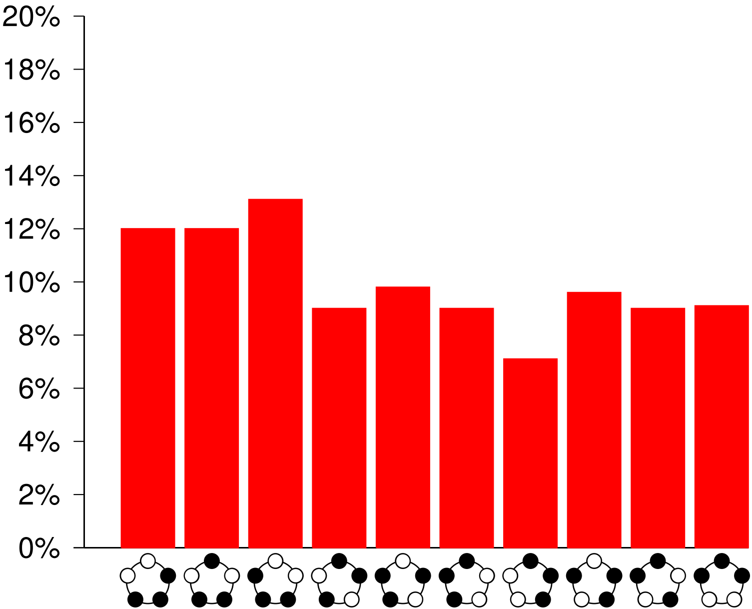

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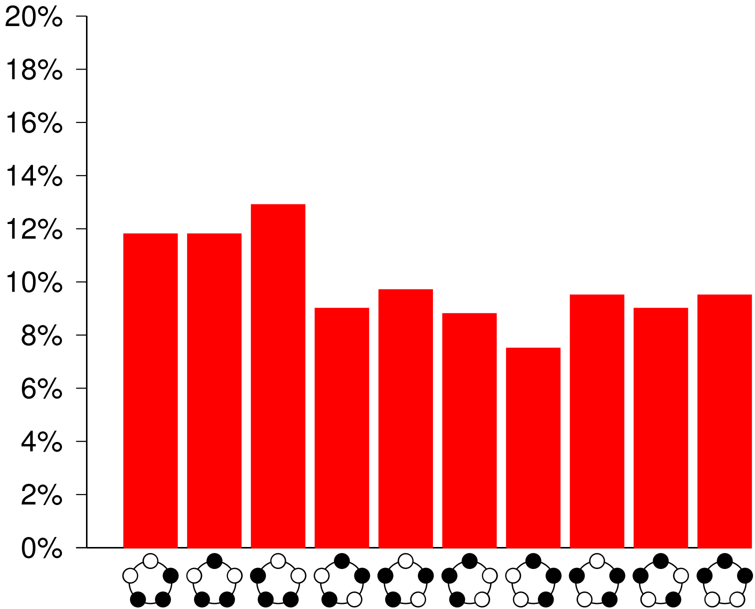

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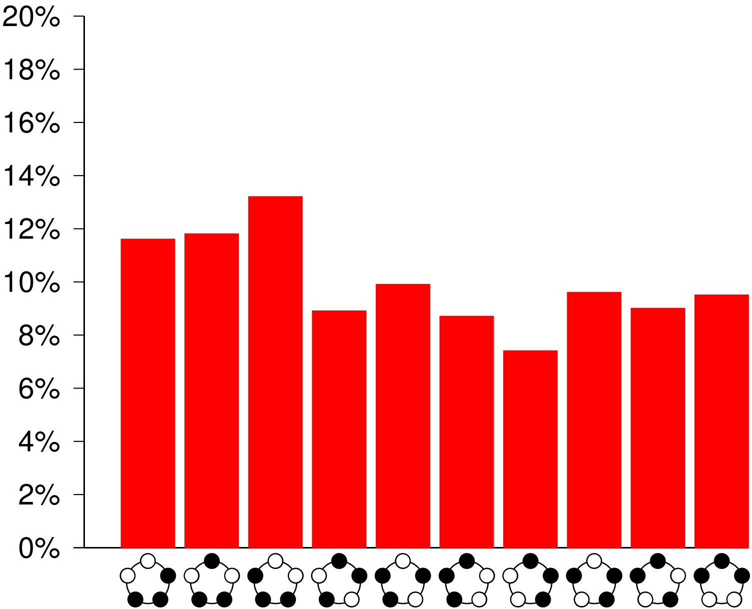

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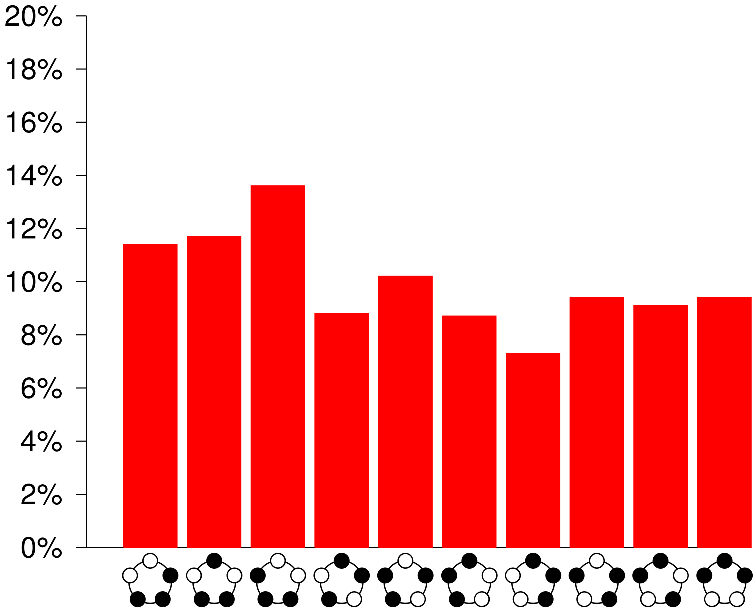

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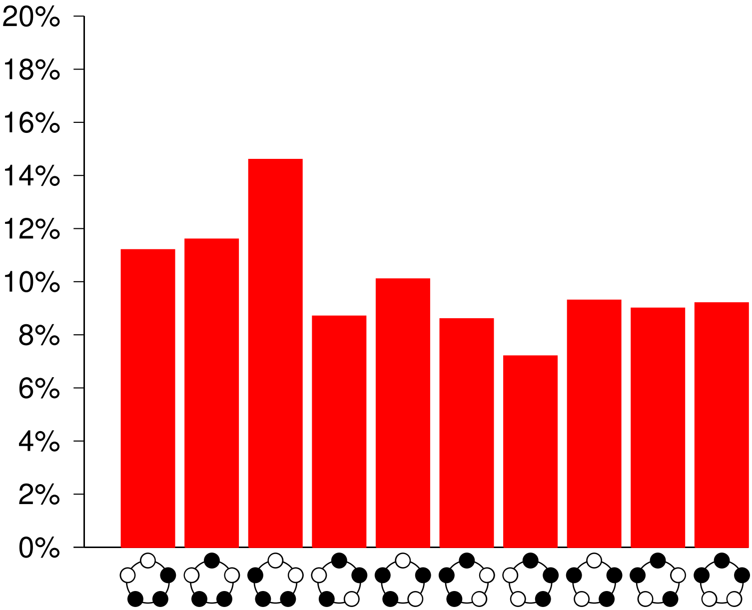

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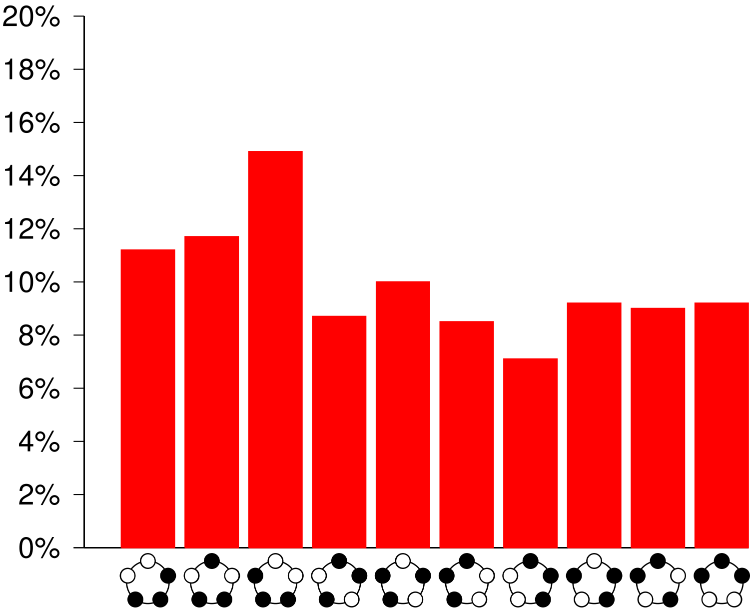

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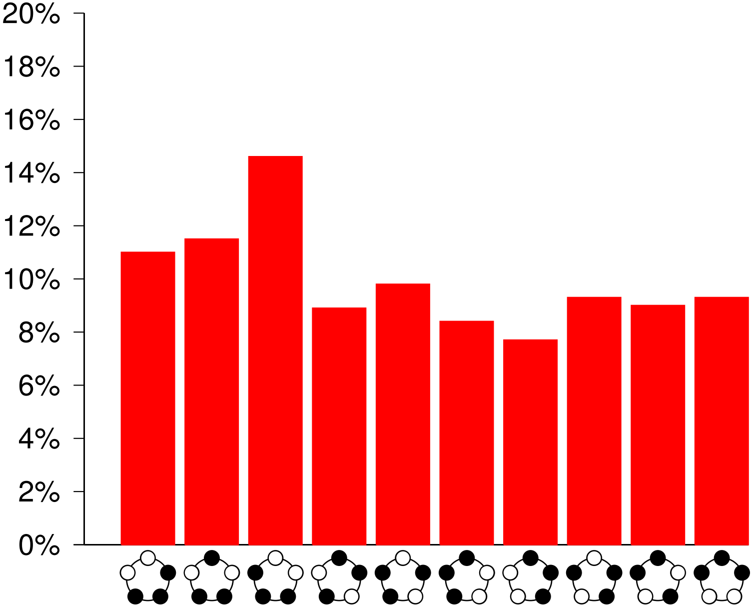

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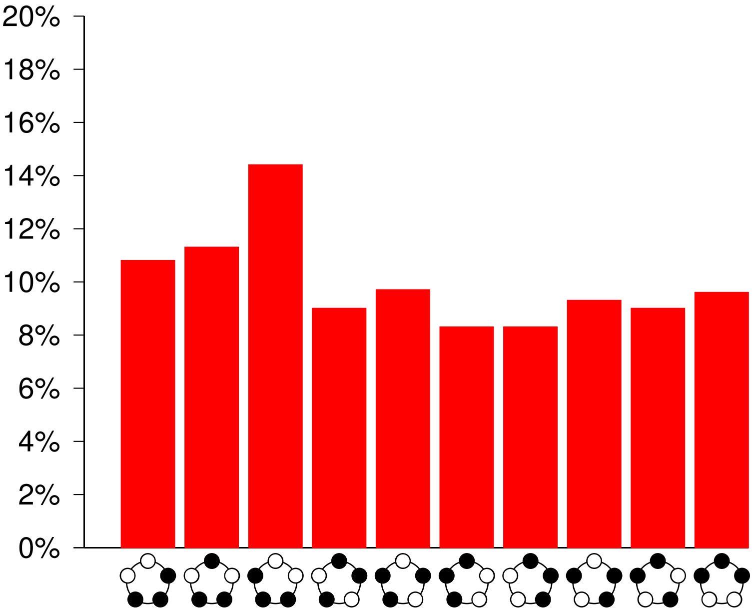

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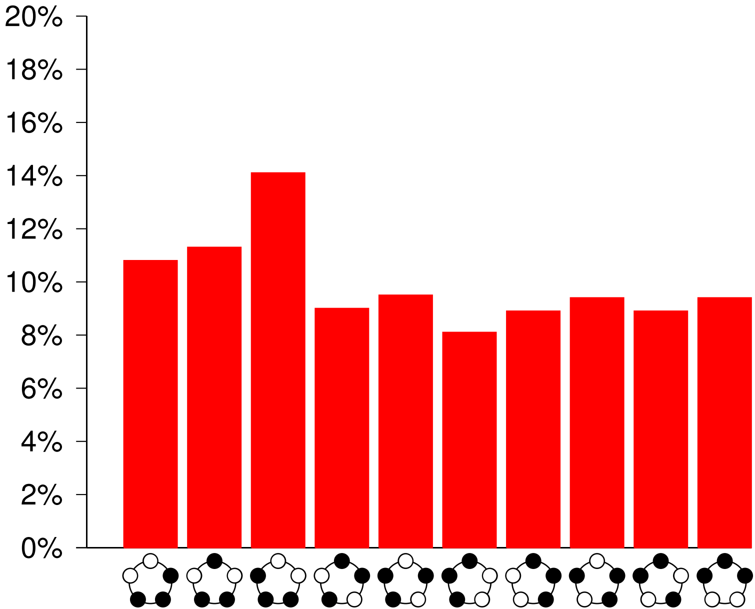

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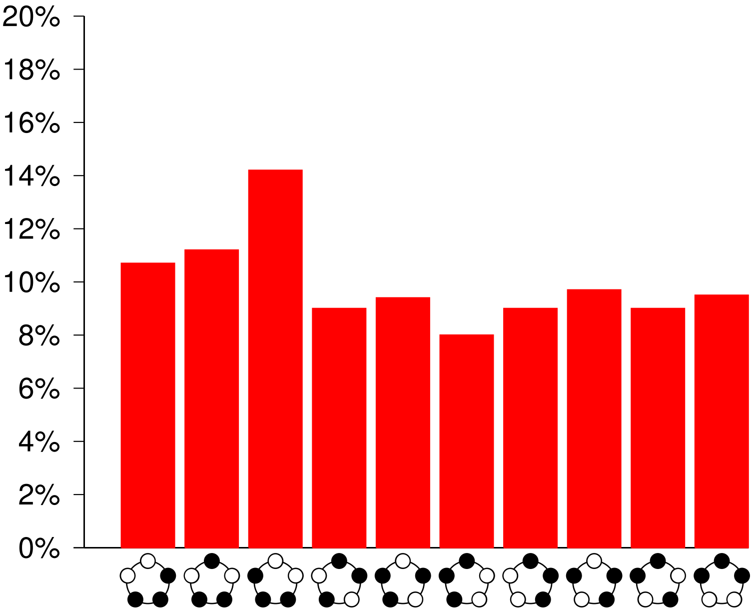

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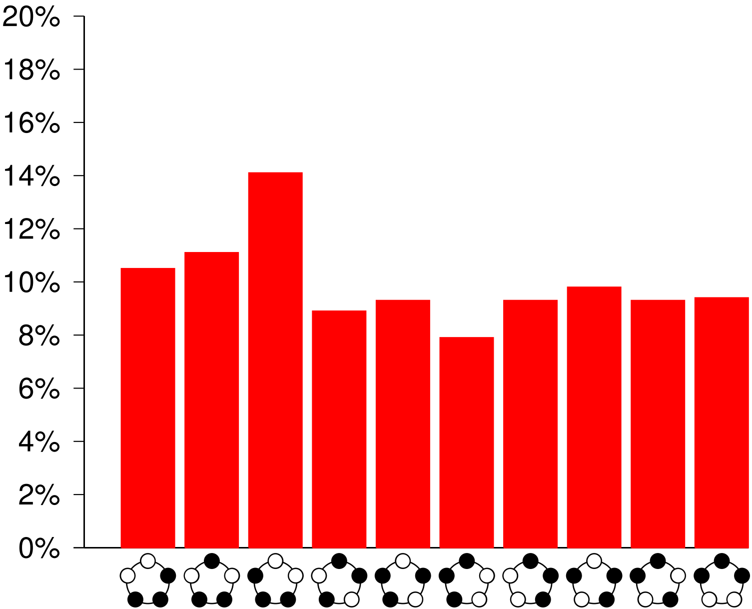

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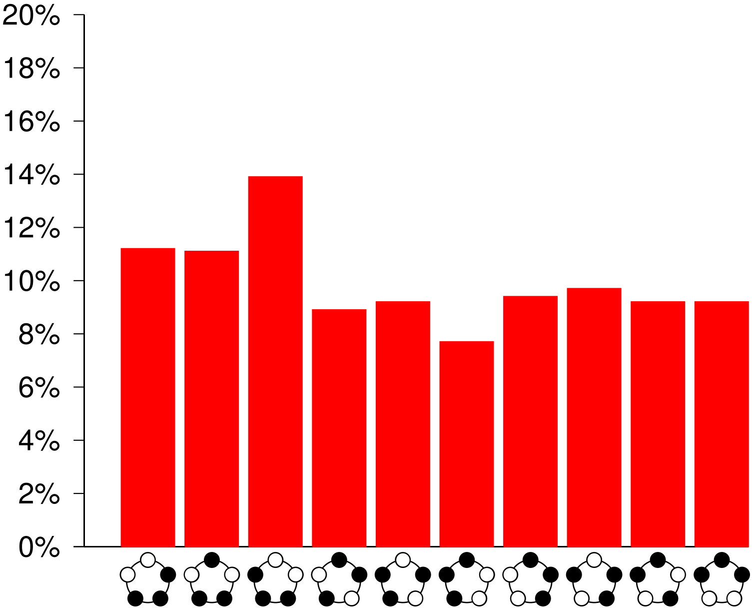

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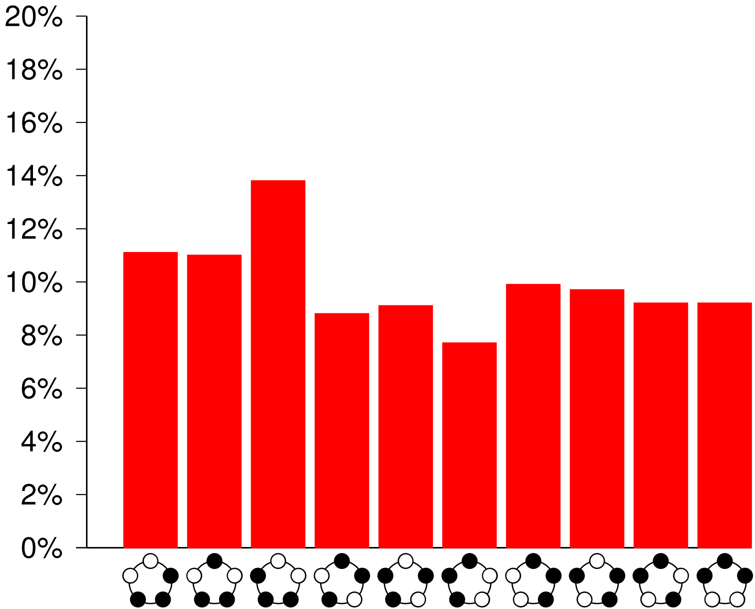

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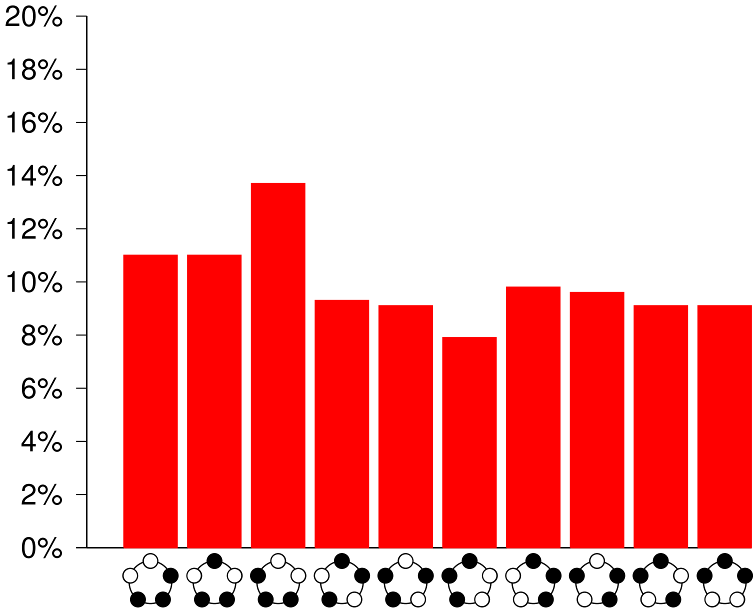

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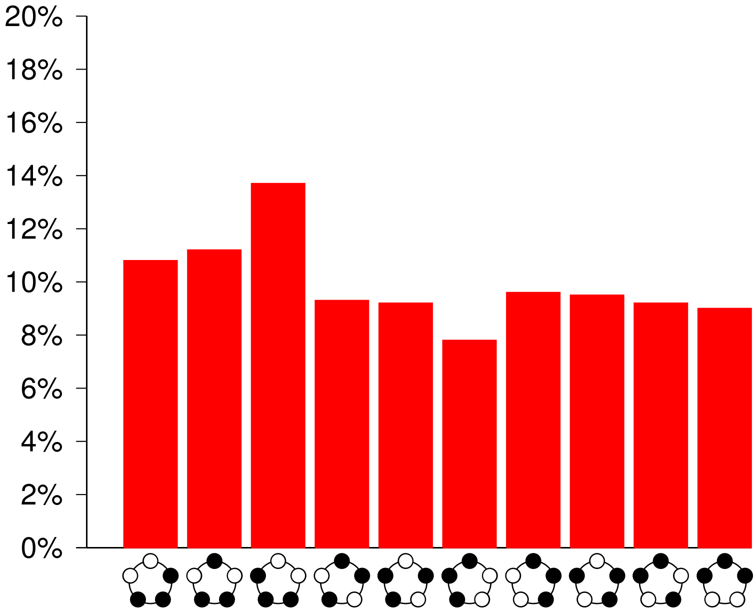

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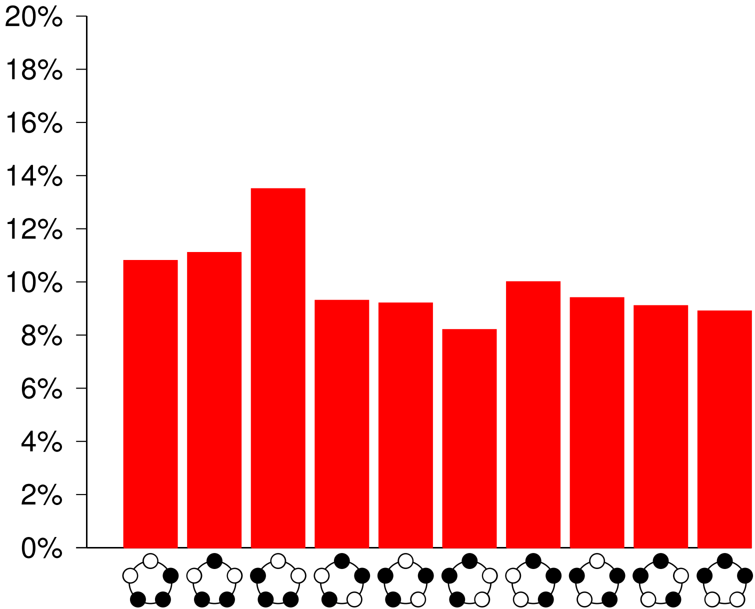

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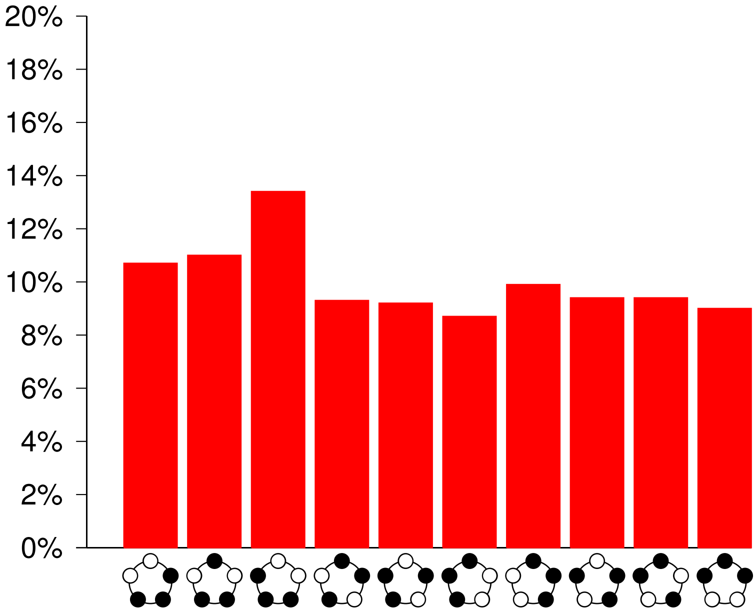

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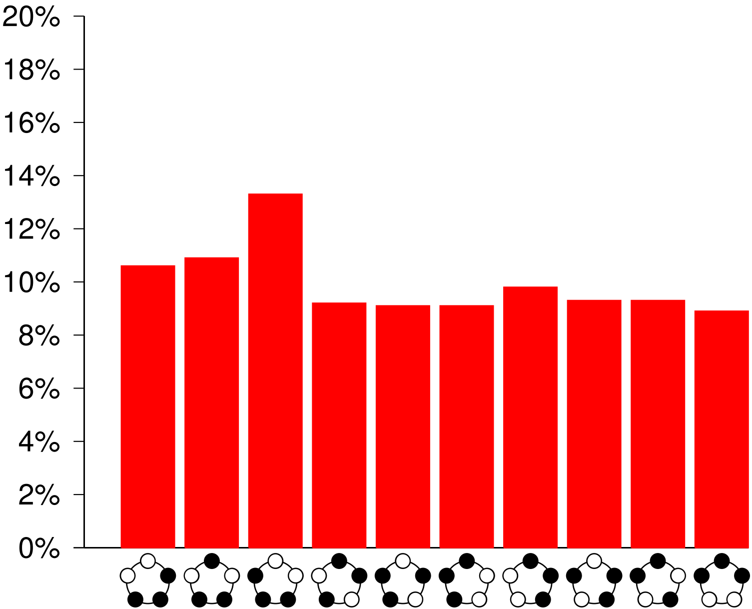

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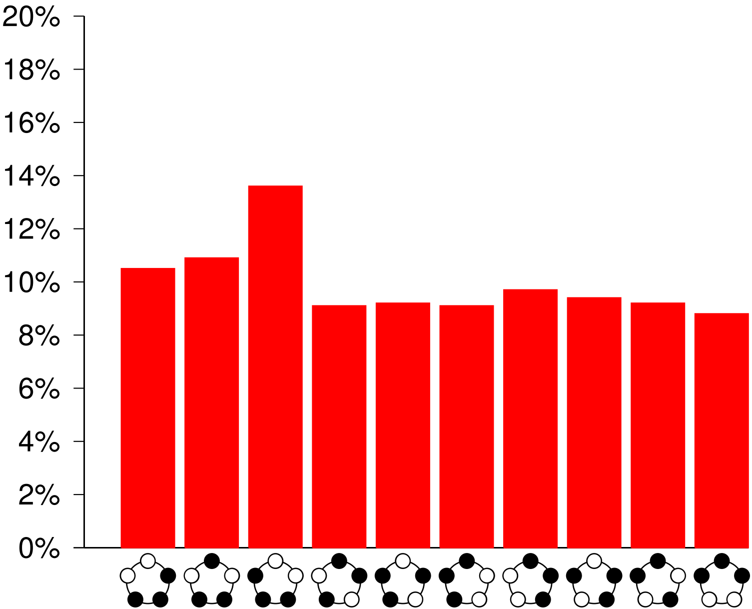

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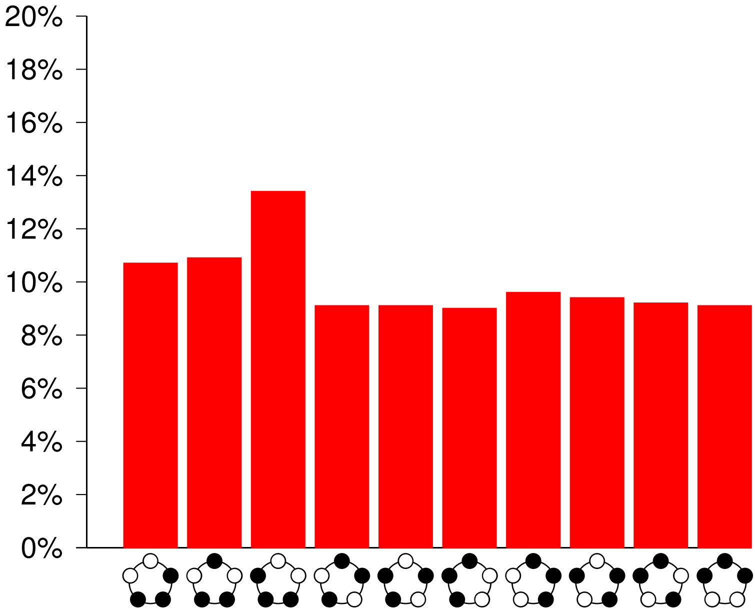

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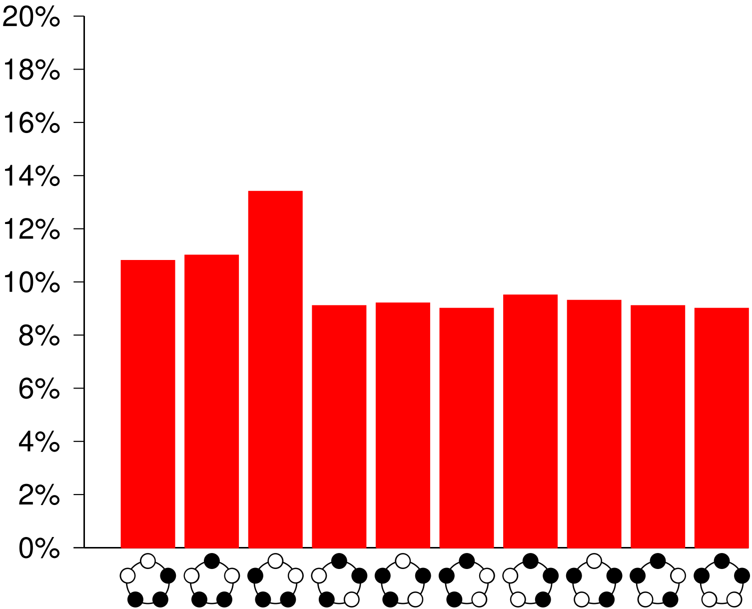

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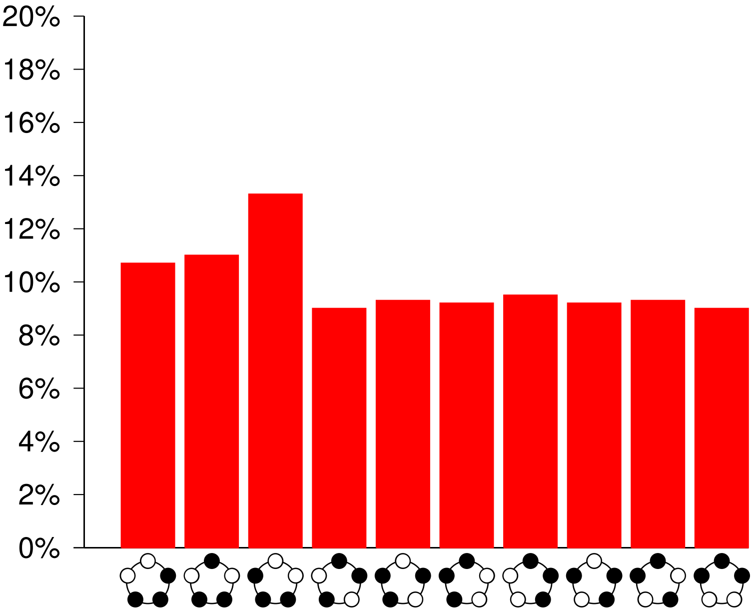

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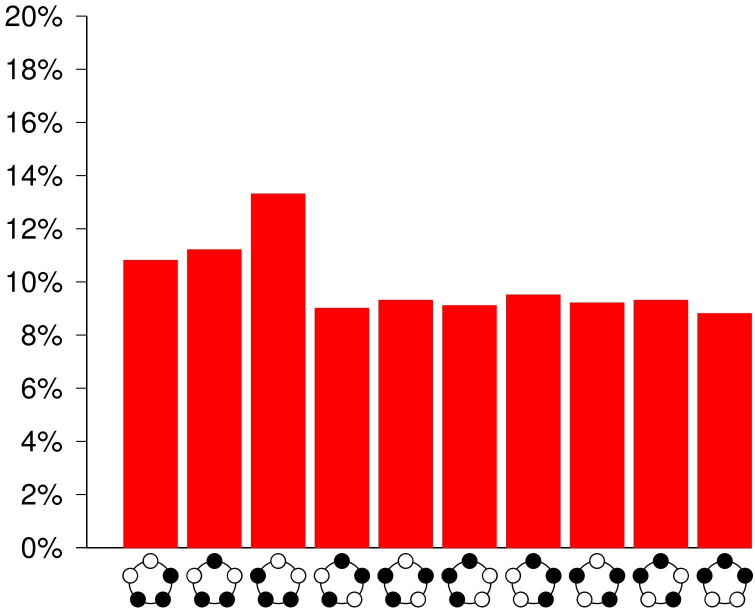

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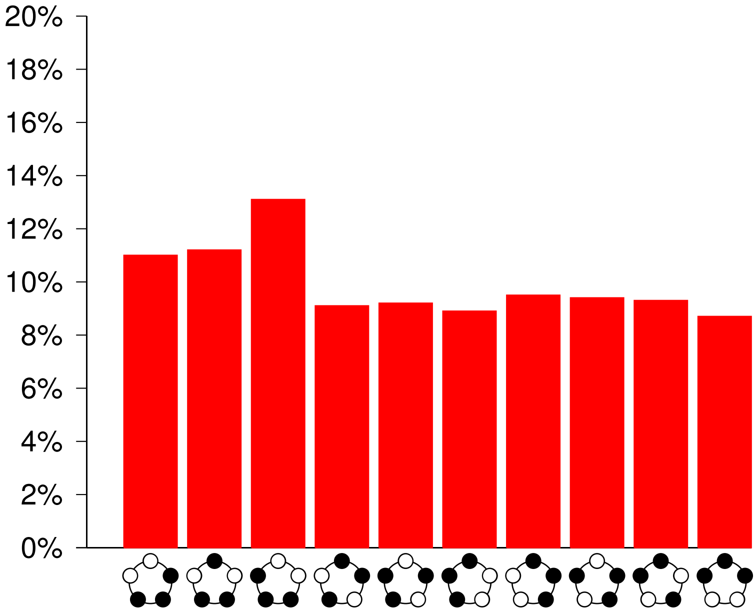

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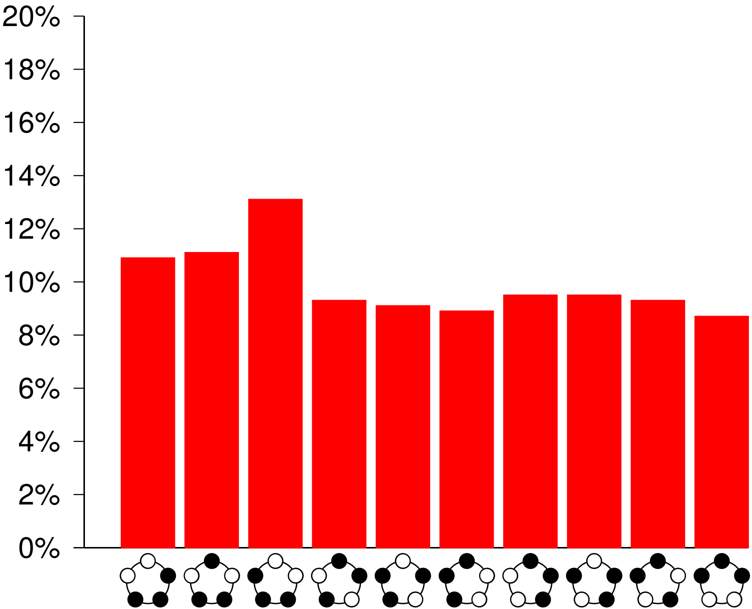

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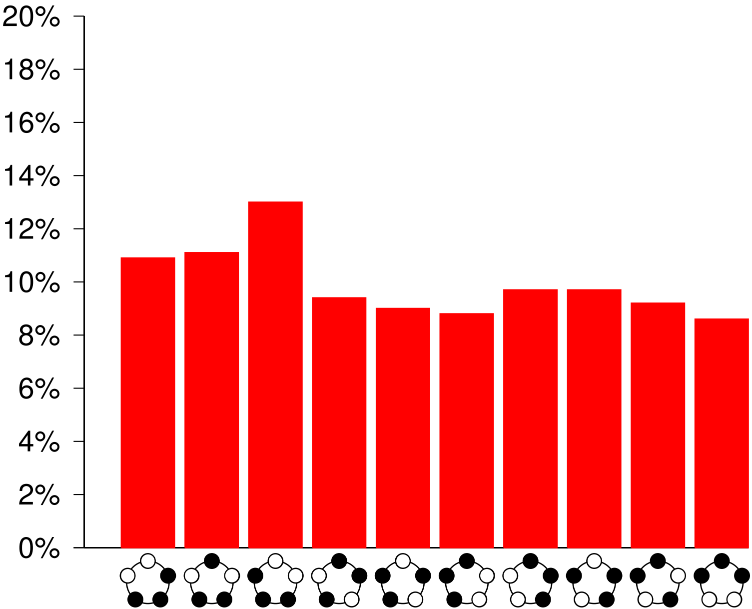

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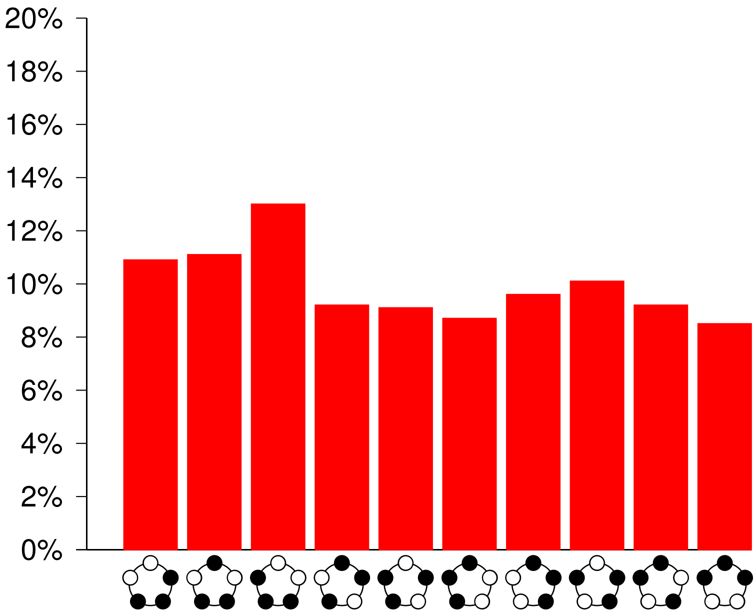

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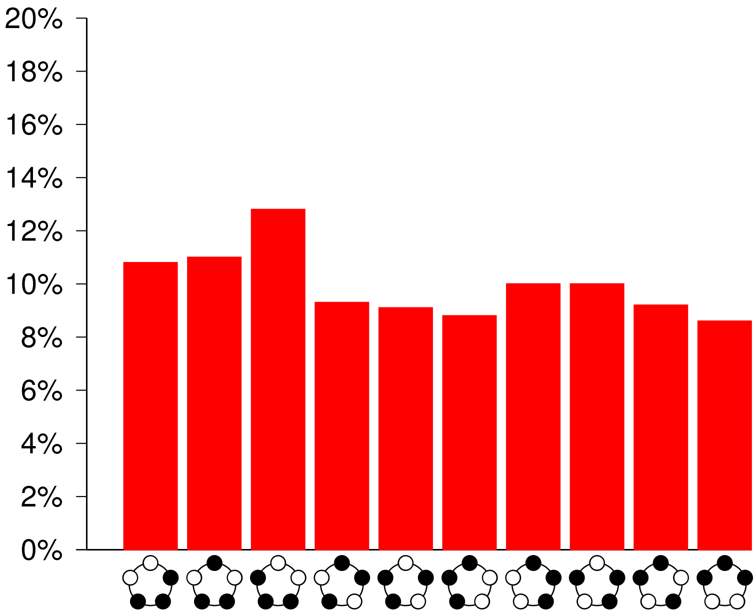

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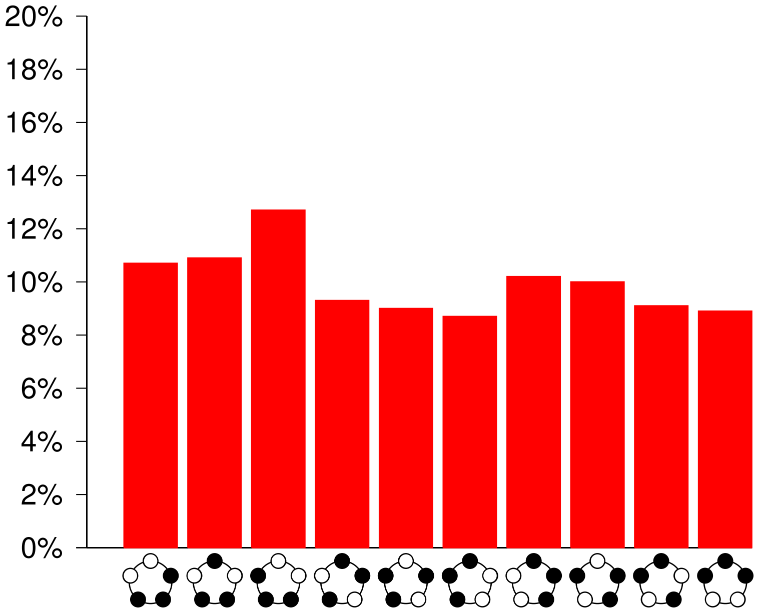

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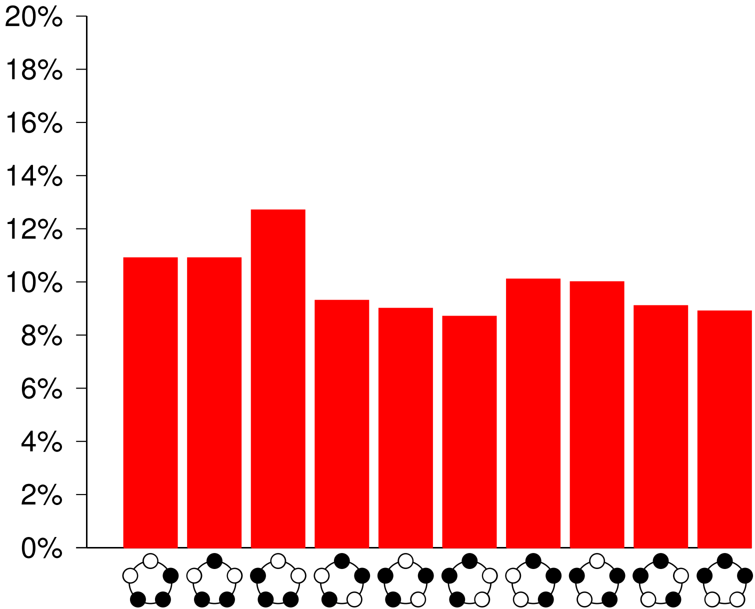

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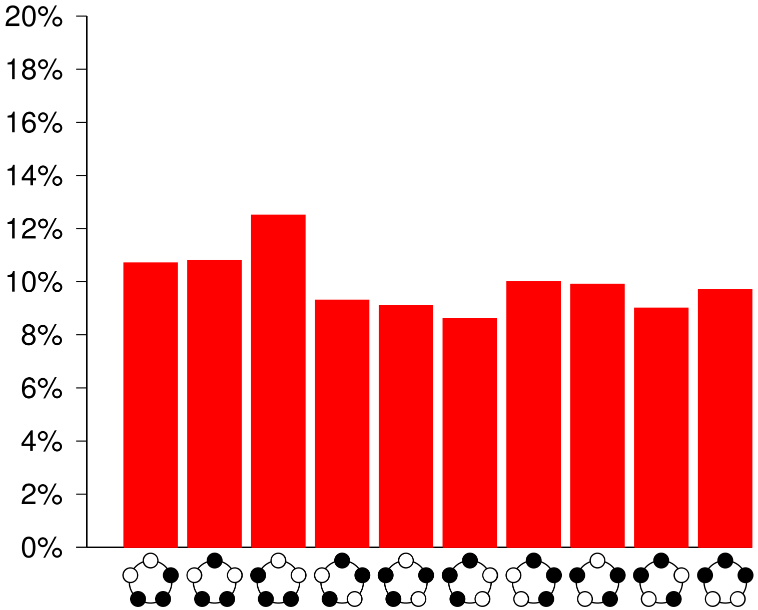

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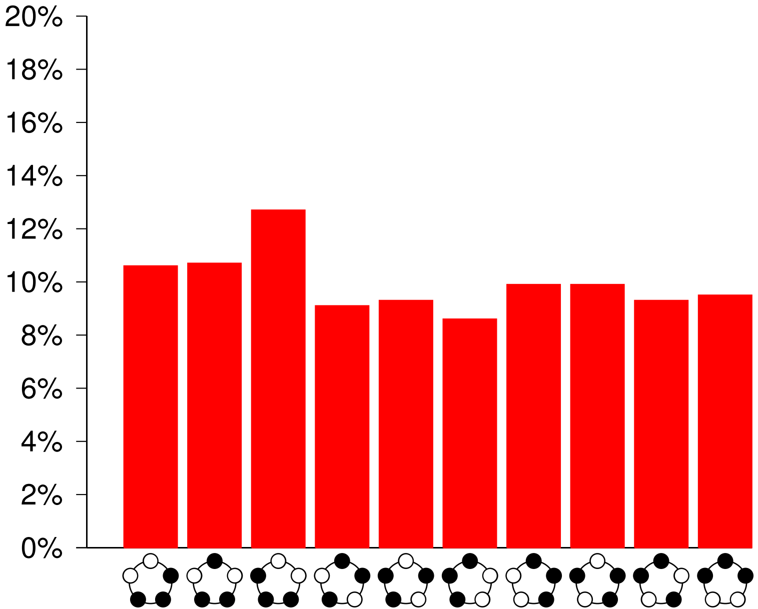

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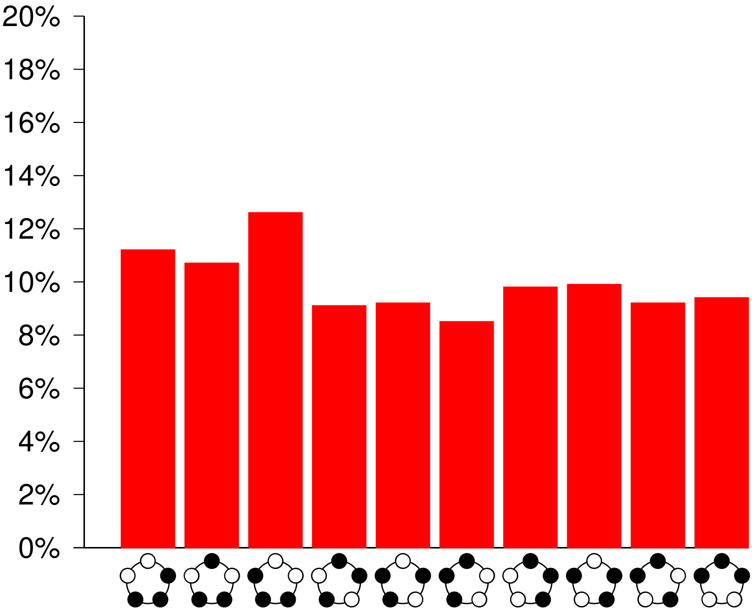

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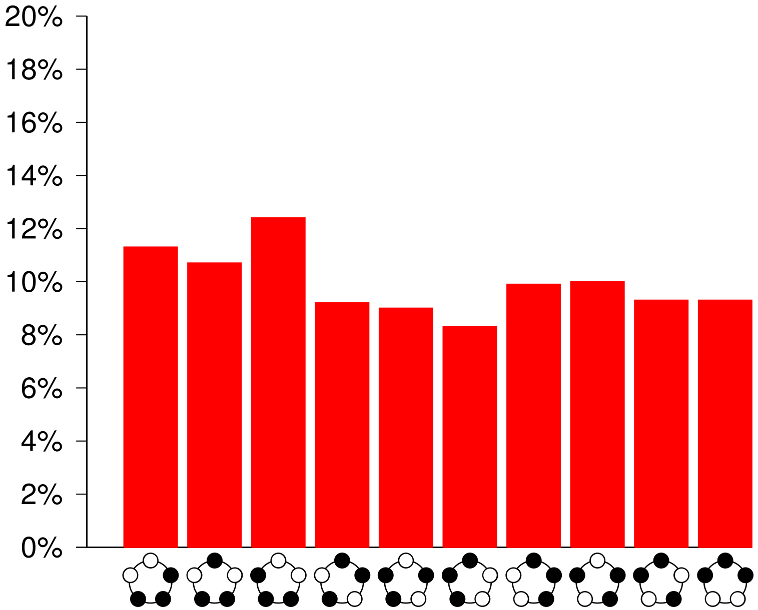

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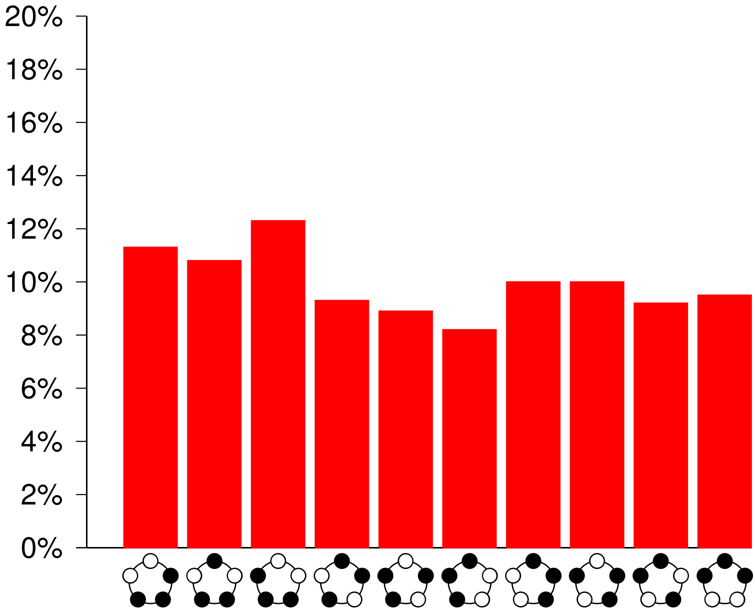

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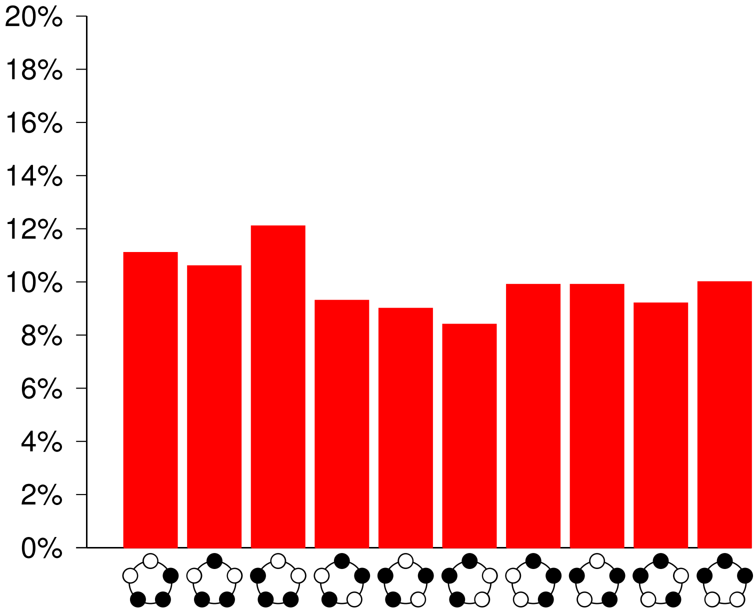

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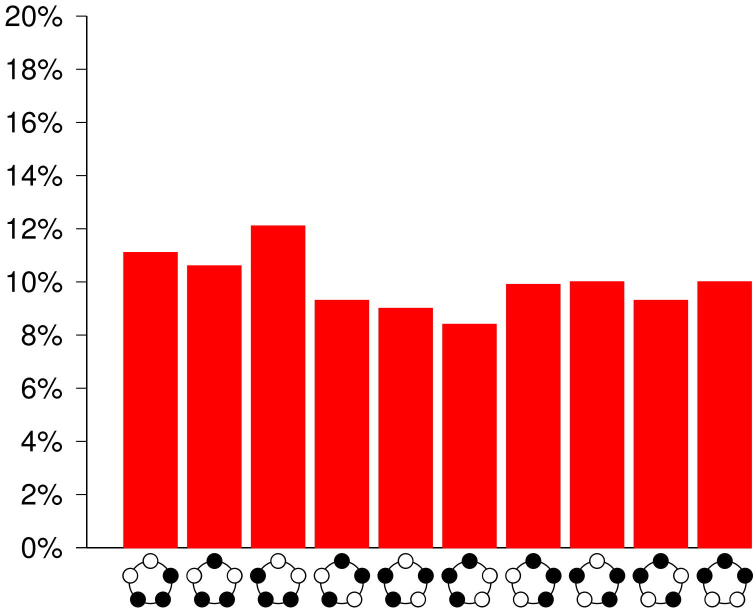

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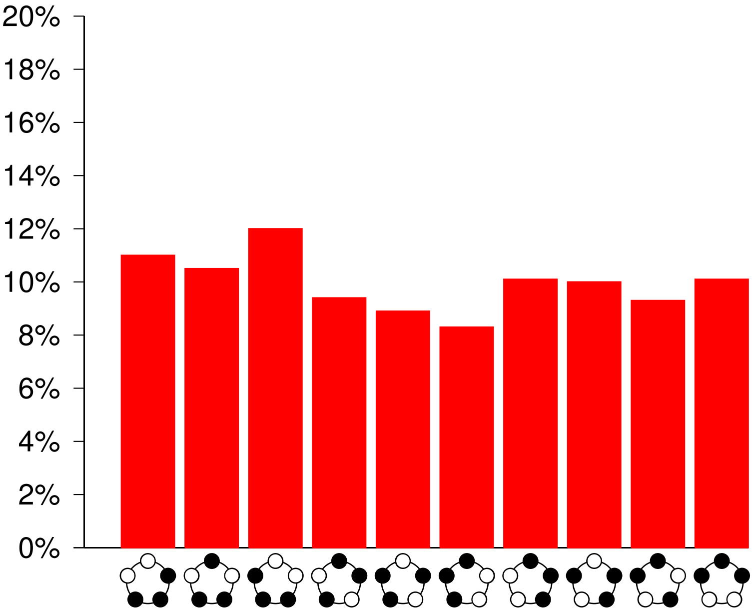

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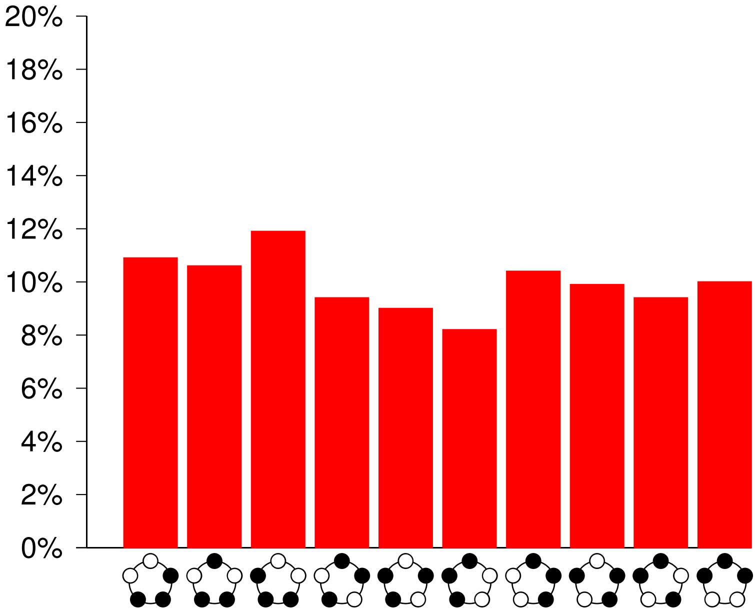

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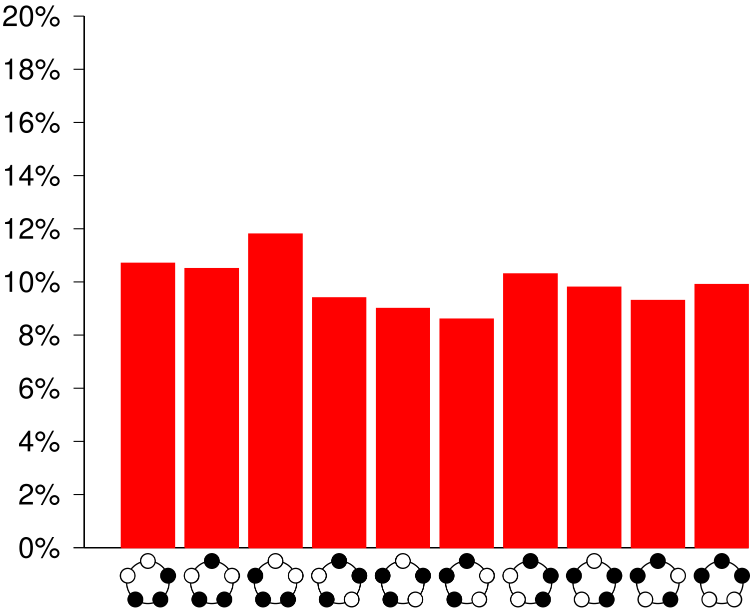

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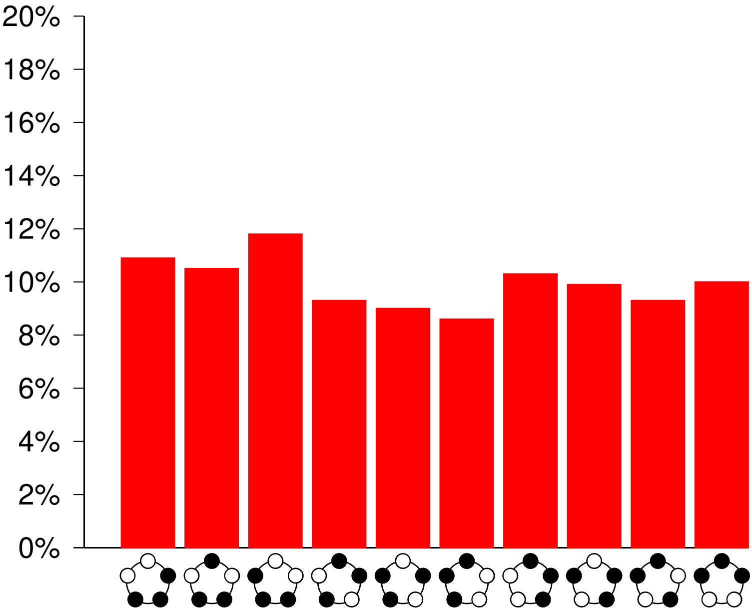

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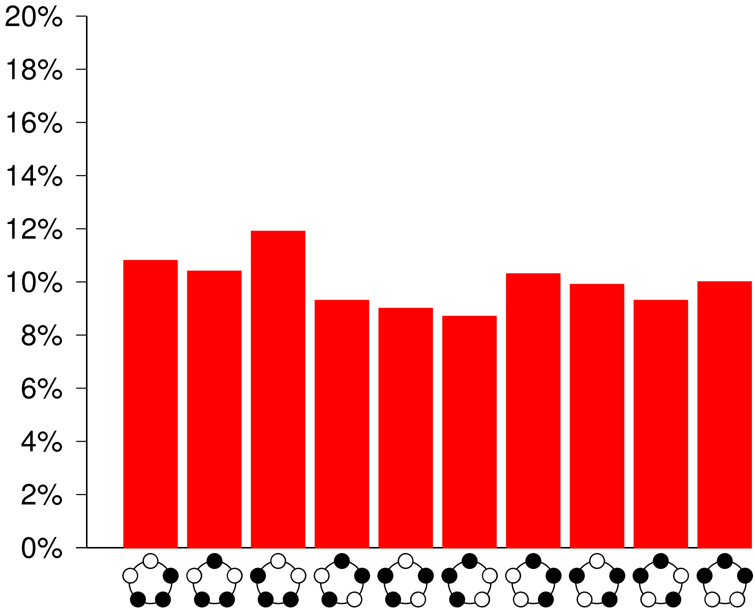

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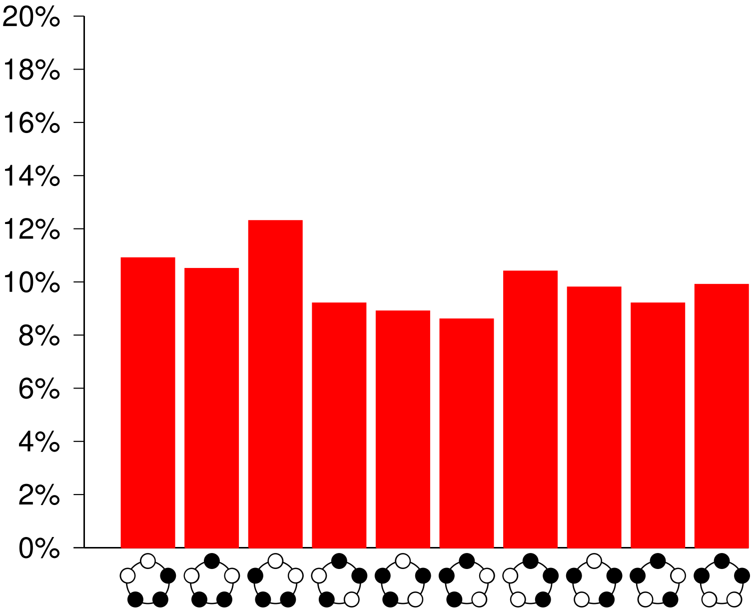

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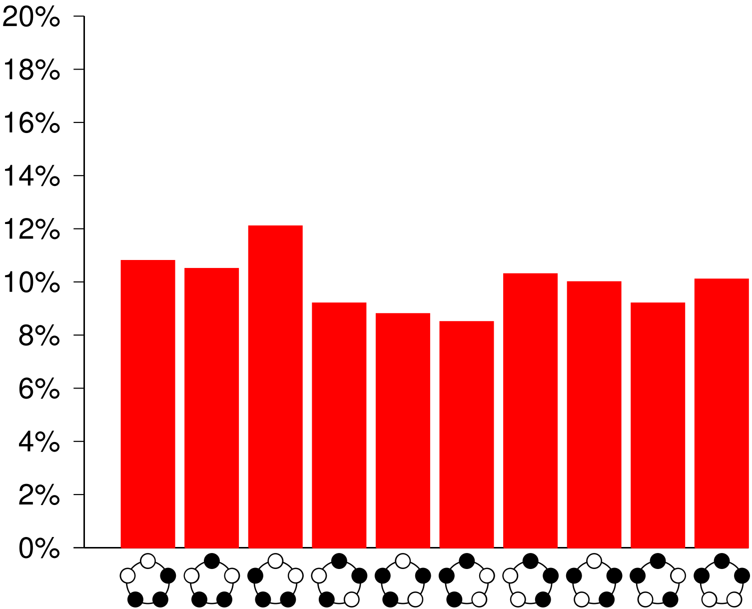

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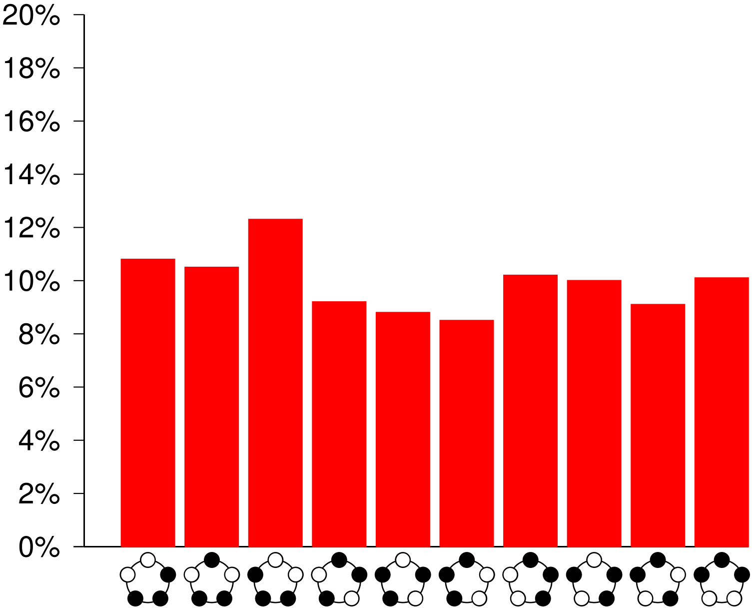

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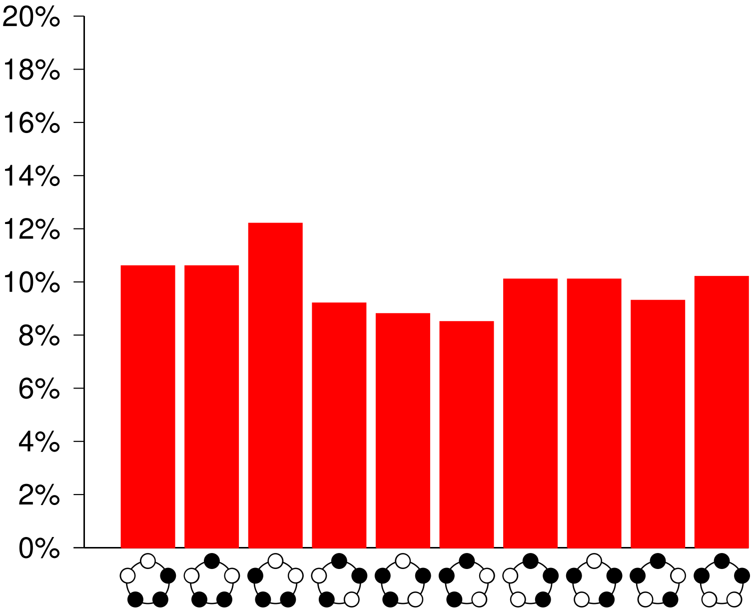

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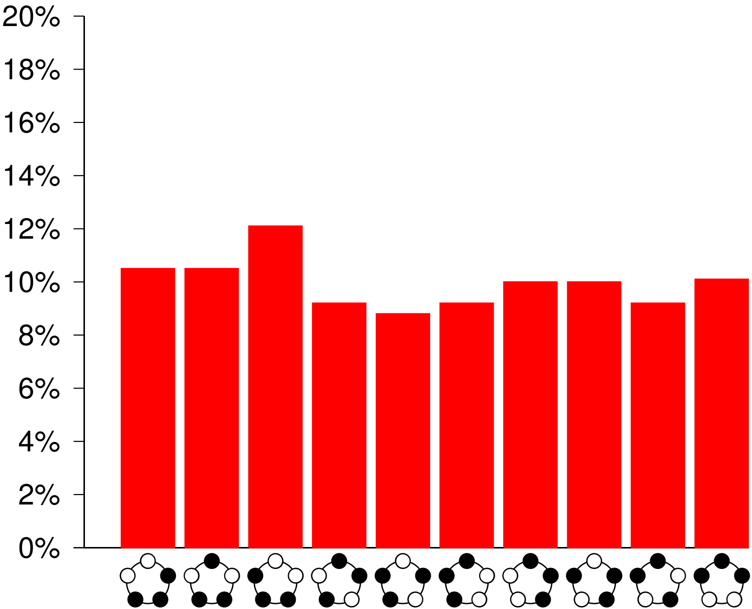

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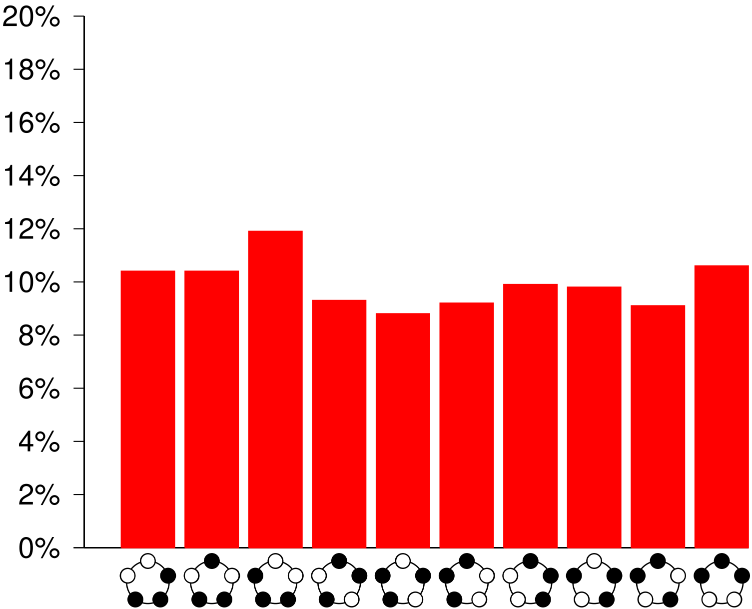

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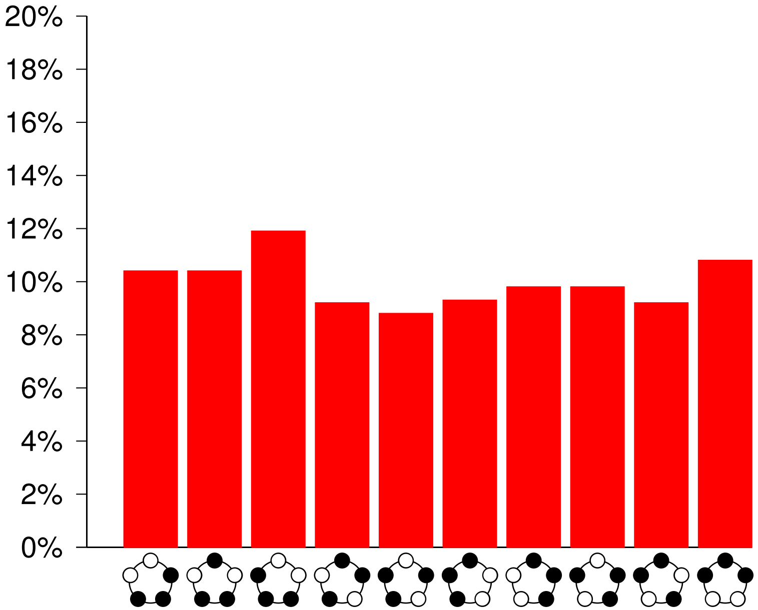

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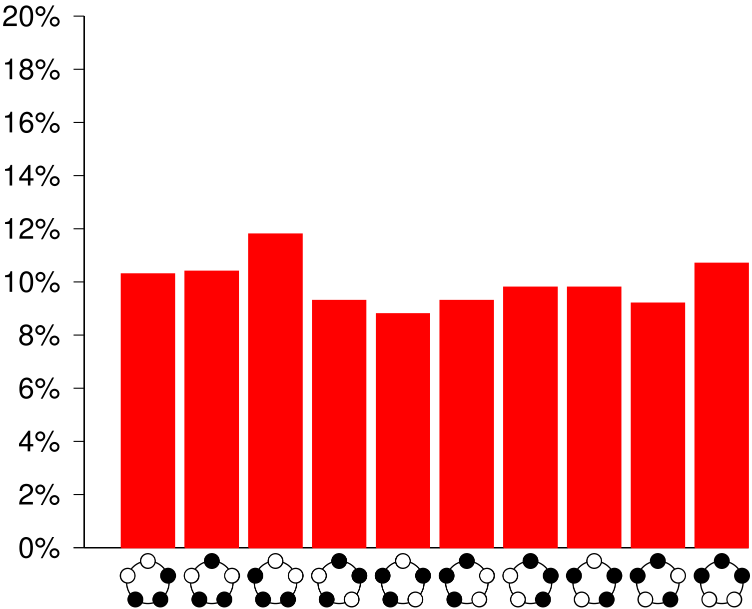

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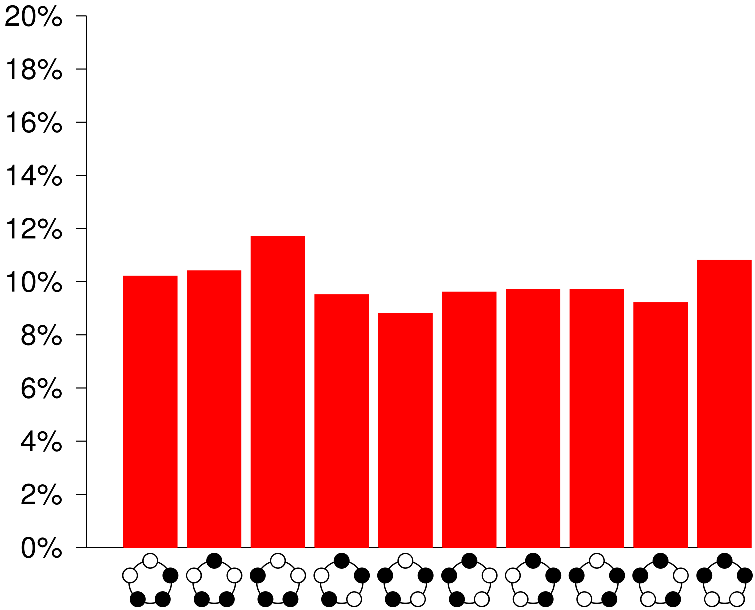

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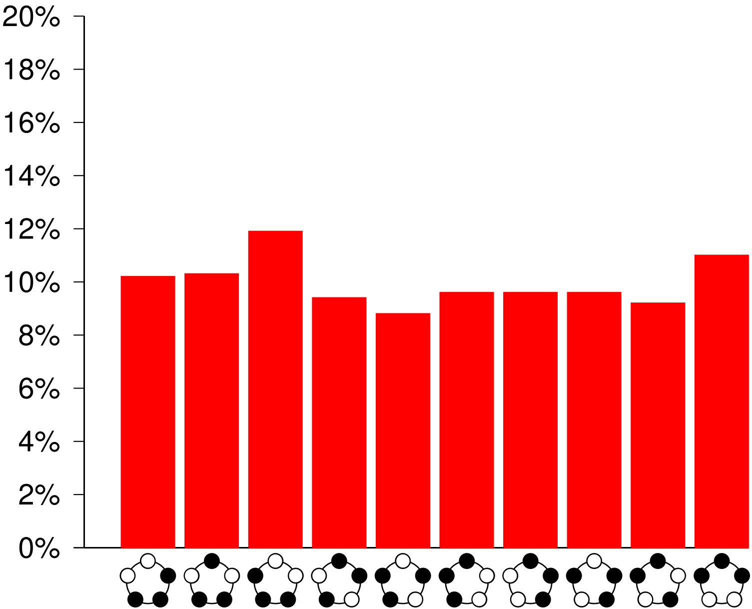

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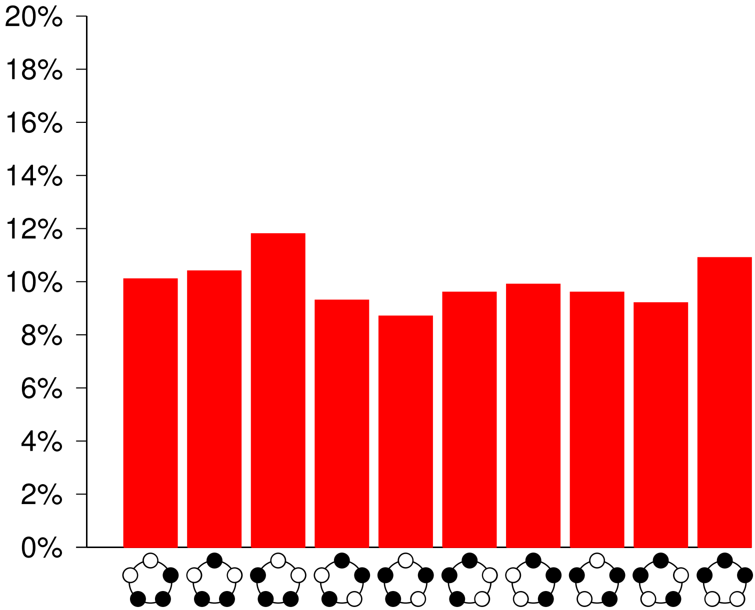

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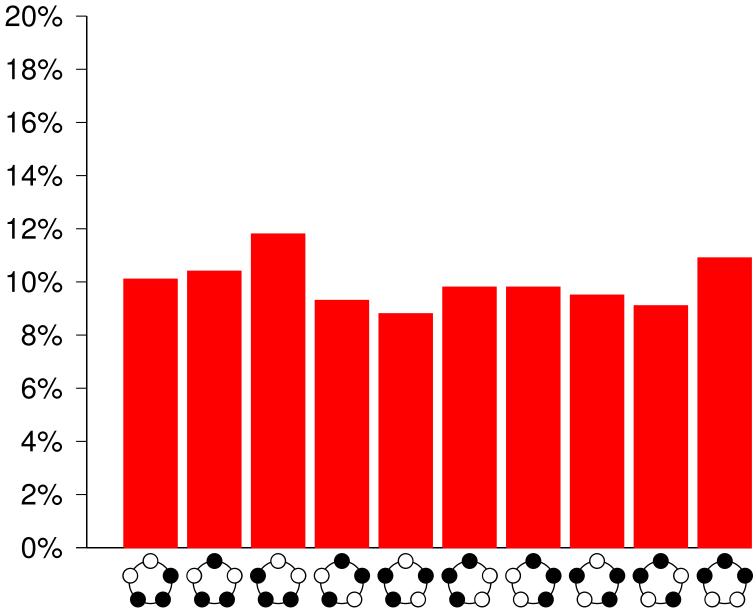

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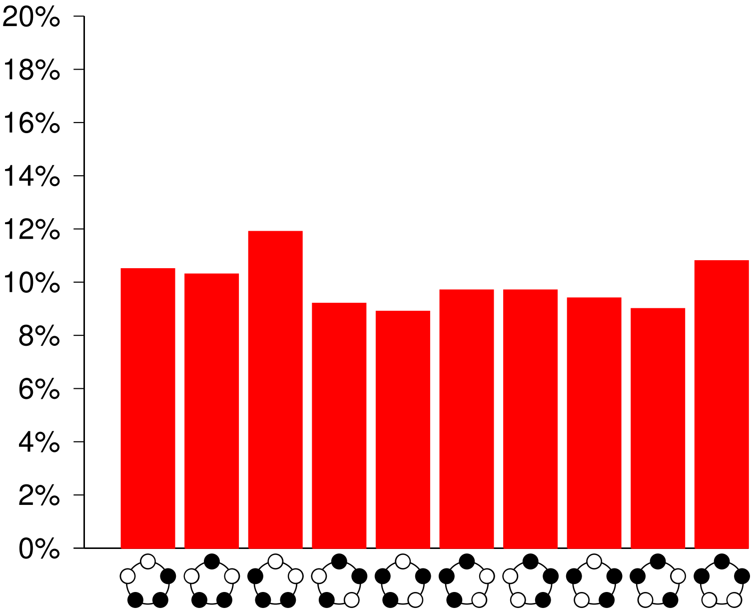

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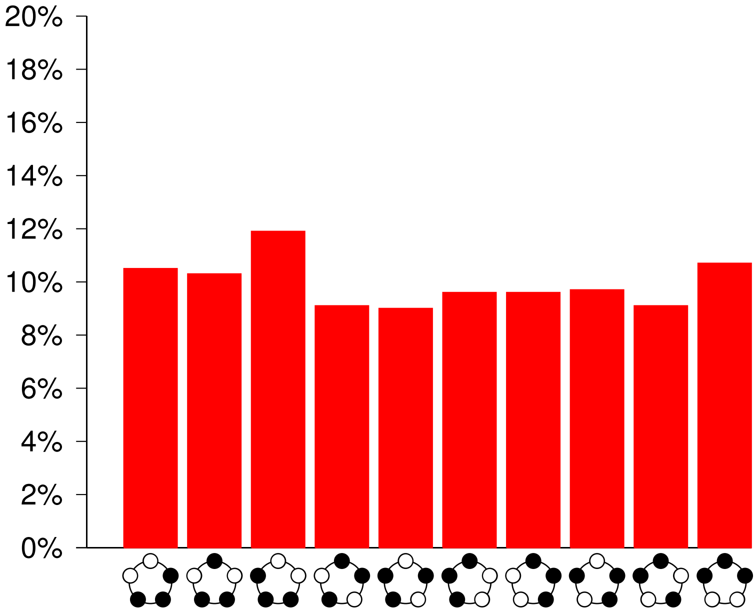

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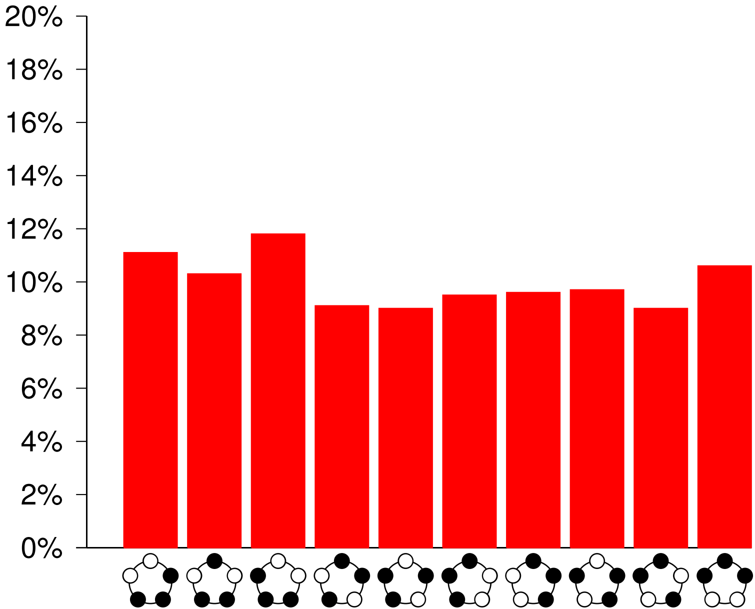

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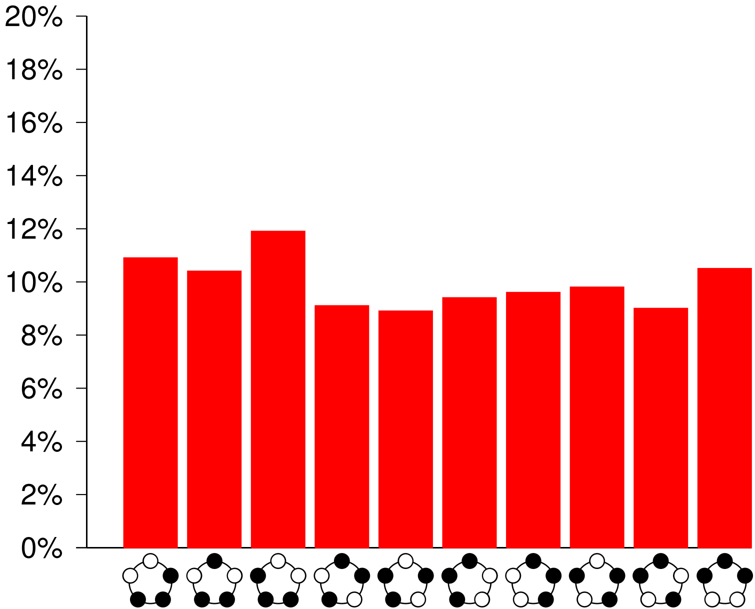

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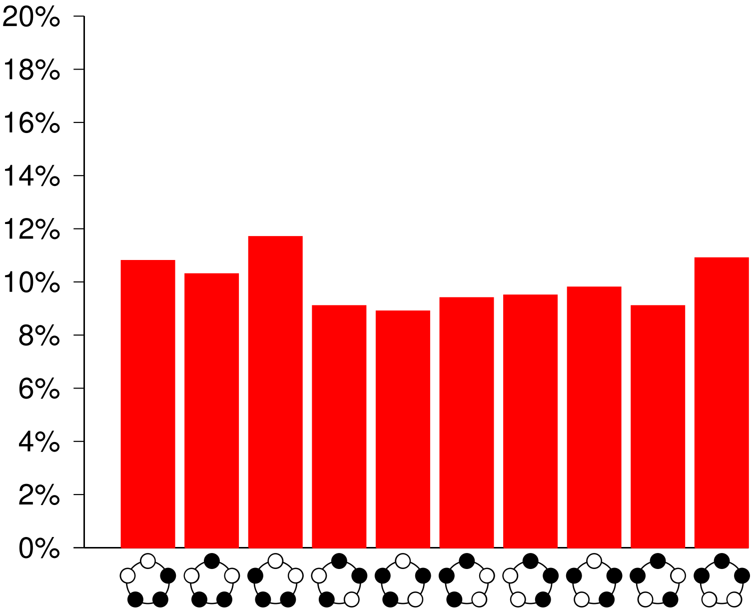

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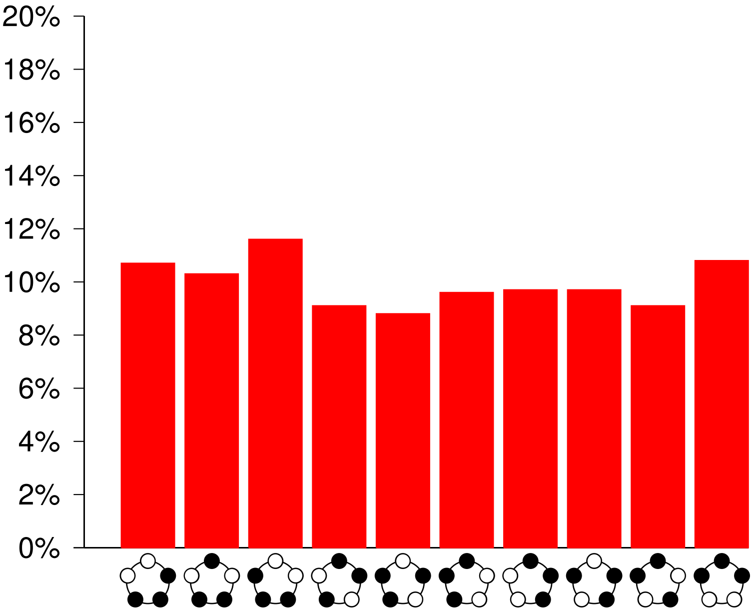

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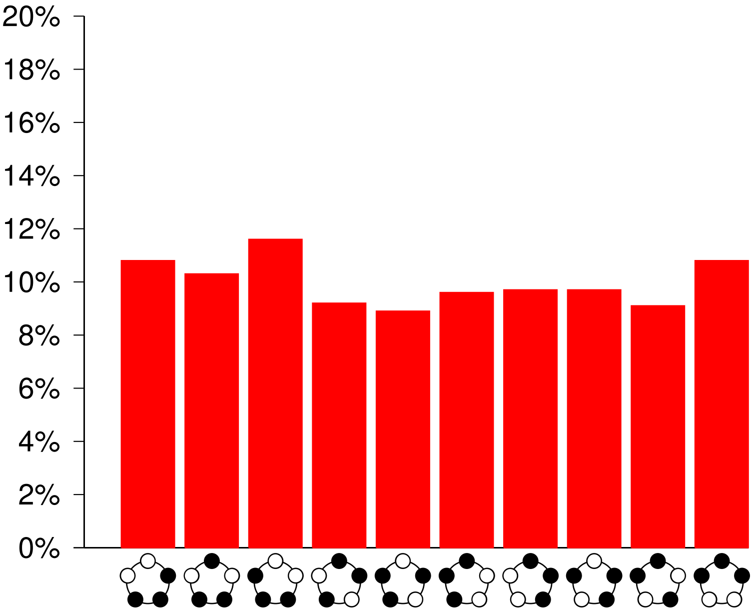

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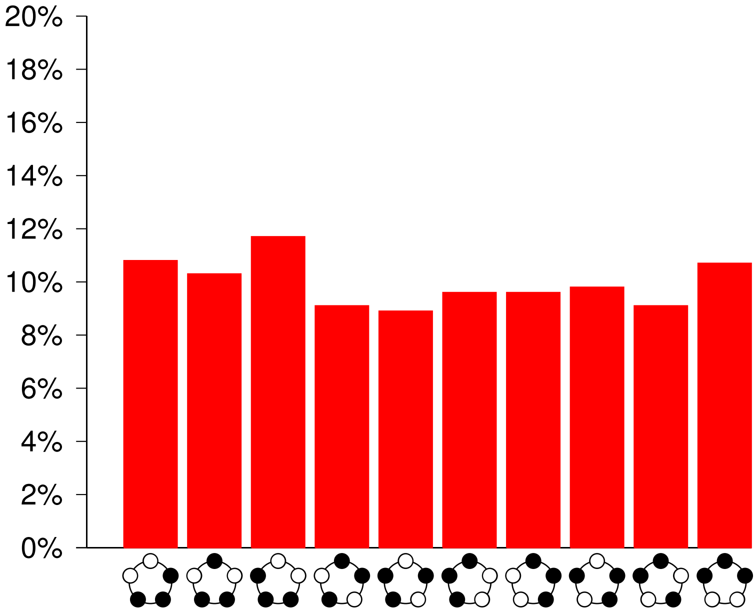

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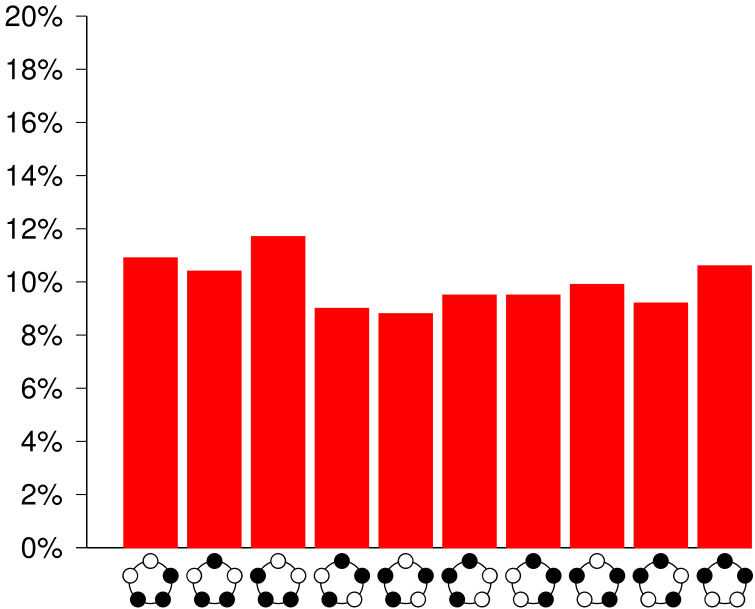

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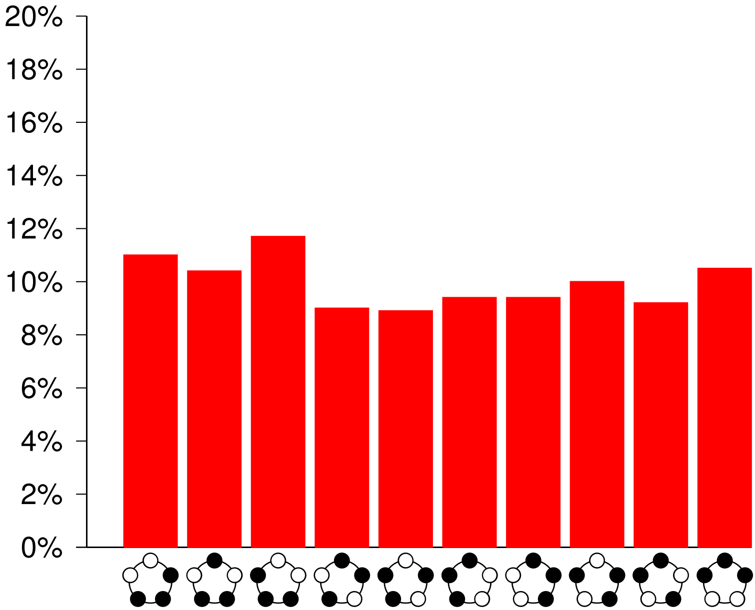

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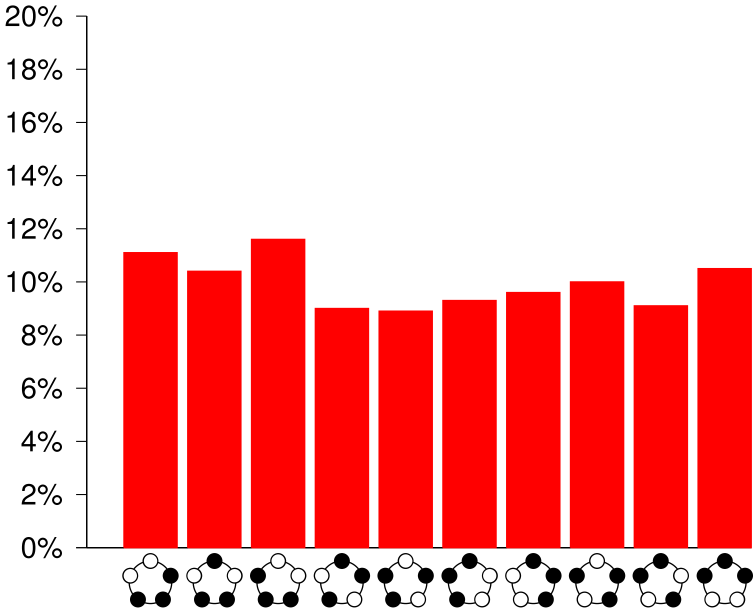

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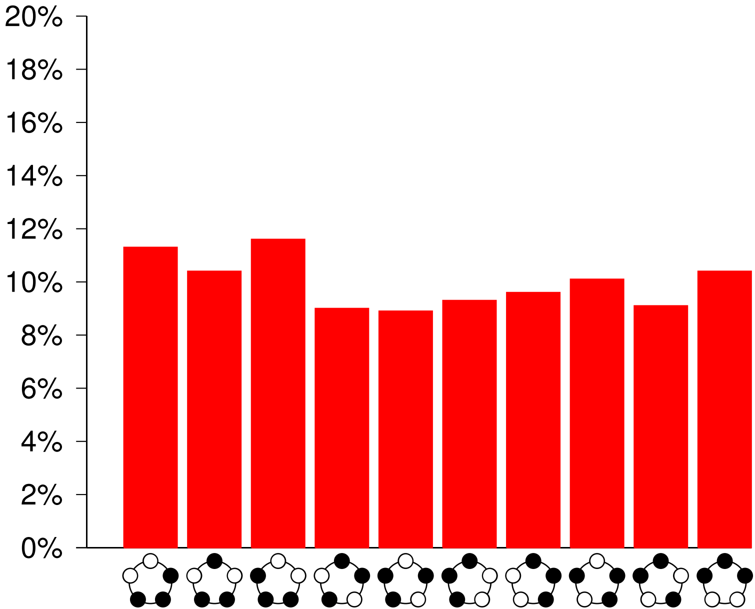

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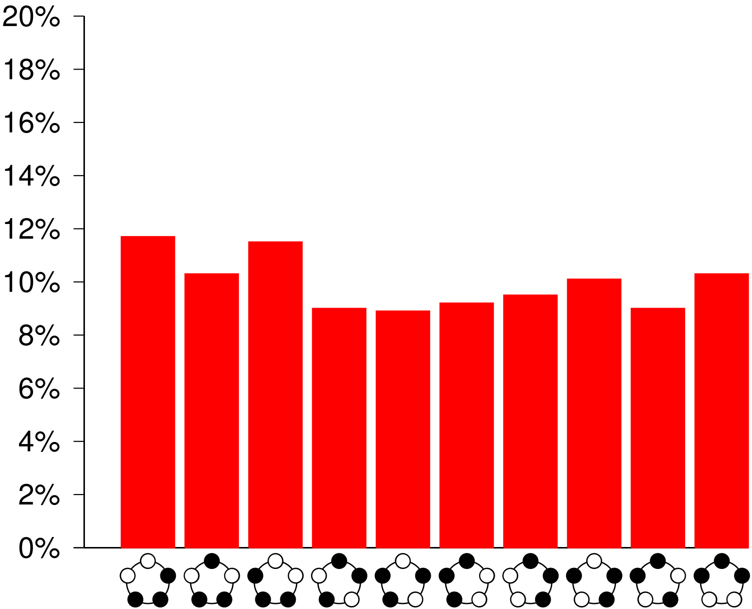

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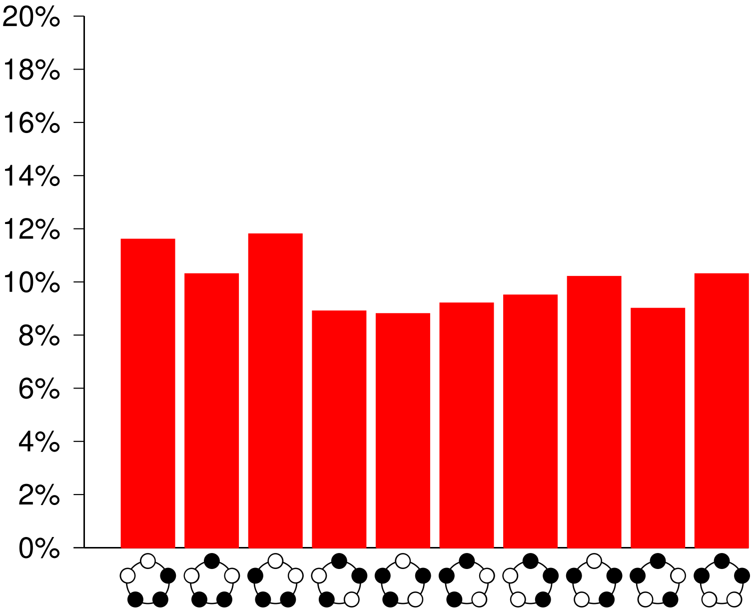

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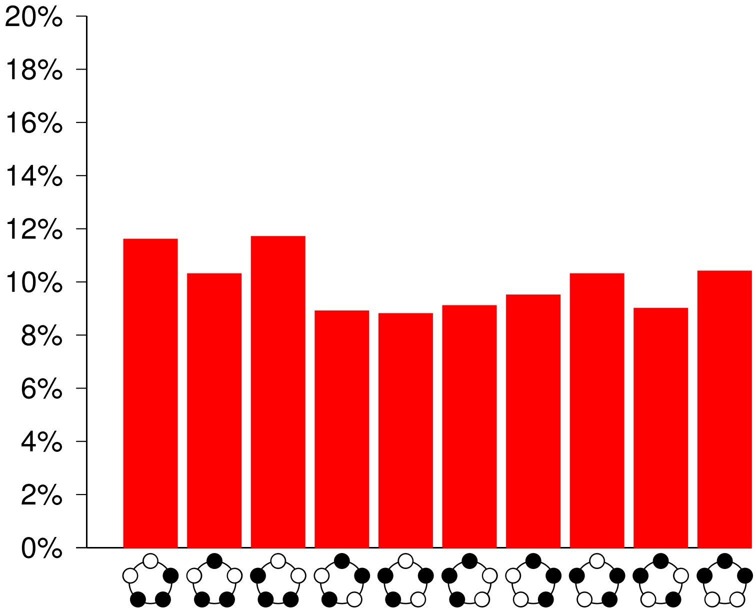

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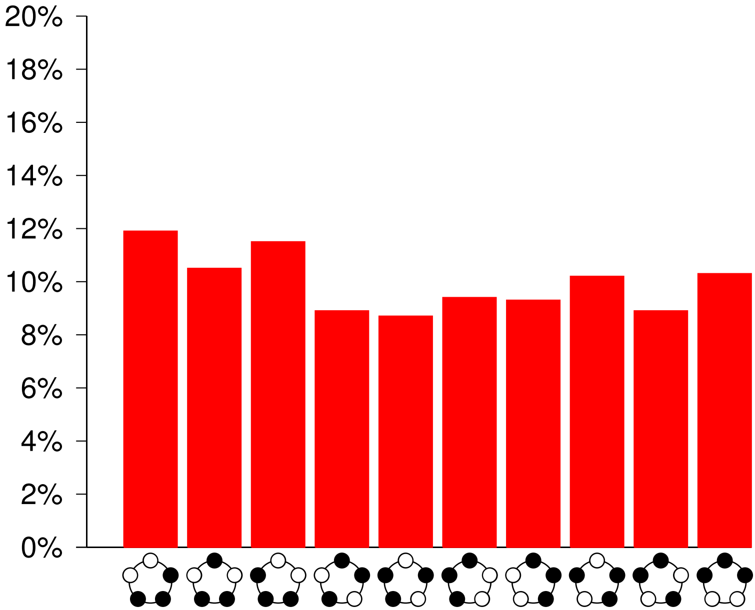

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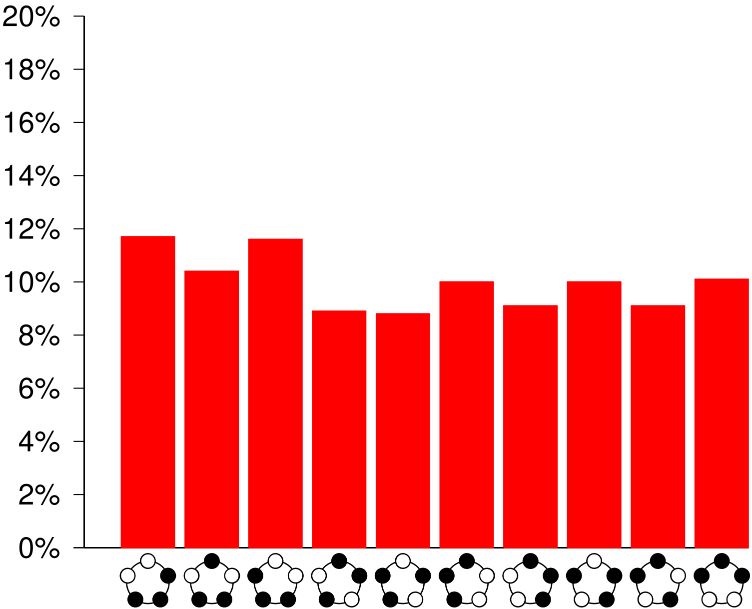

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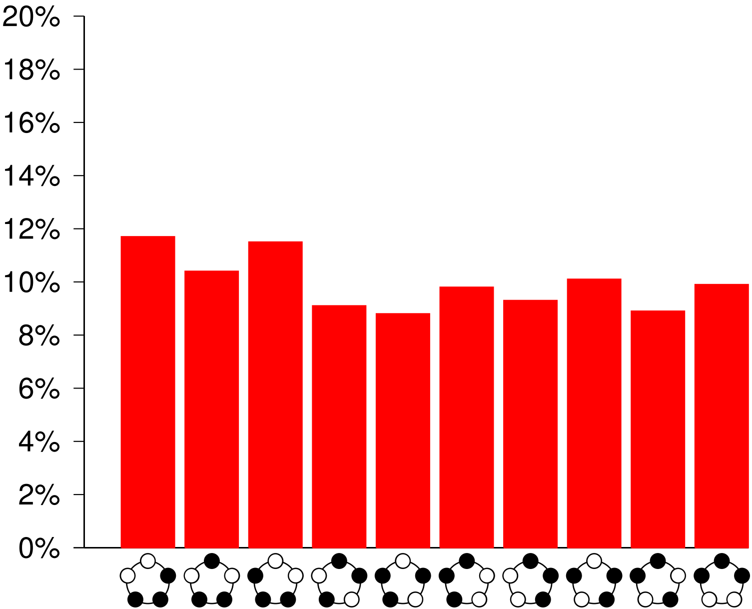

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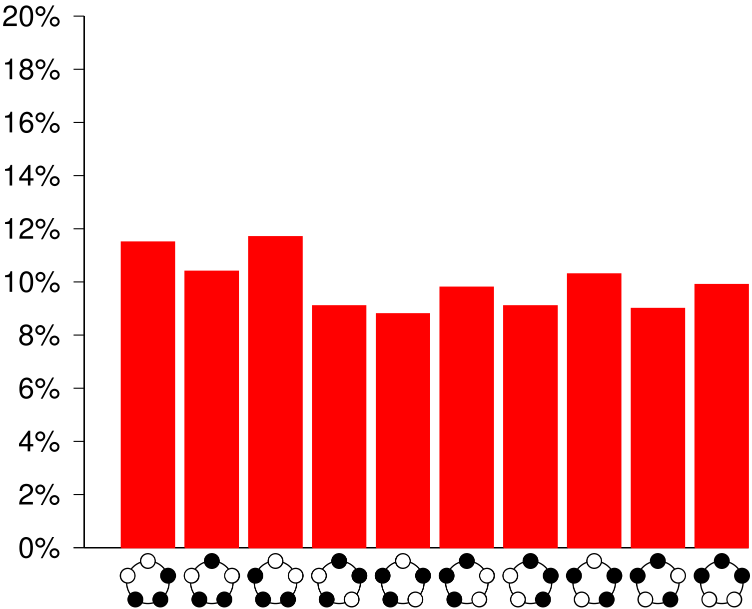

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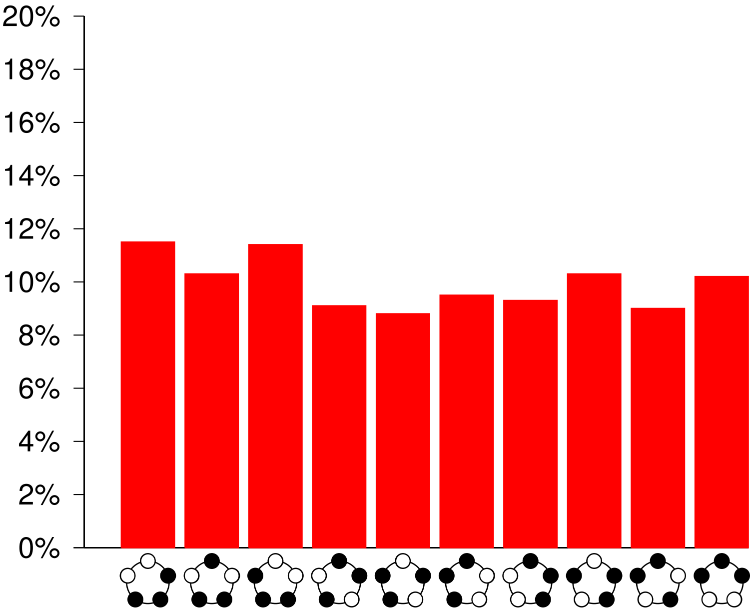

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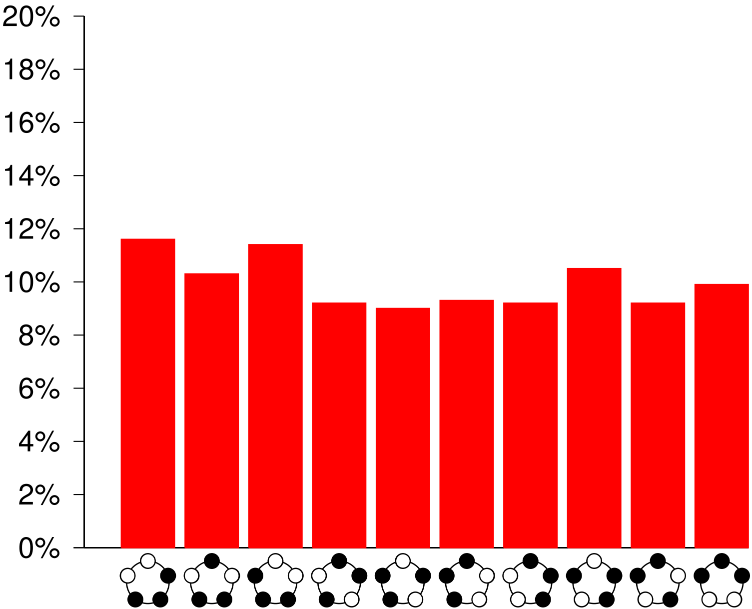

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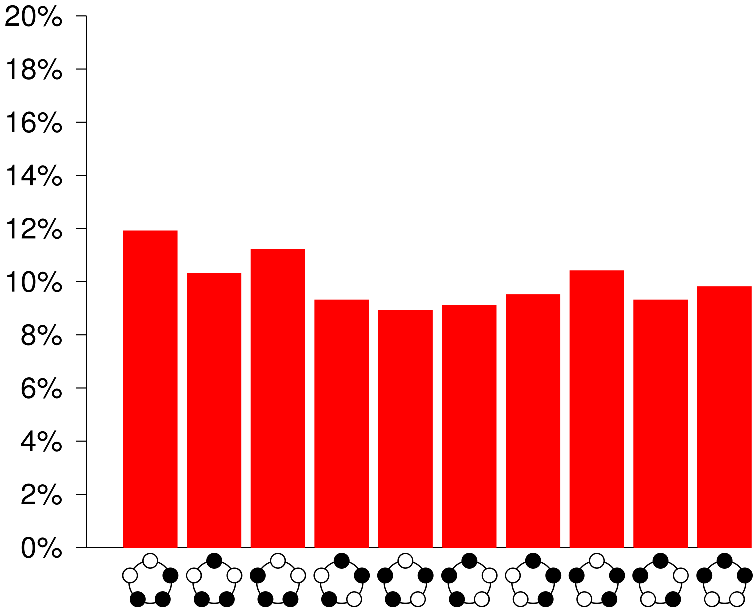

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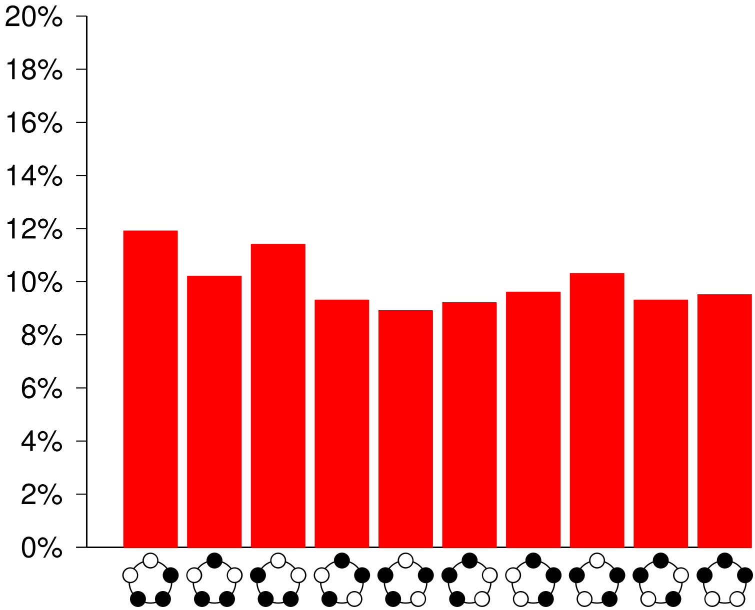

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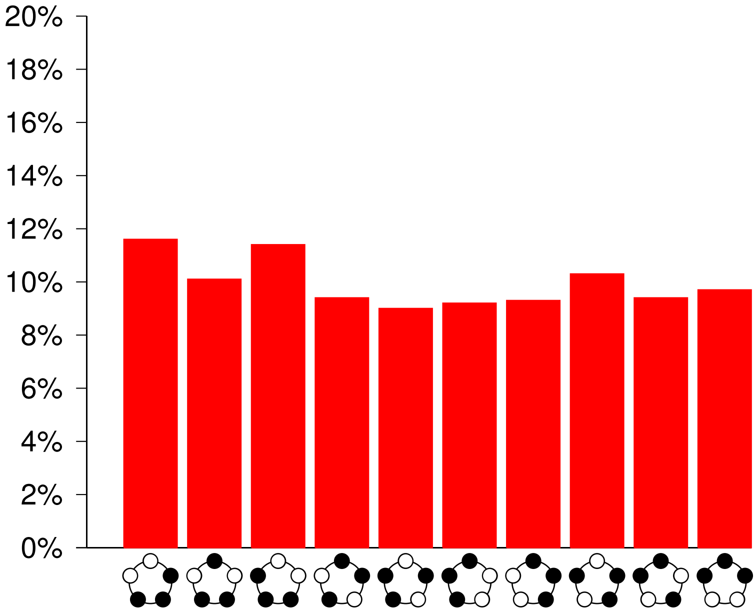

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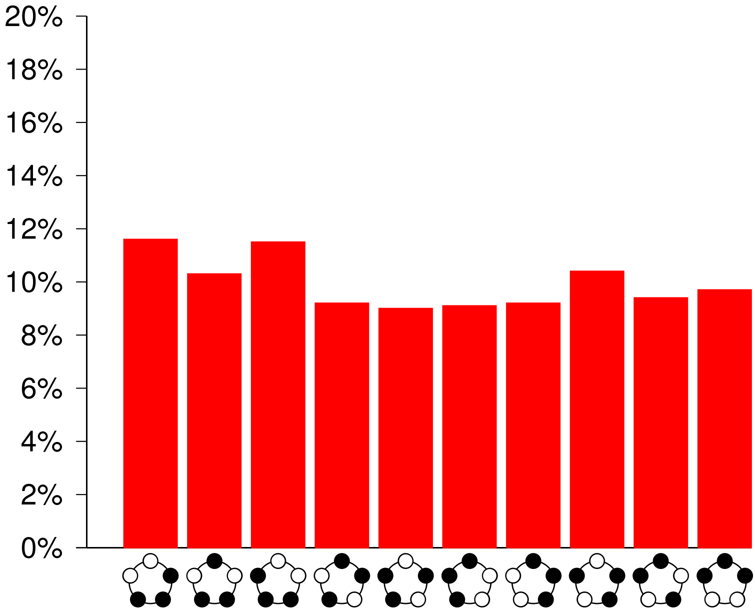

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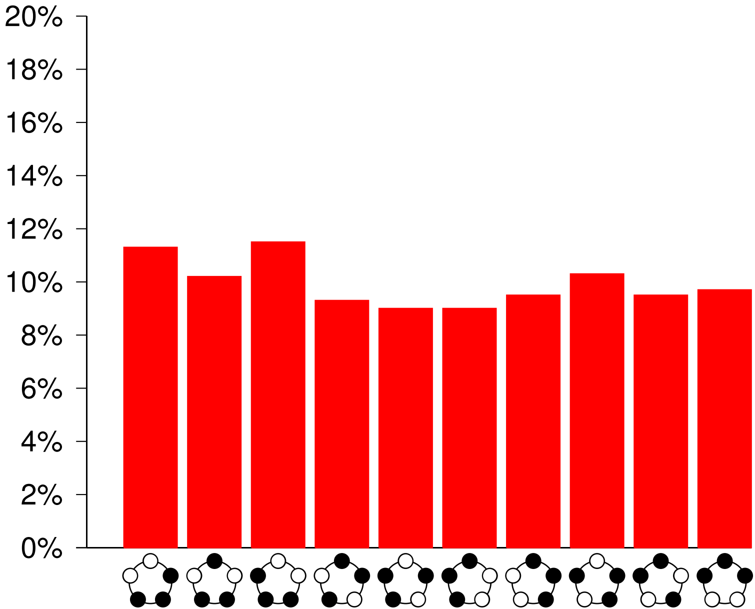

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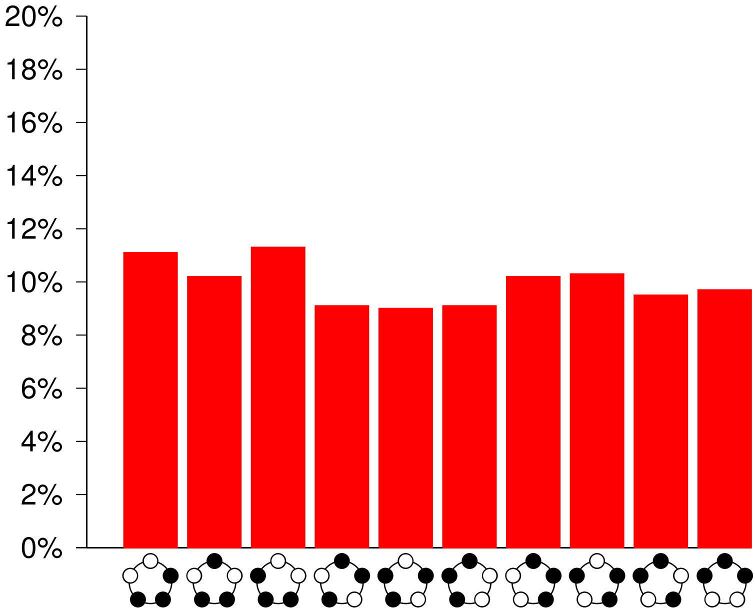

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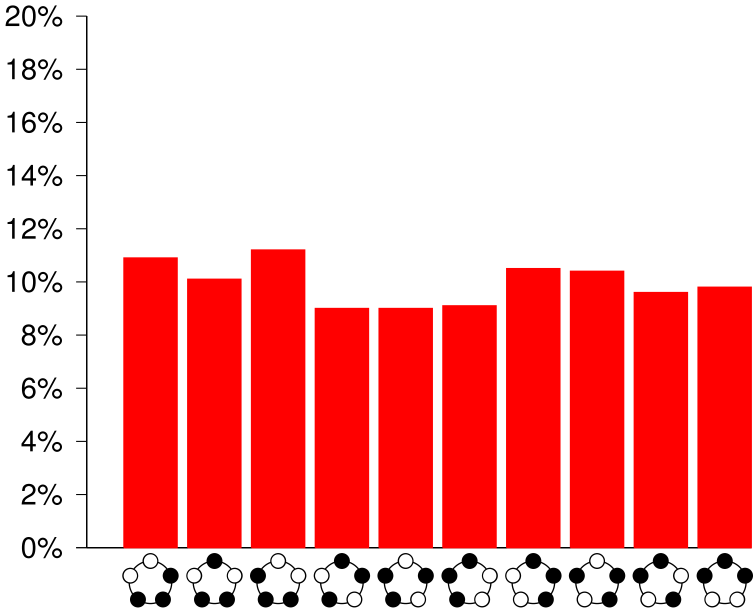

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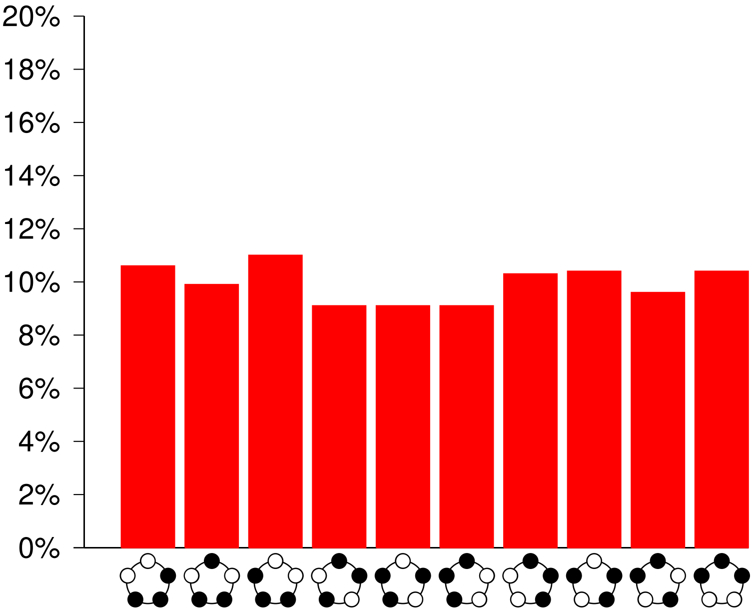

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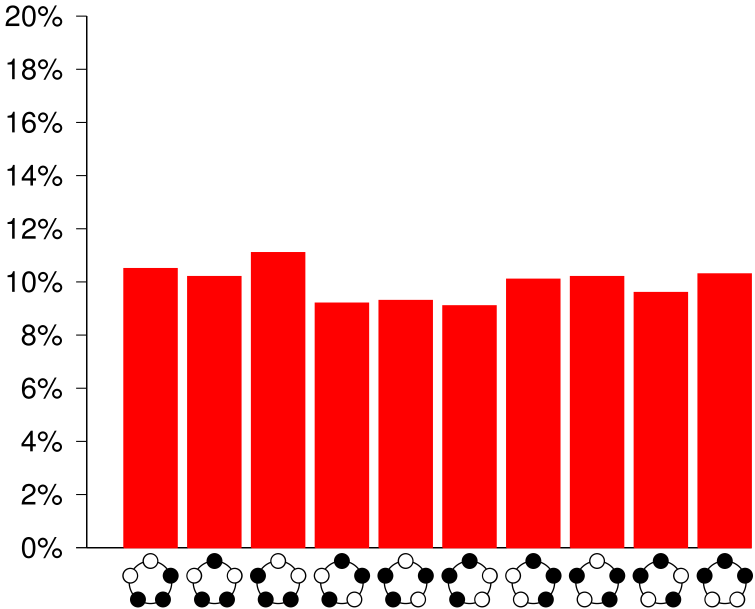

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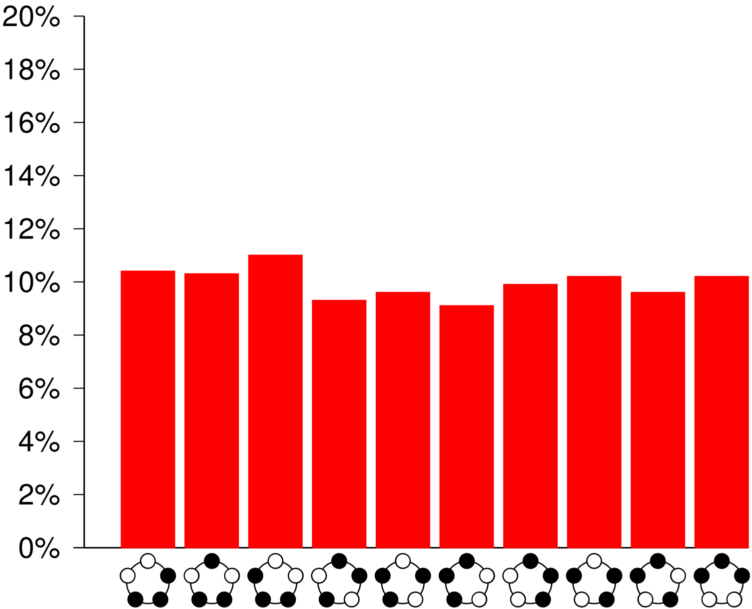

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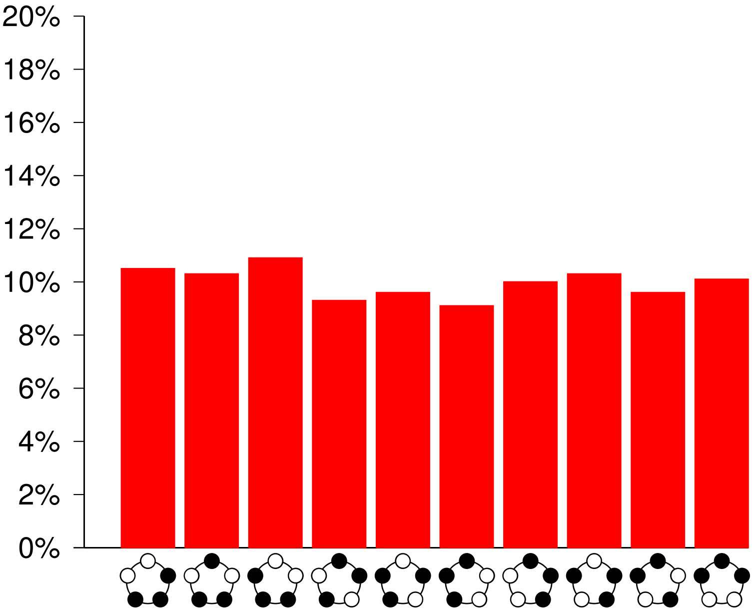

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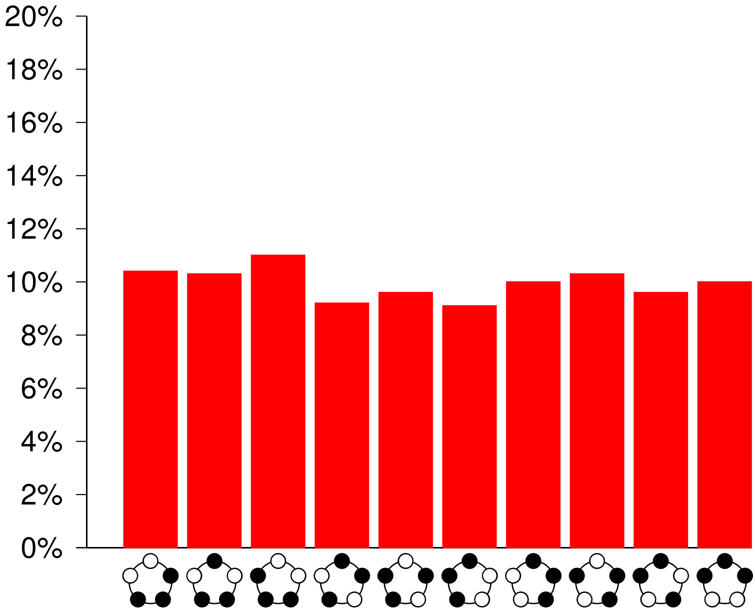

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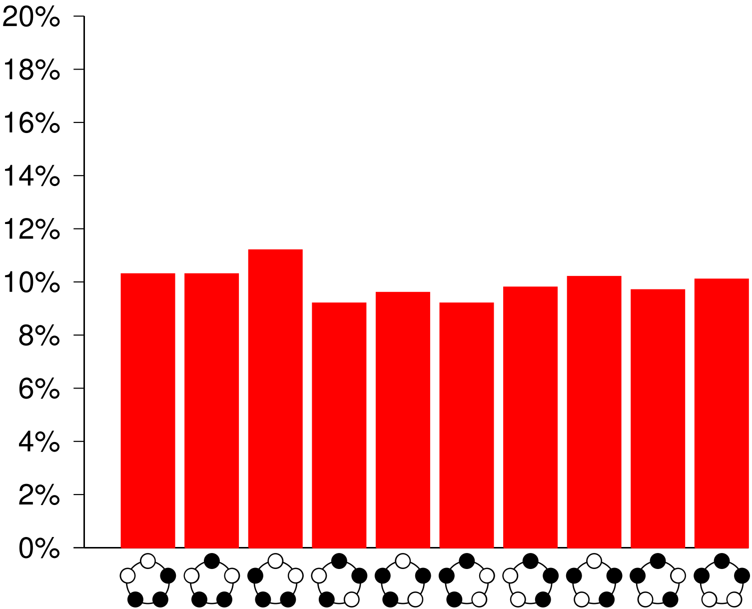

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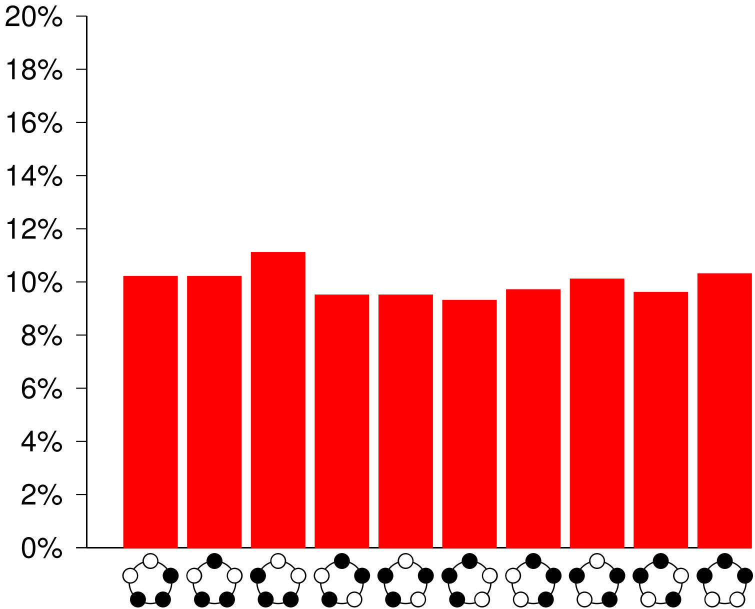

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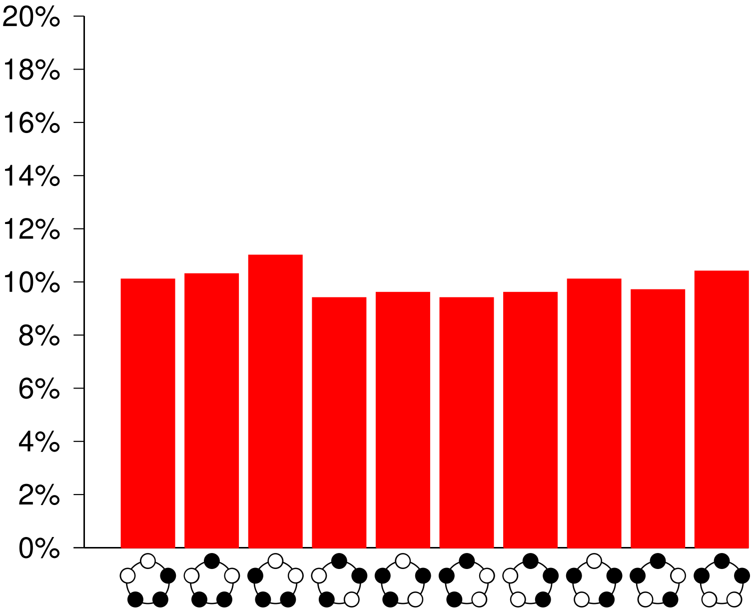

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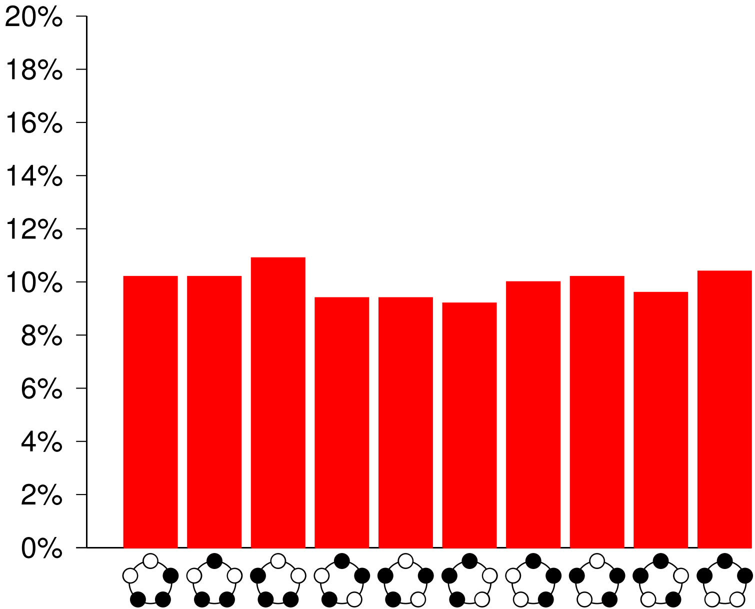

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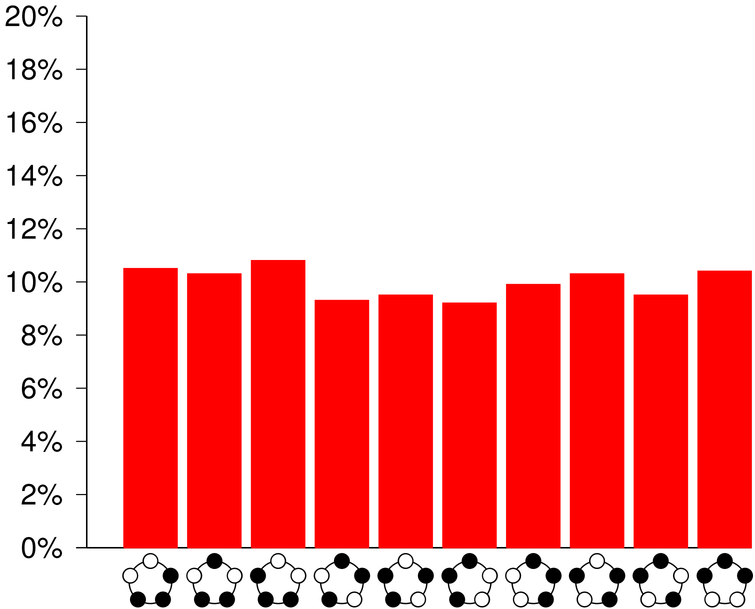

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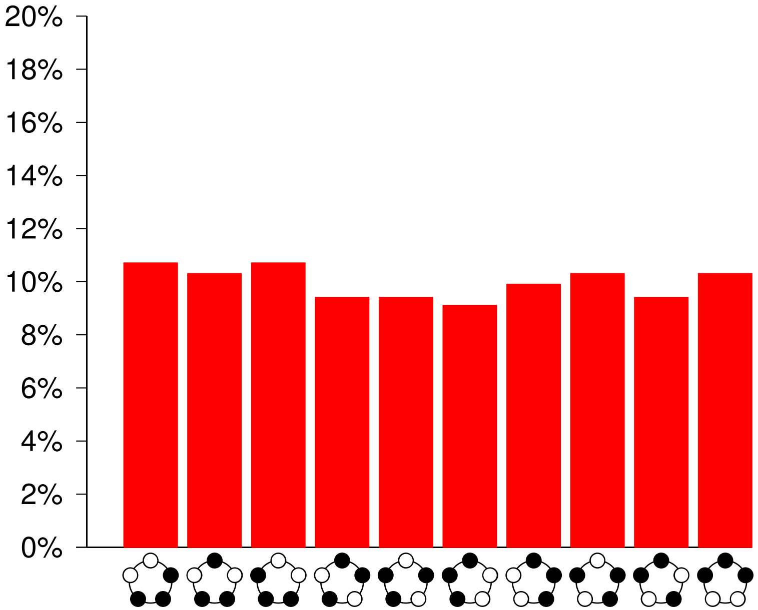

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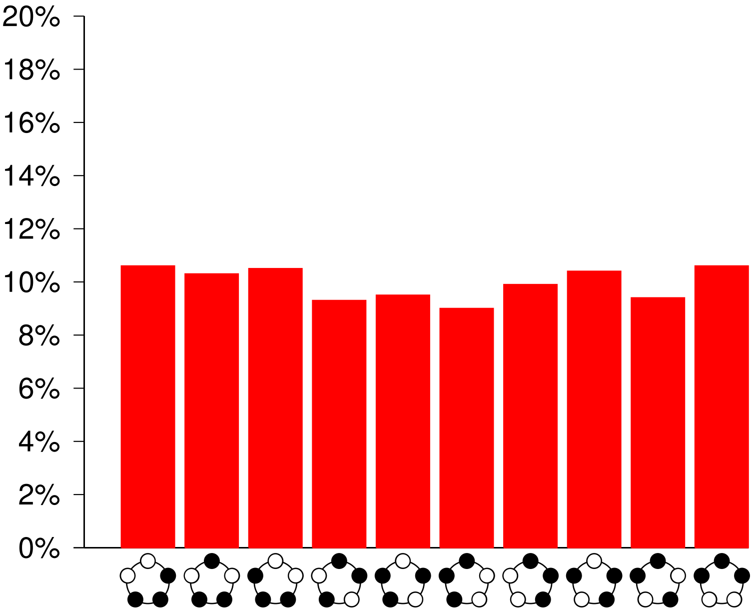

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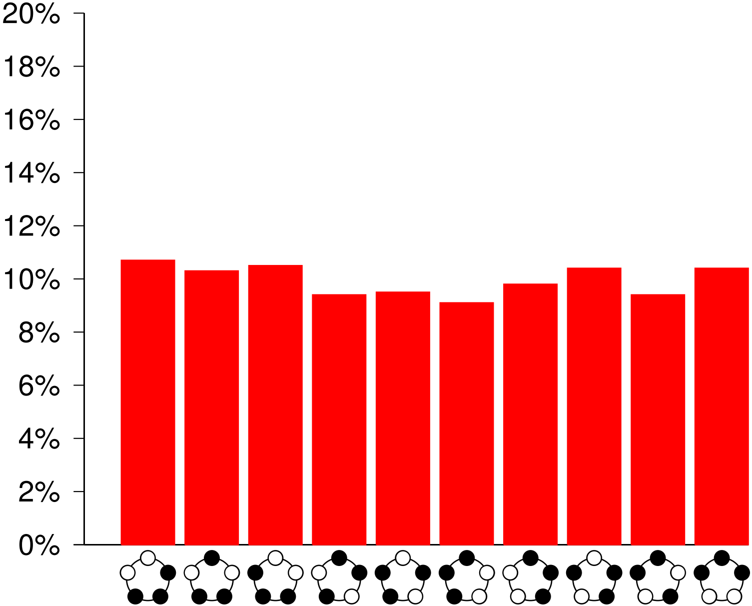

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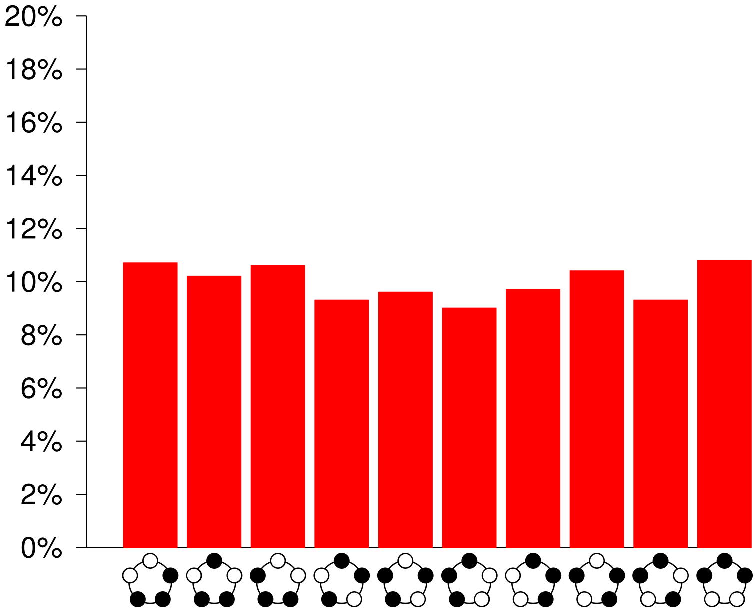

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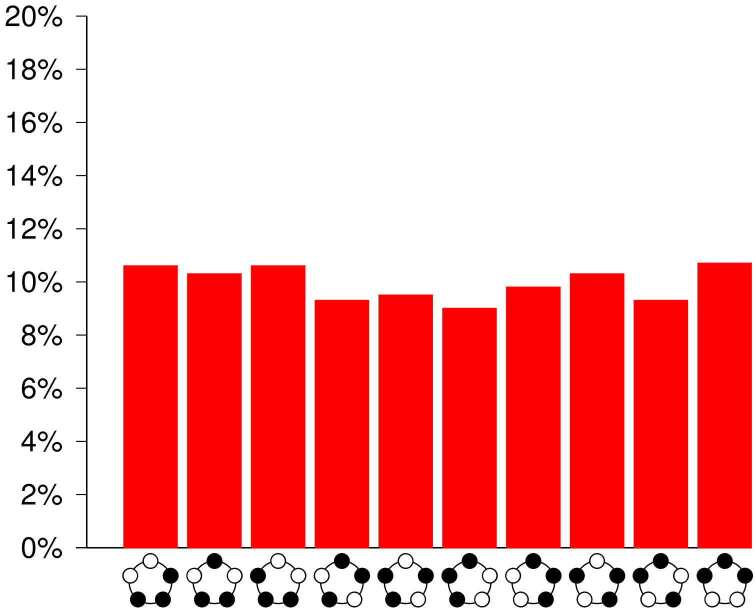

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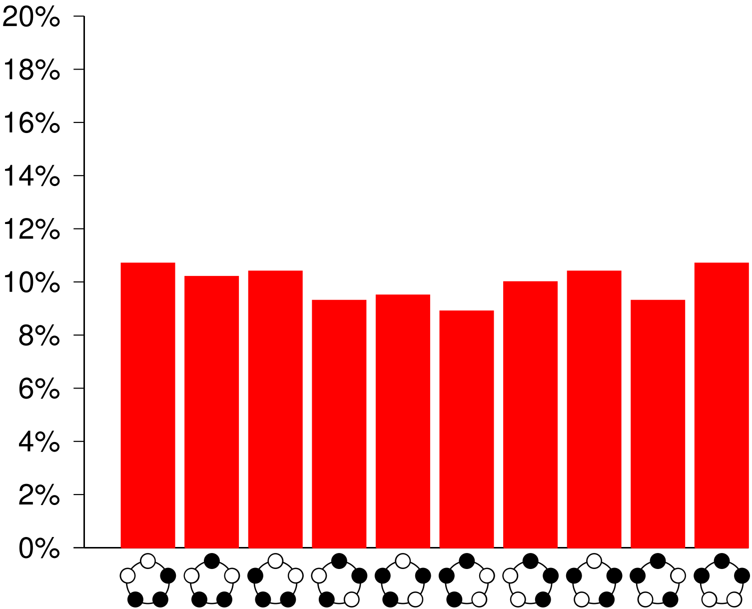

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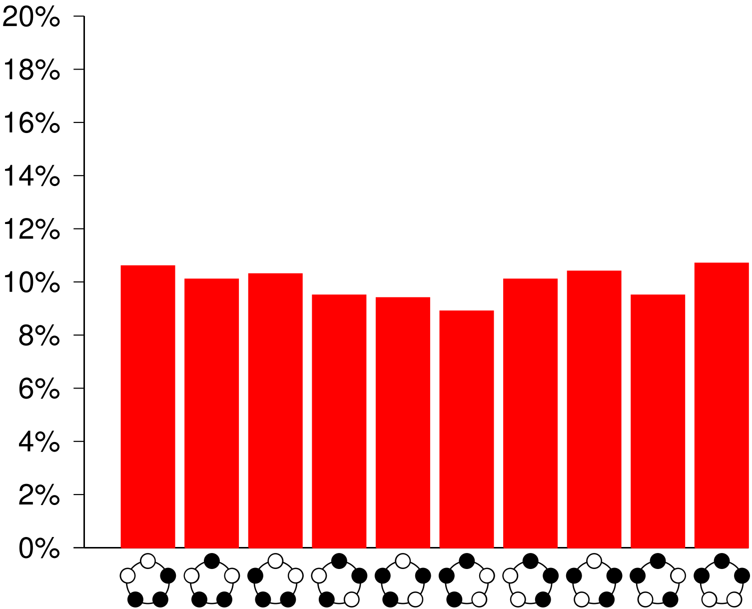

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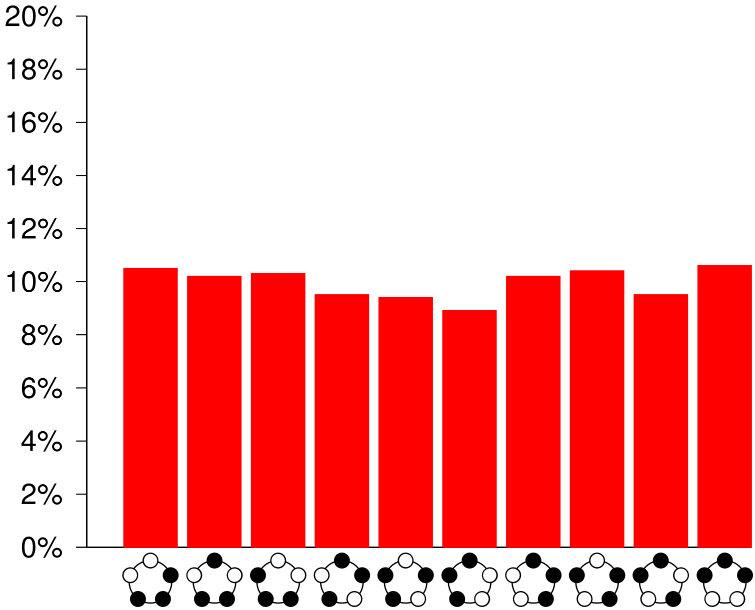

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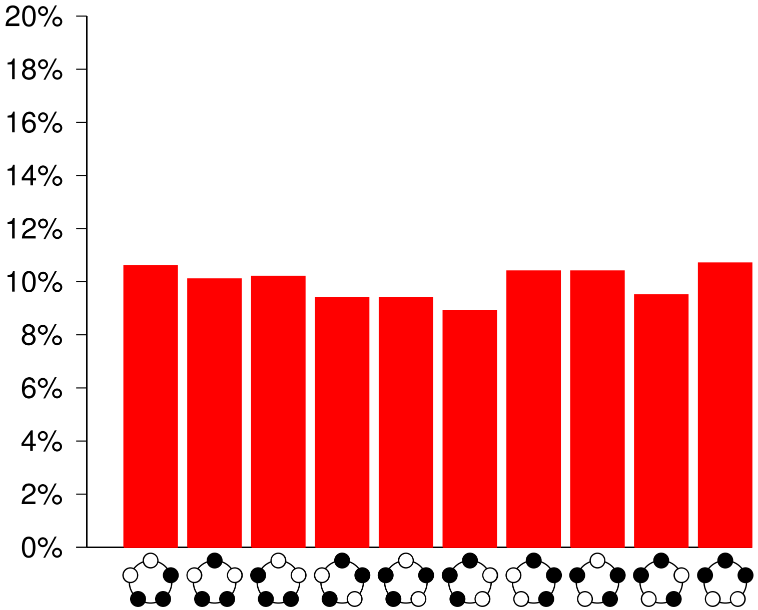

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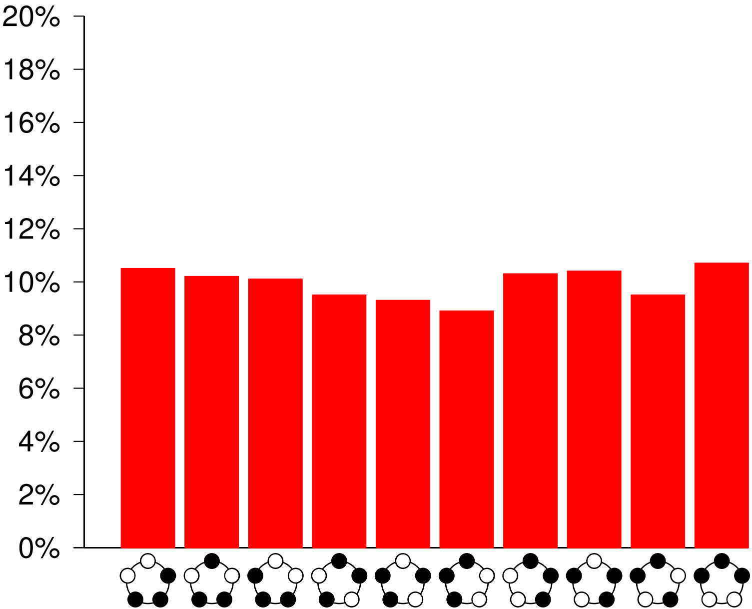
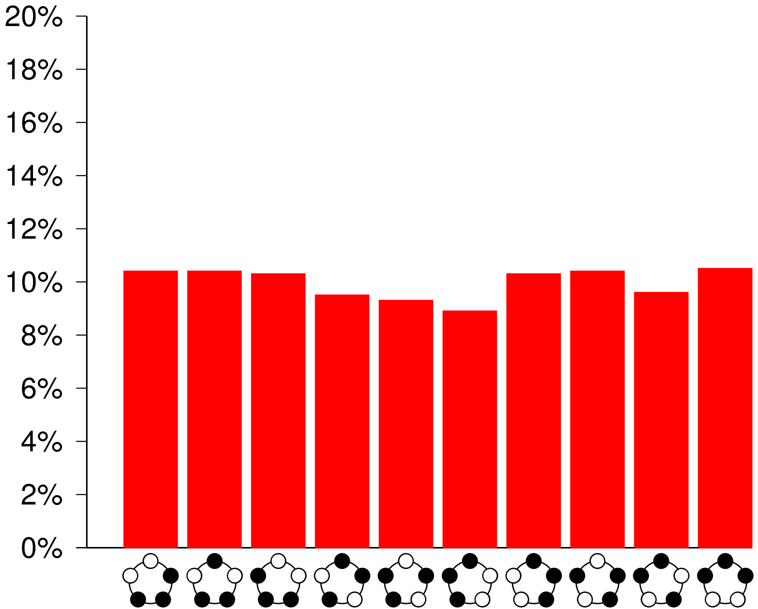


Stationary distribution



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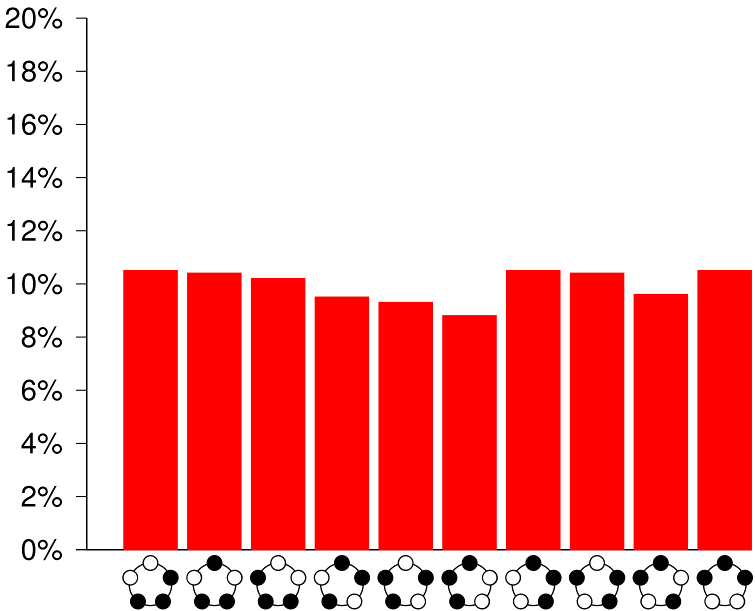

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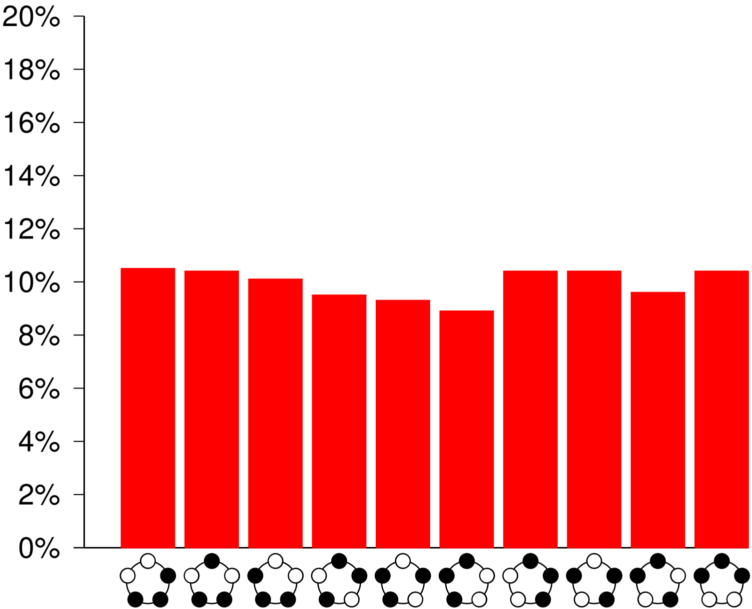

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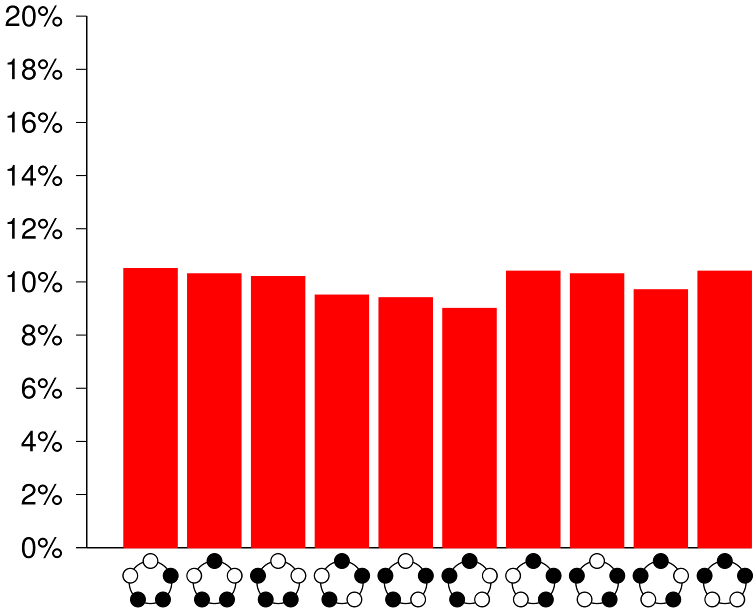

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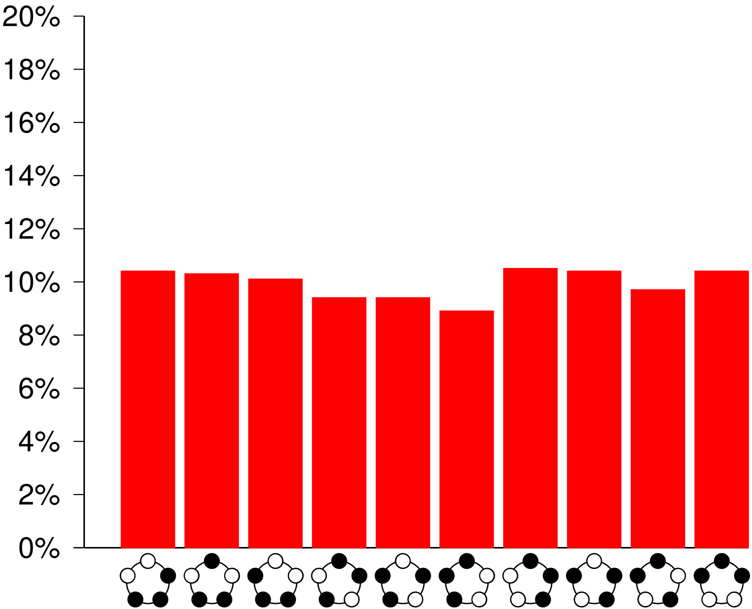

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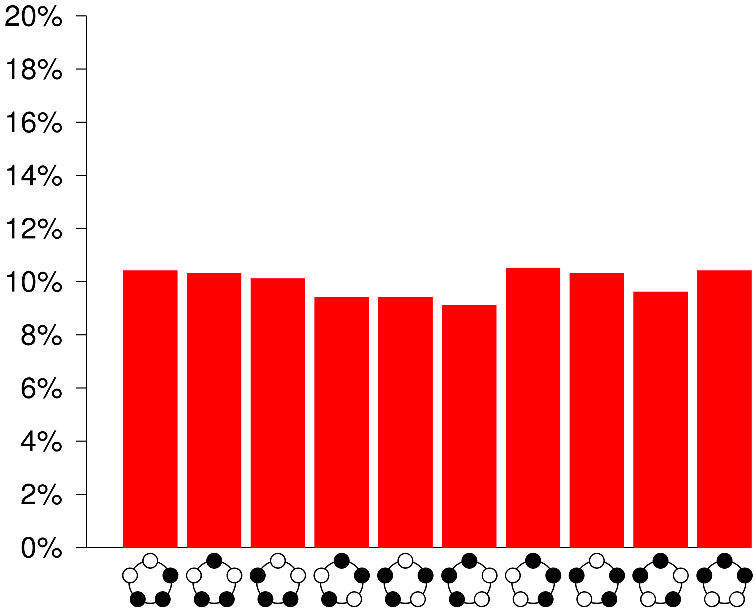

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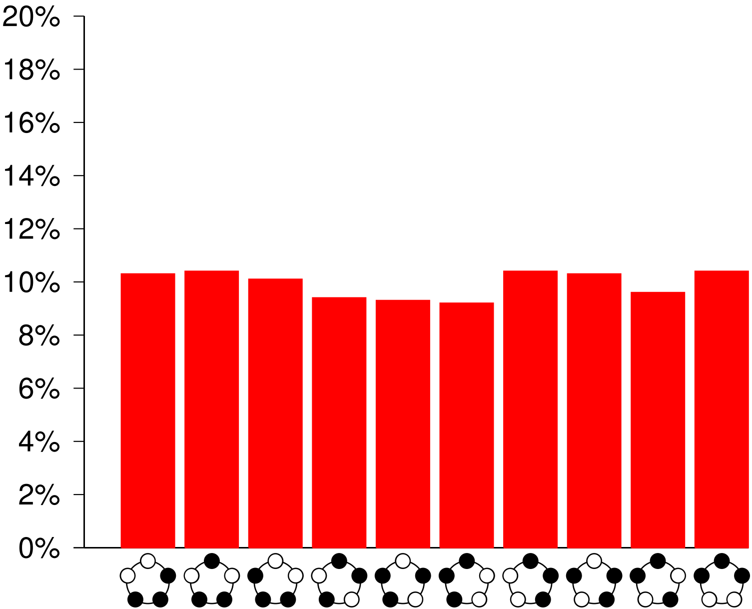

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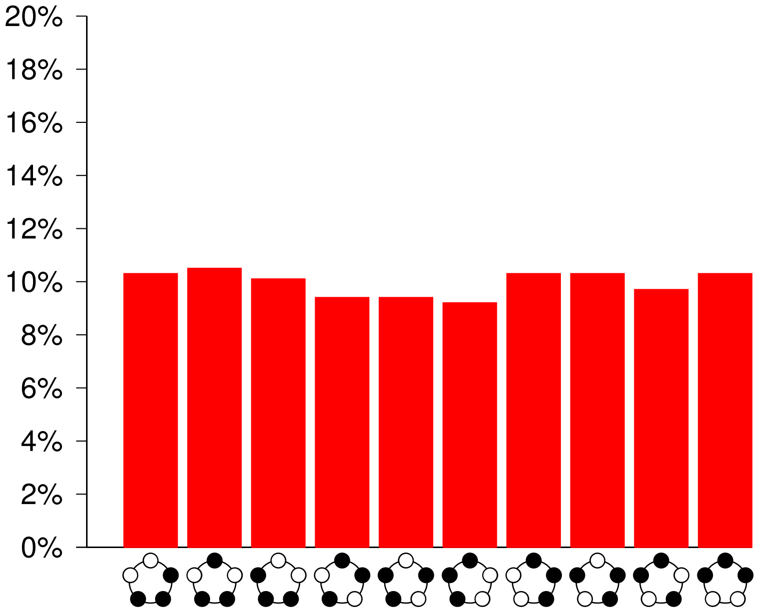

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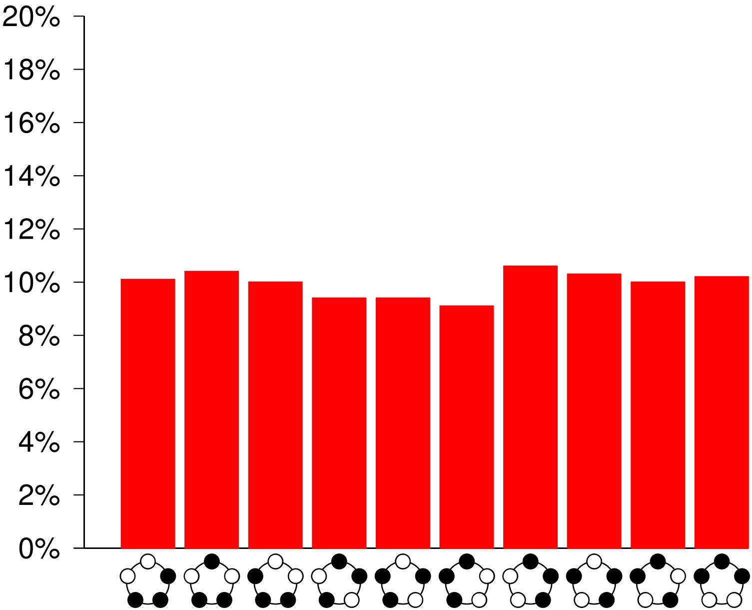

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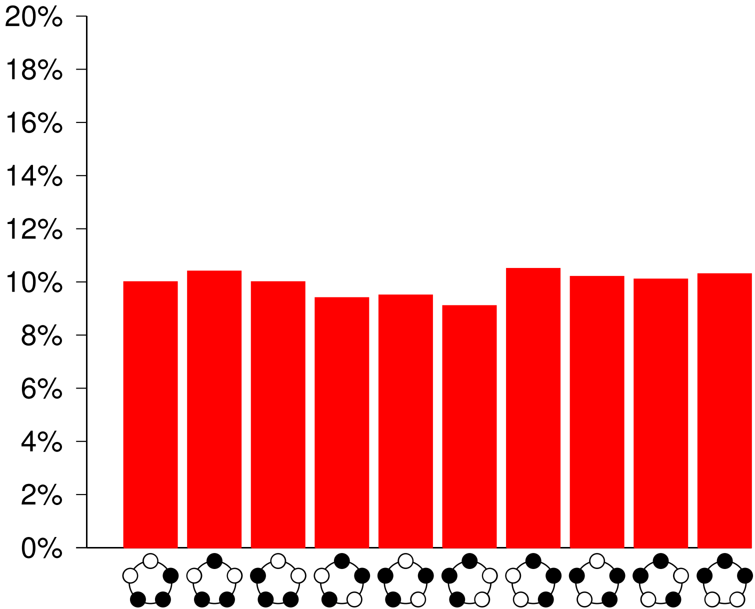

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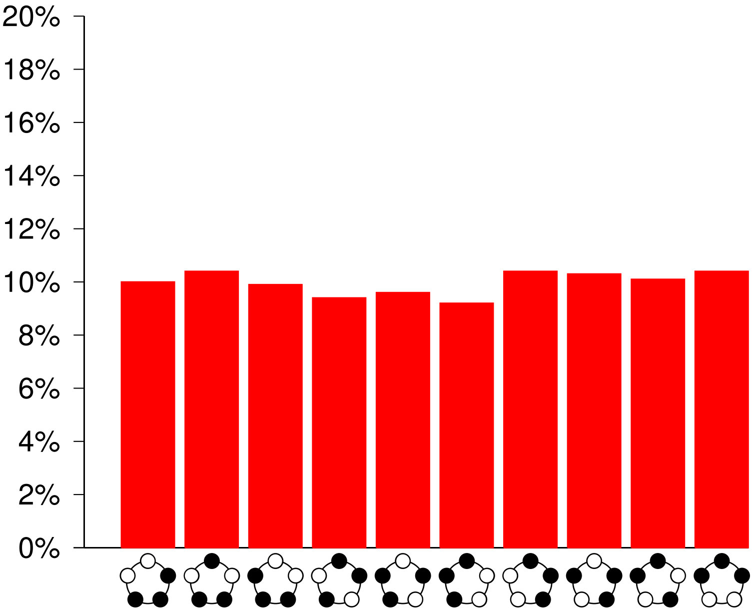

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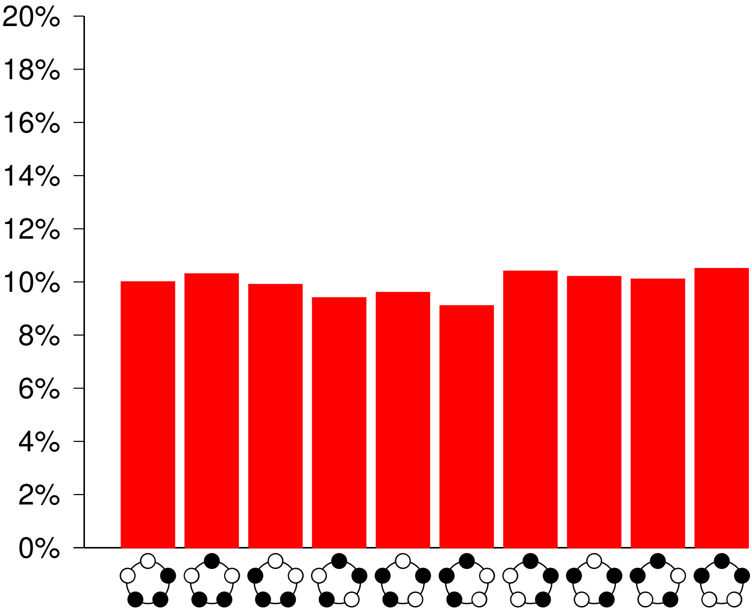

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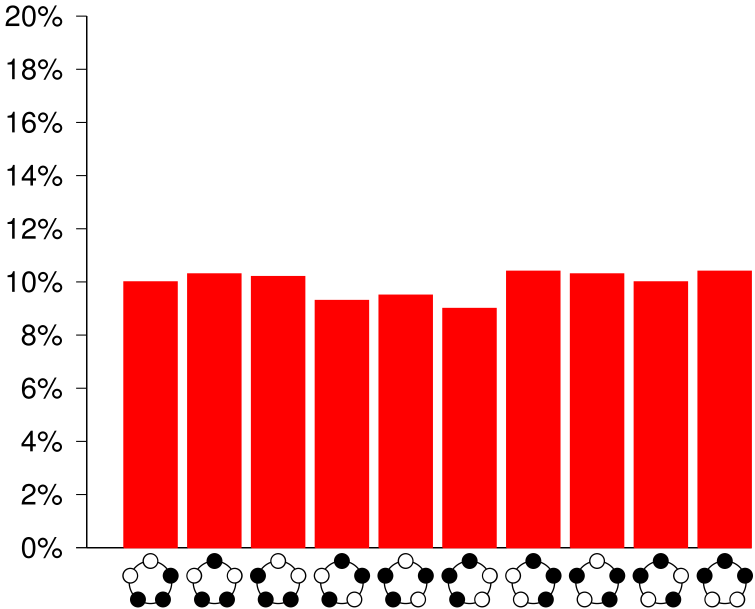

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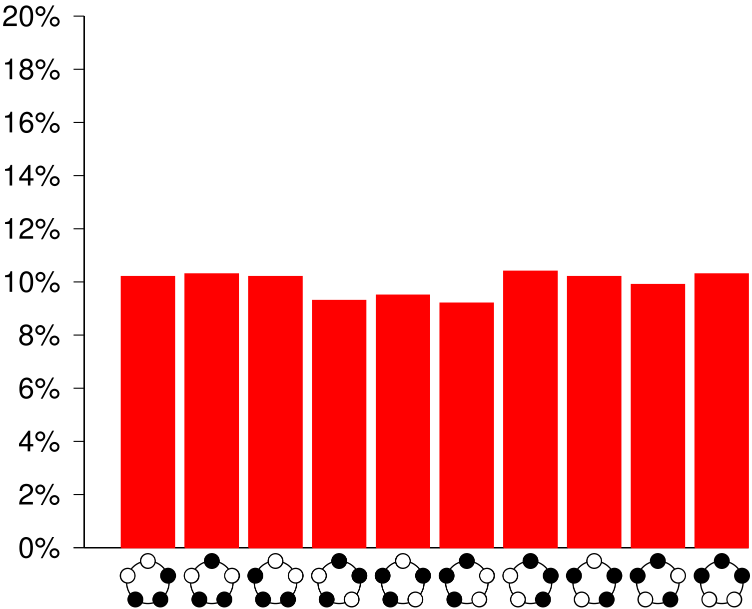

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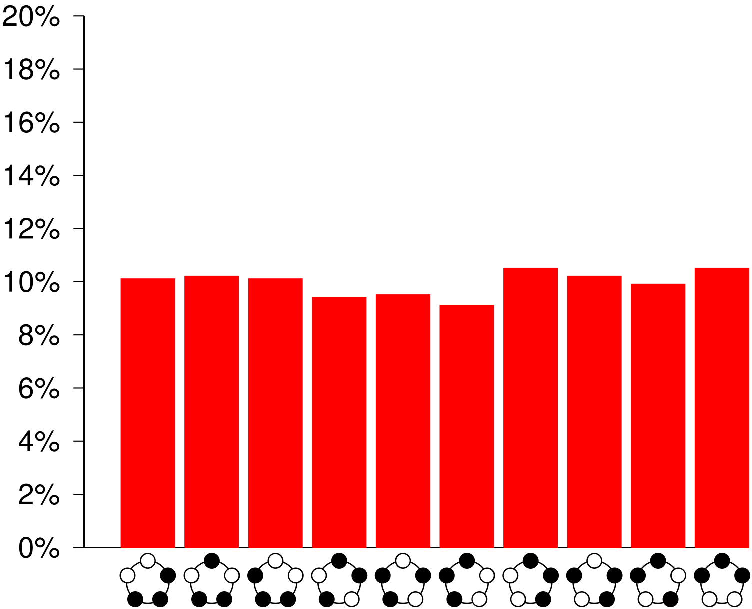

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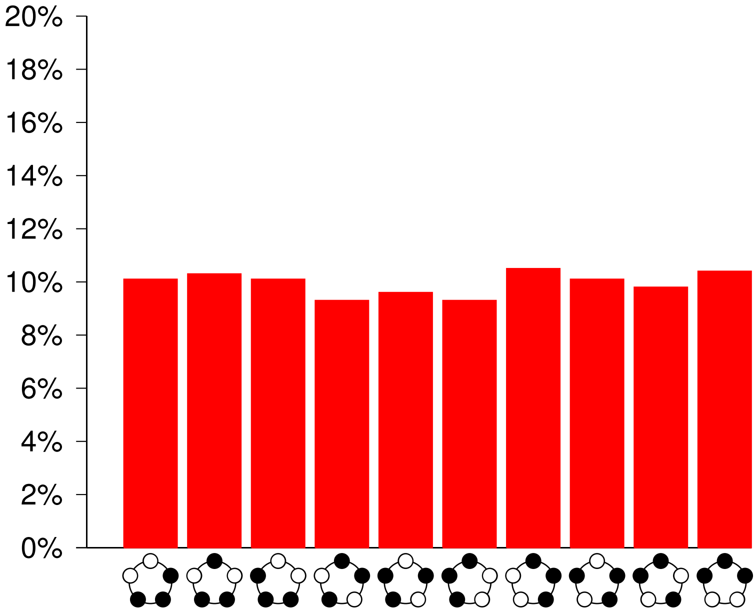

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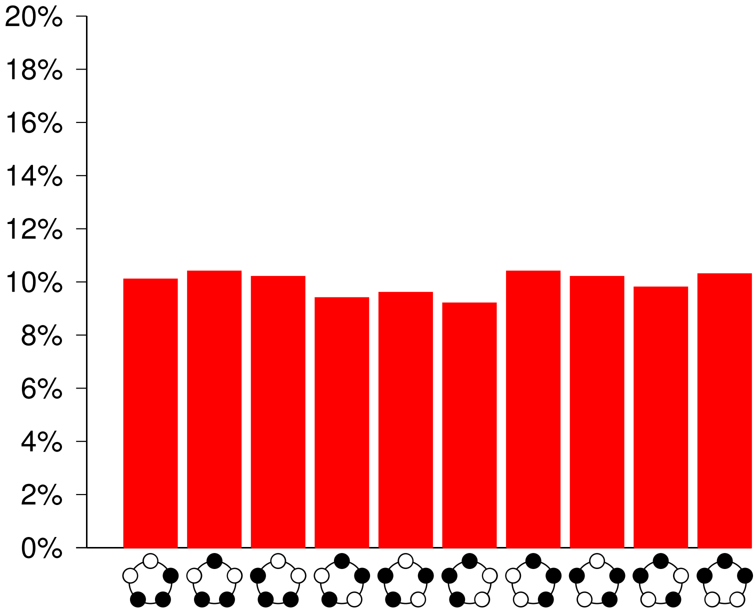

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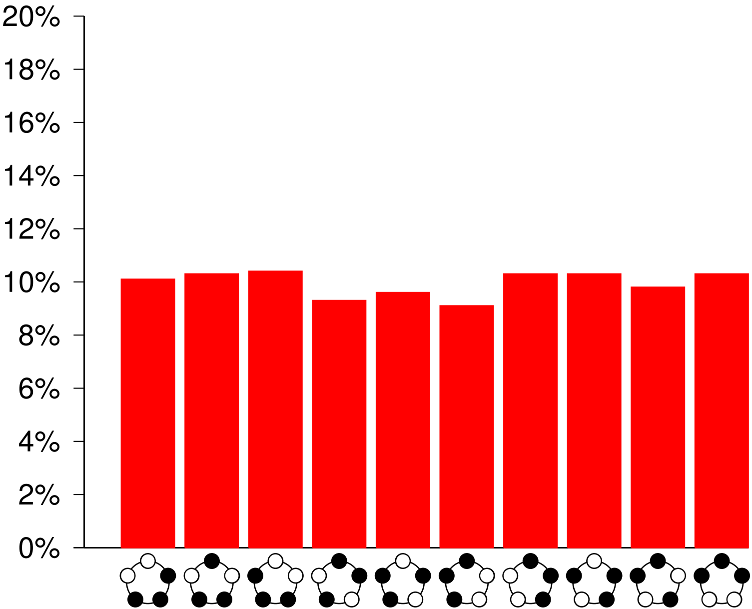

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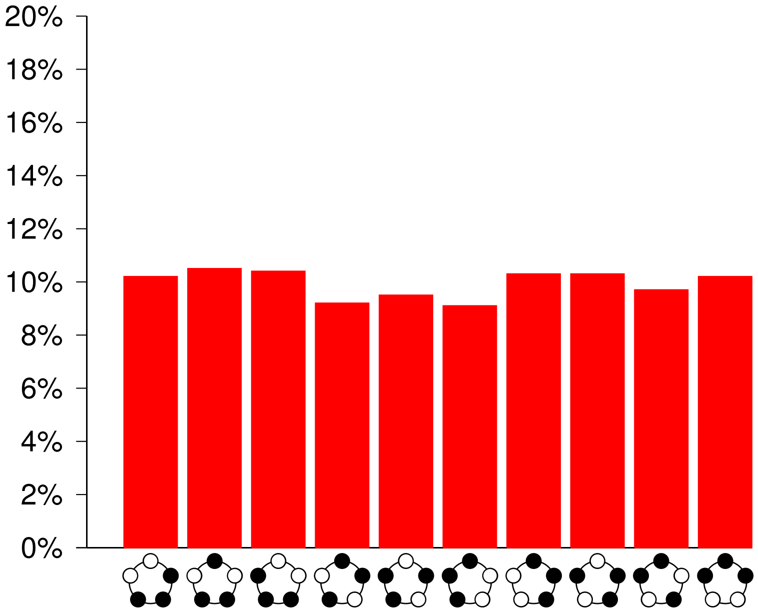

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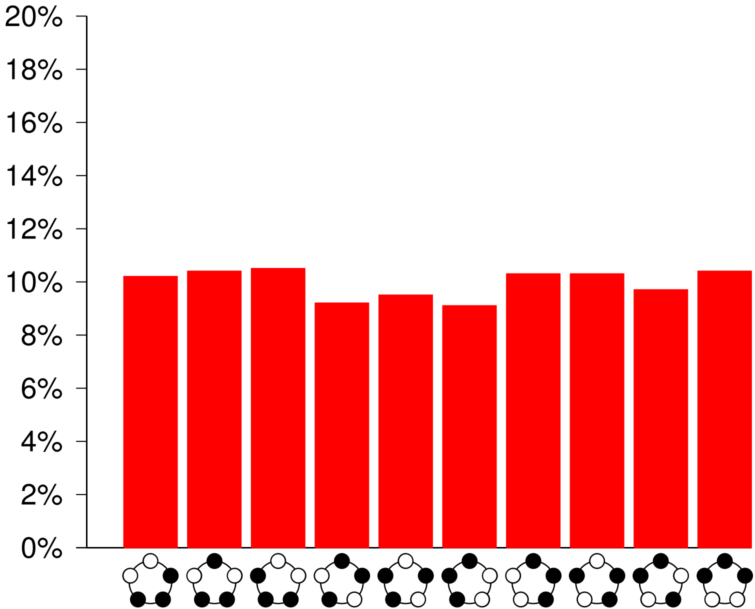

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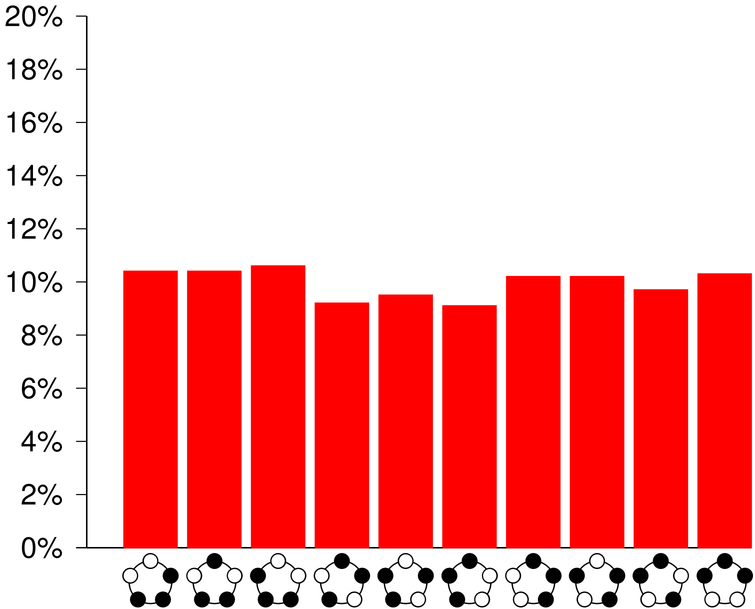

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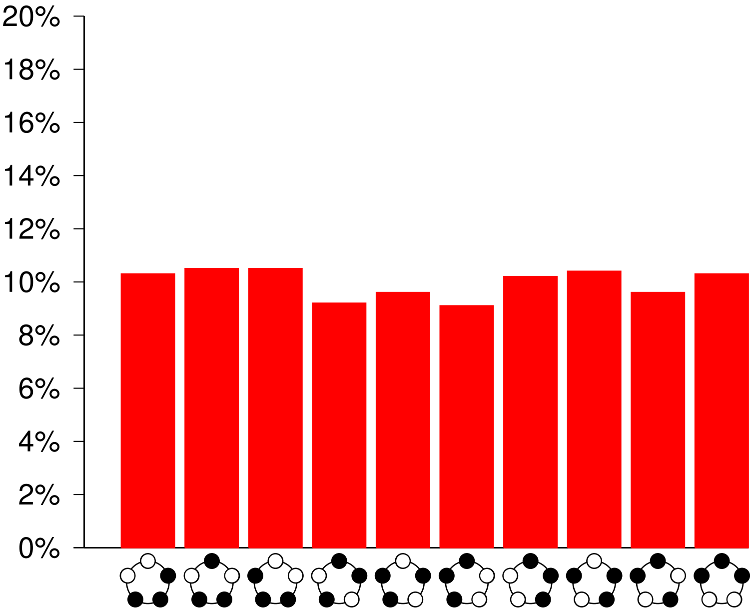

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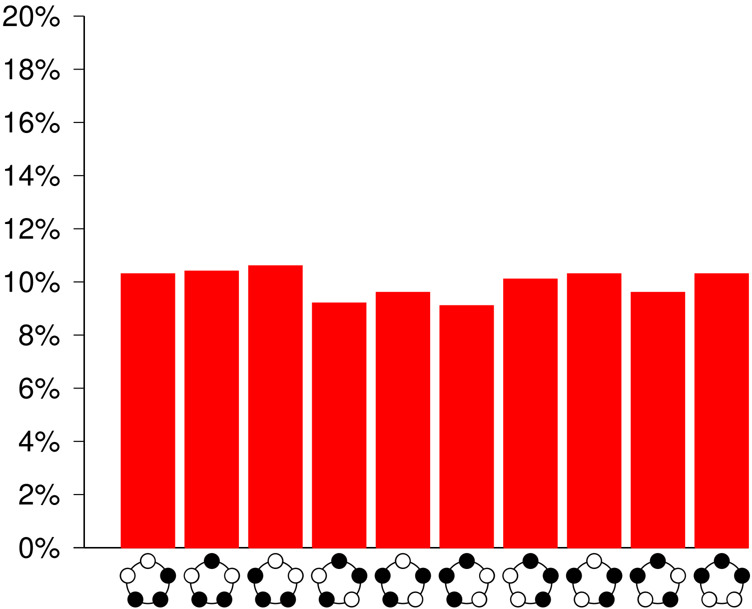

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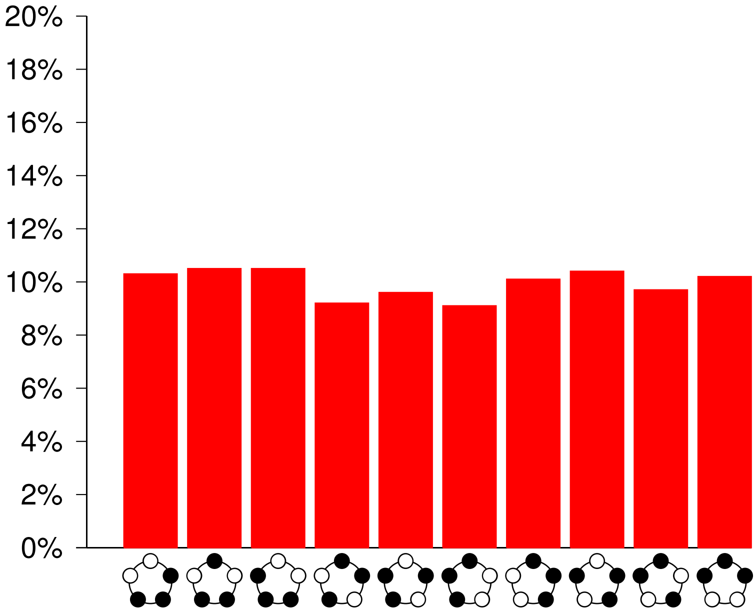

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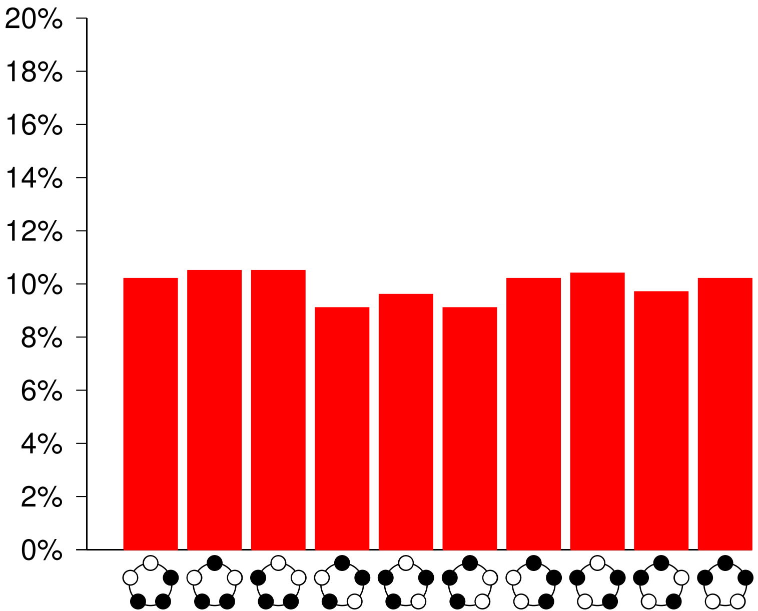

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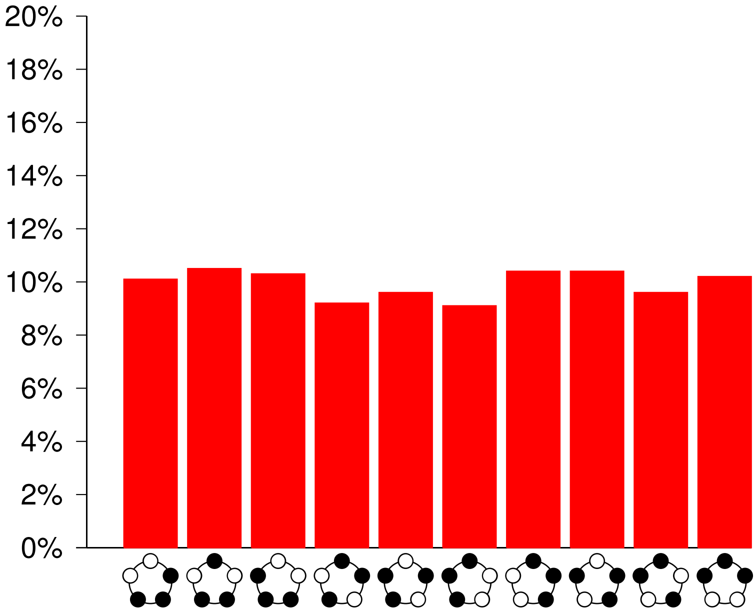

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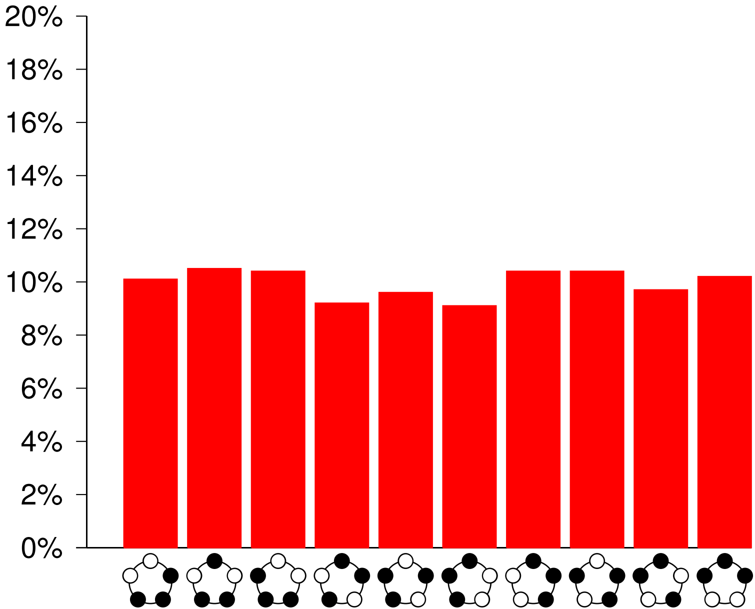

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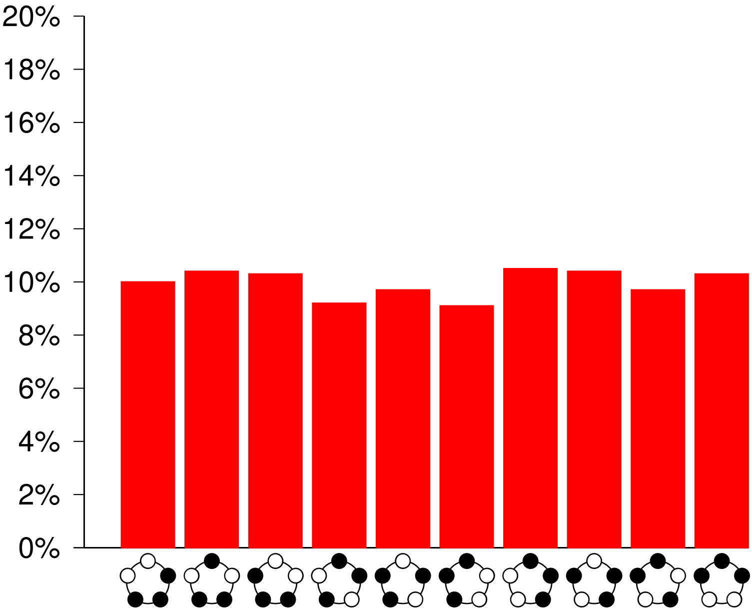

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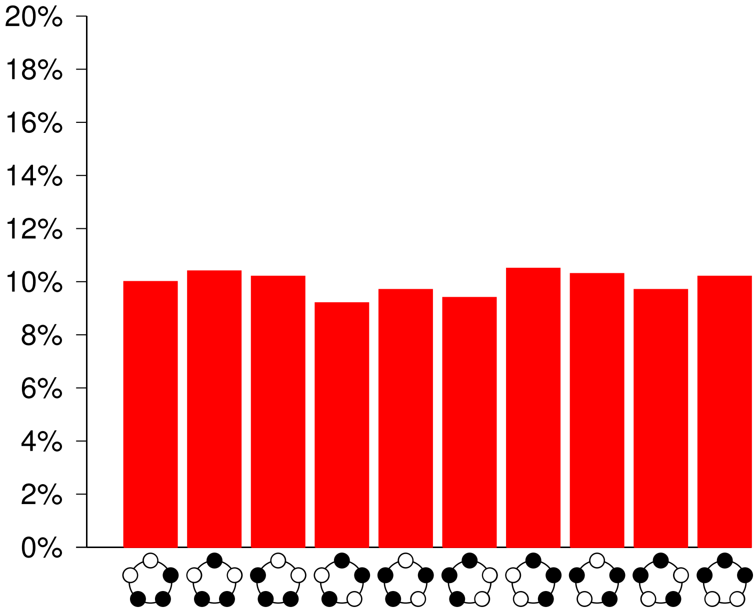

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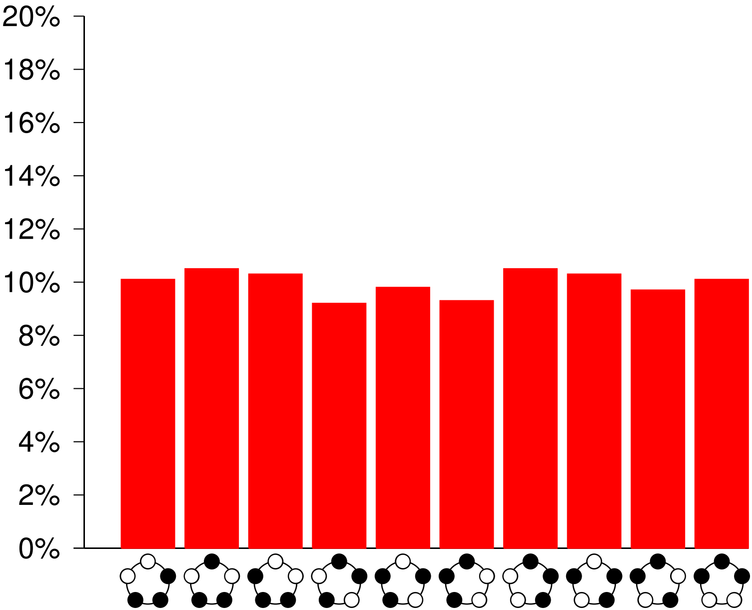

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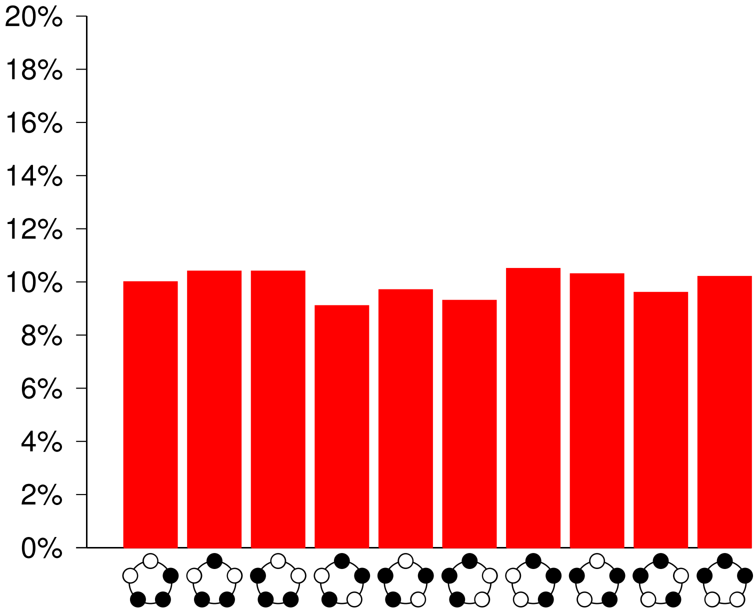

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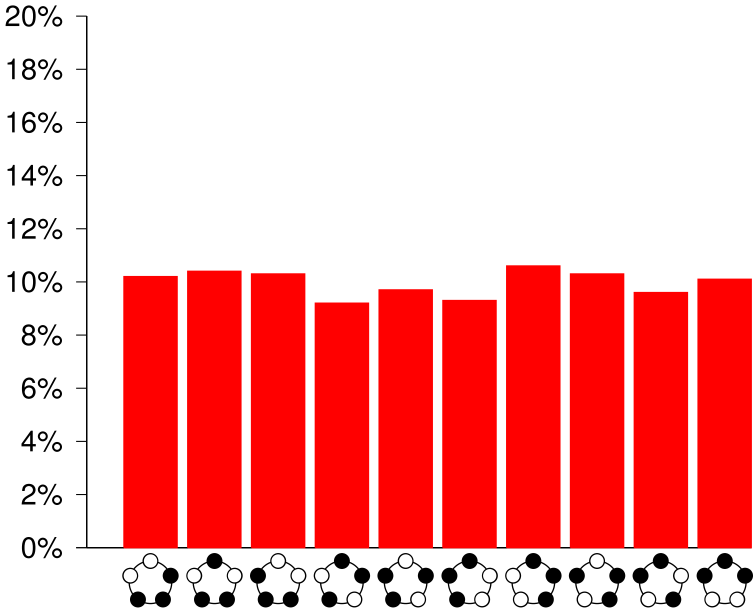

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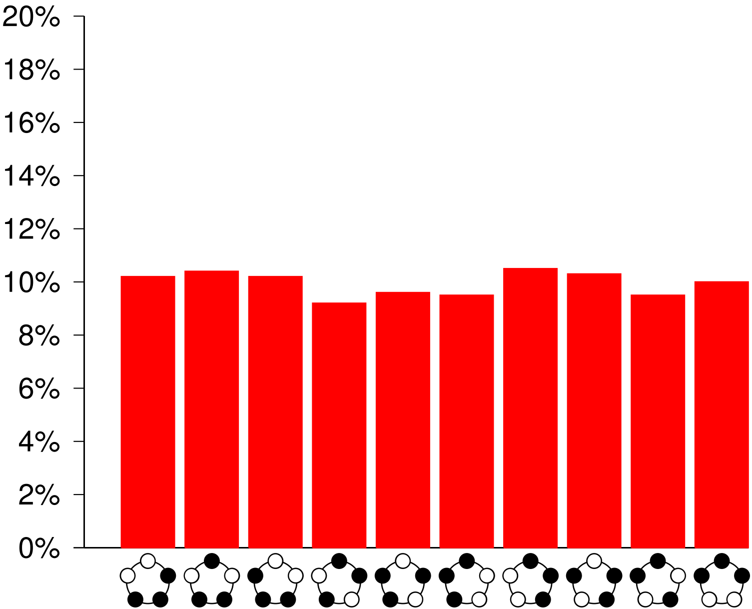

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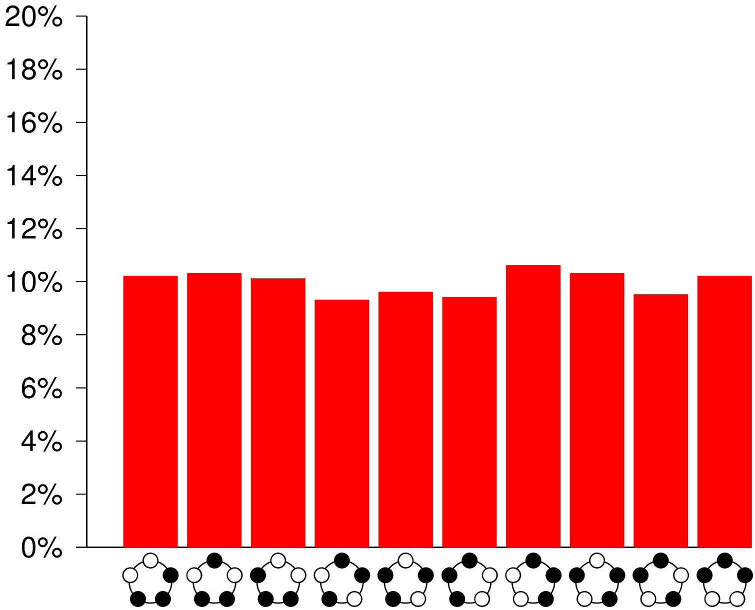

The infinite model

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In the limit we obtain a model on $\mathbb{Z}$. In its stationary distribution we have a ball with probability ϱ, and don't have one with probability $1 - \varrho$ independently for each slot.

On large scales

Let us now look at the infinite model on the large scale, and let it evolve for a long time. If we change the initial density ϱ on the large (X) scale, then the process will not be stationary anymore. Instead, its density will change on the large time scale (T).

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The density $\varrho(T, X)$ as a function of the large scale variables satisfies the differential equation

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X}[\varrho(1 - \varrho)] = 0$$

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The following are solutions of this equation:

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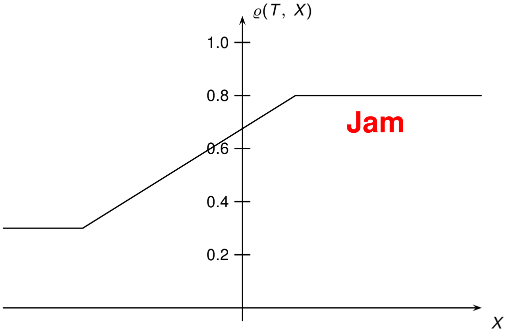

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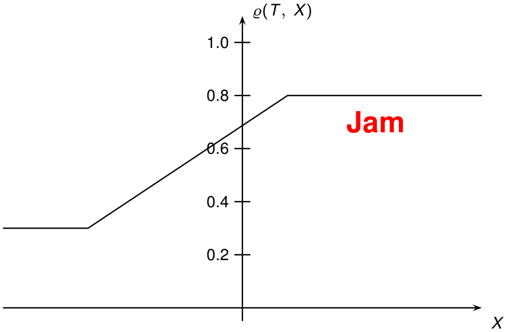

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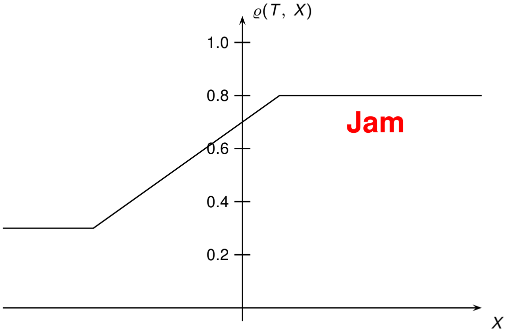

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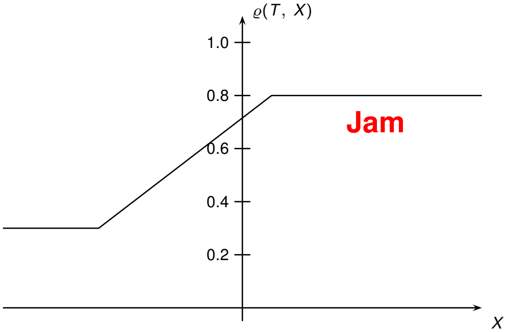

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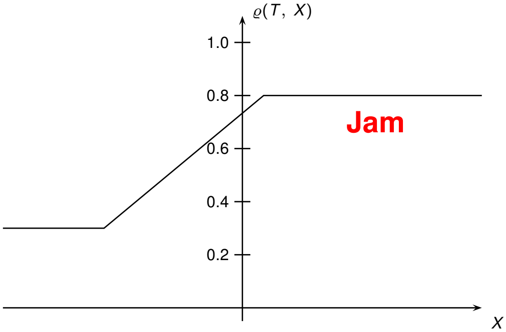

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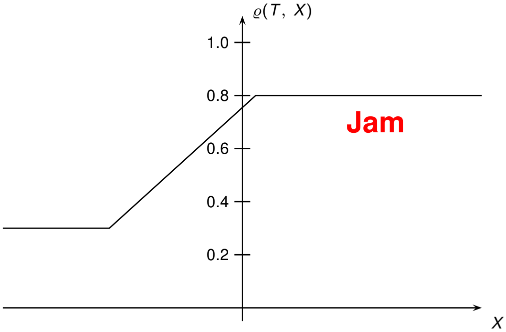

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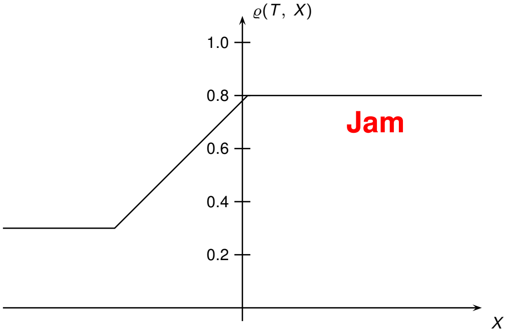

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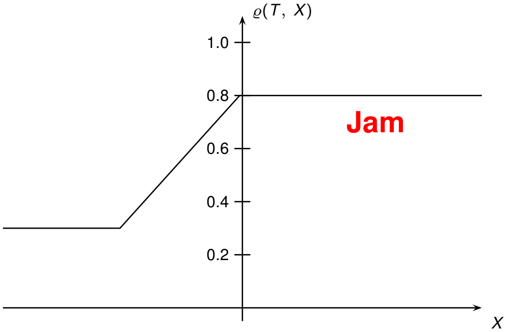

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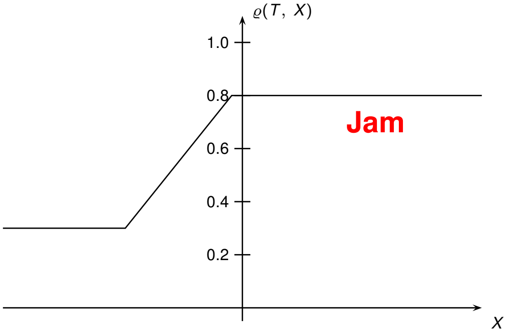

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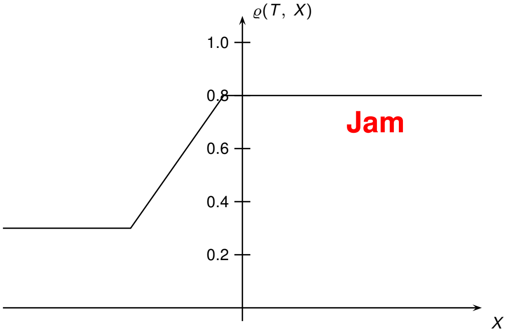

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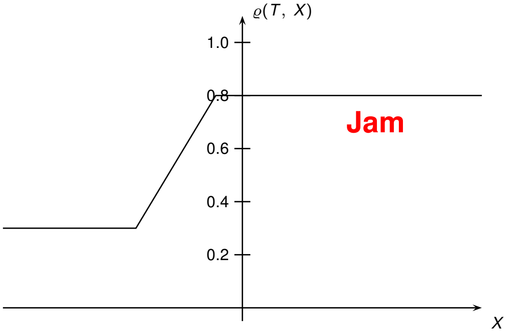

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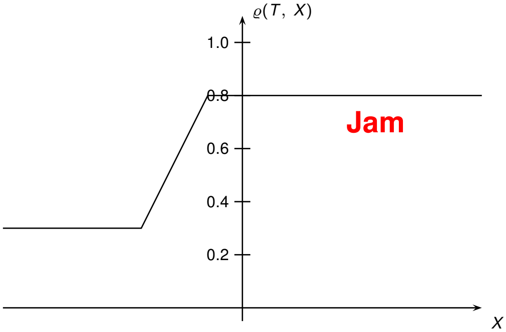

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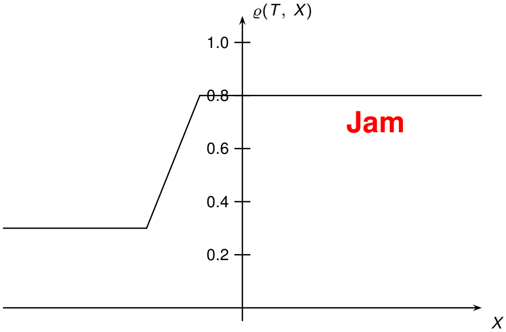

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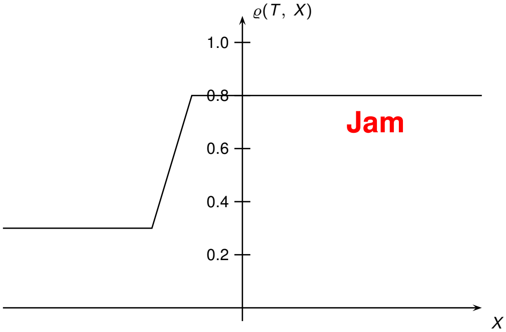

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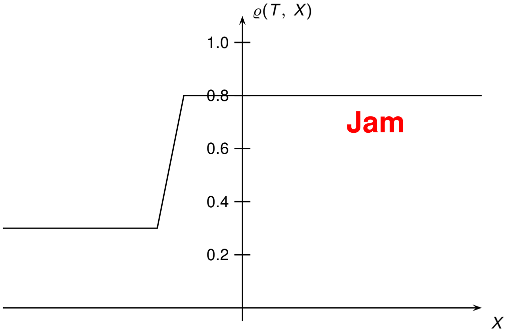

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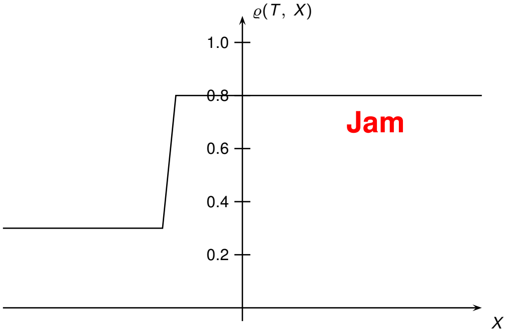

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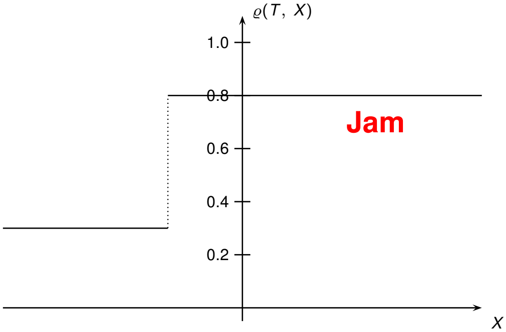

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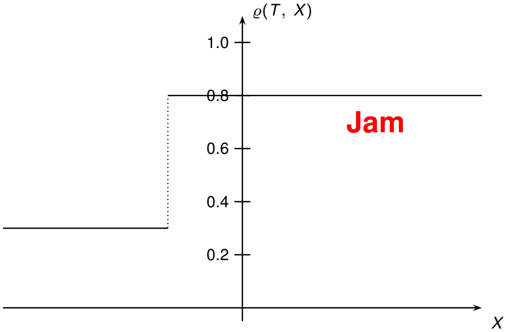

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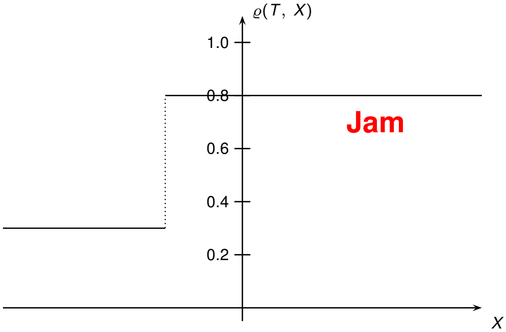

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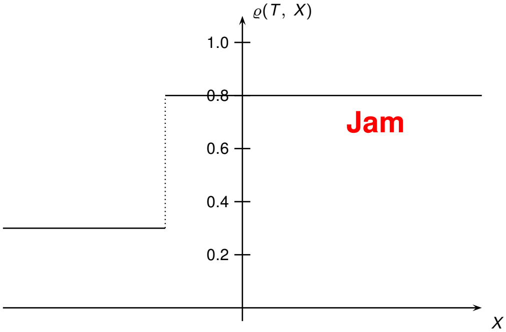

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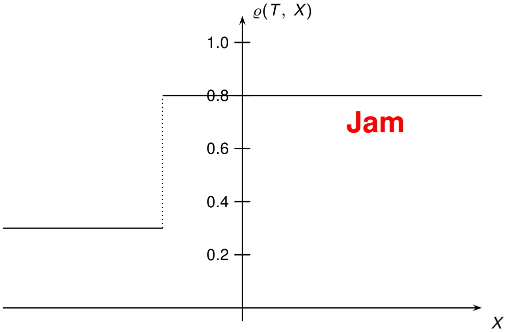

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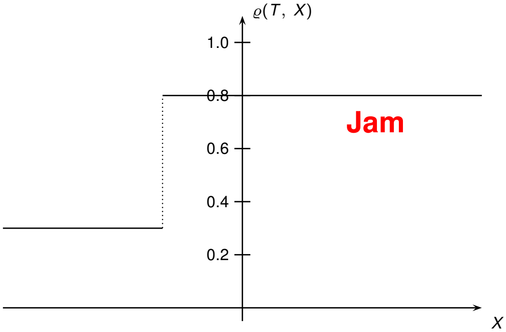

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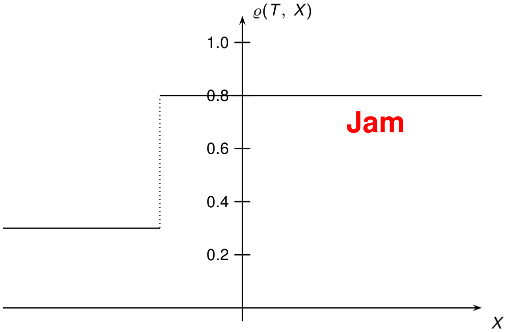

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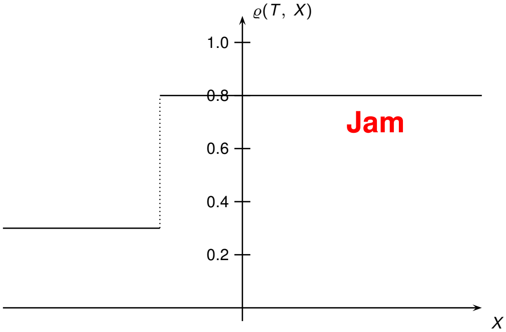

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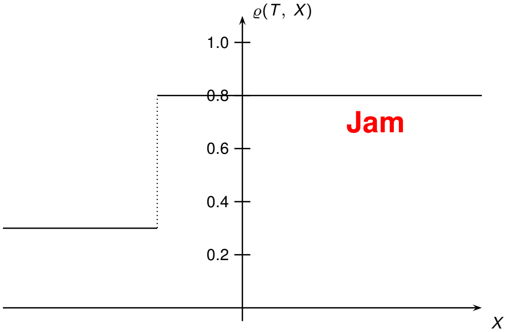

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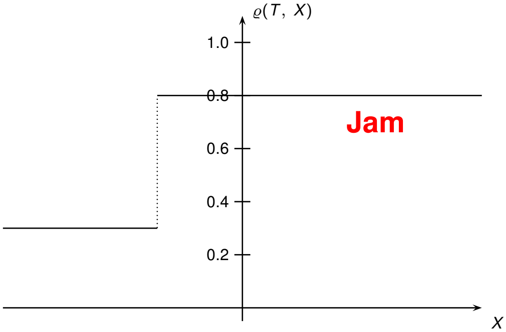

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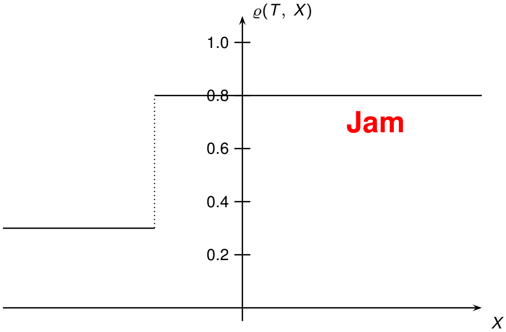

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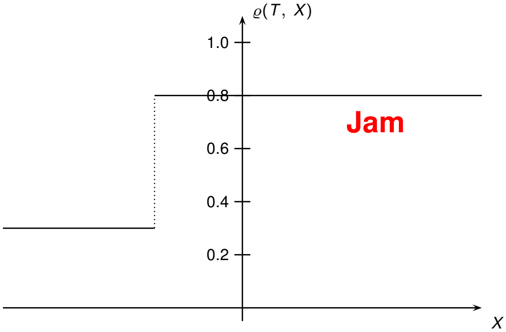

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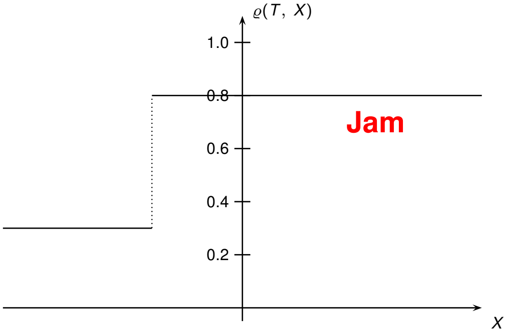

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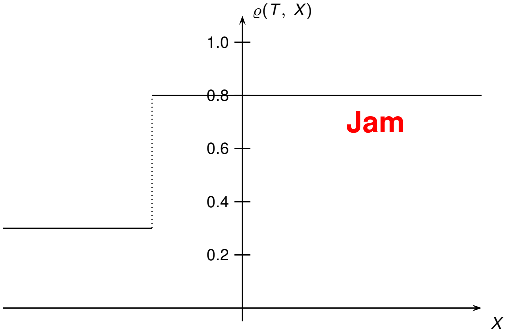

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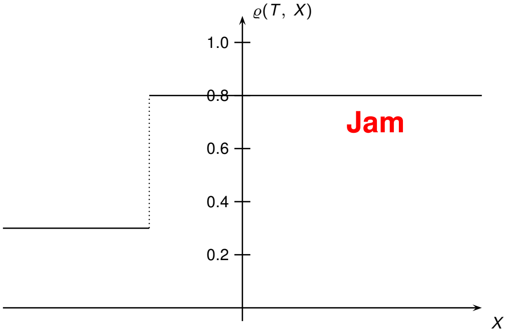

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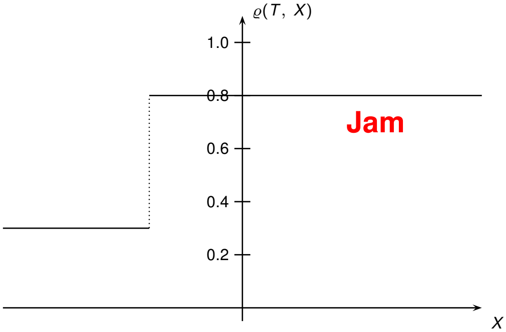

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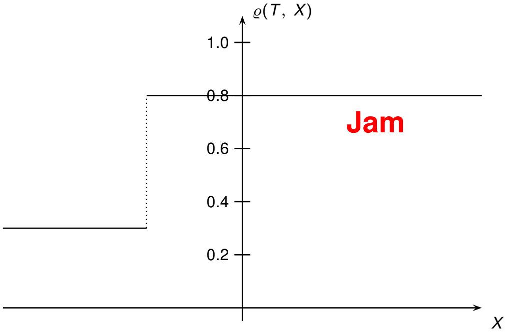

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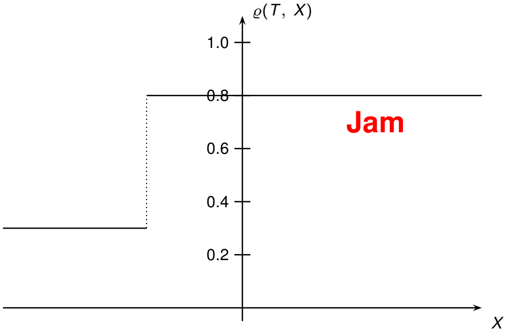

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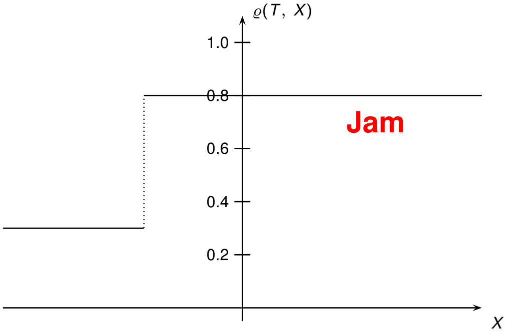

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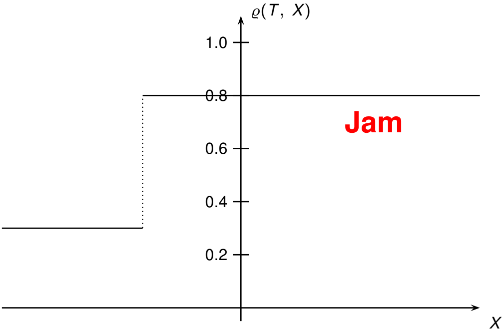

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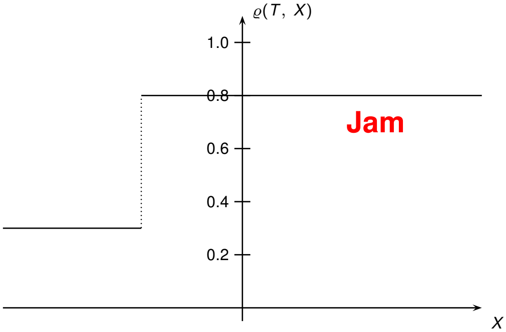

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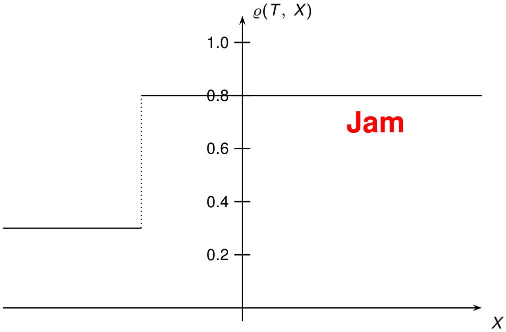

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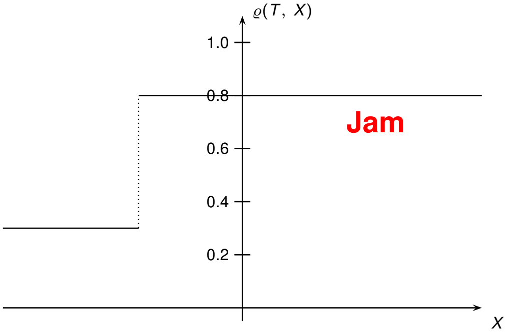

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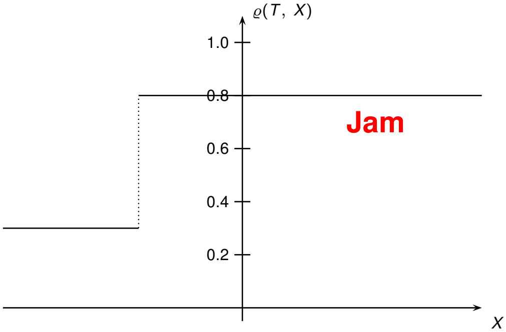

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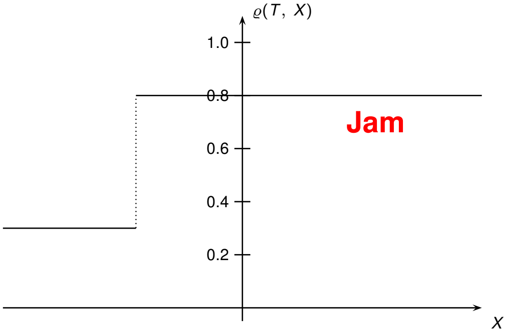

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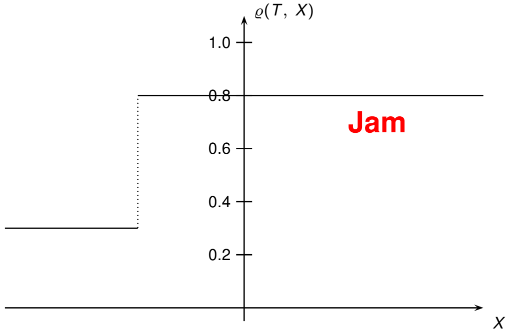

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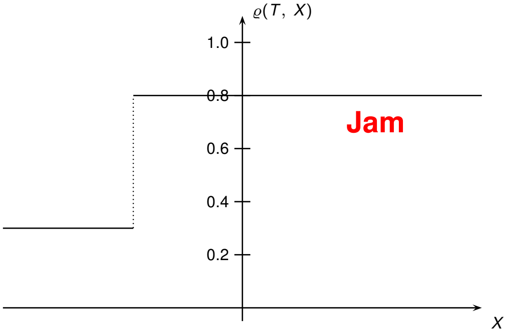

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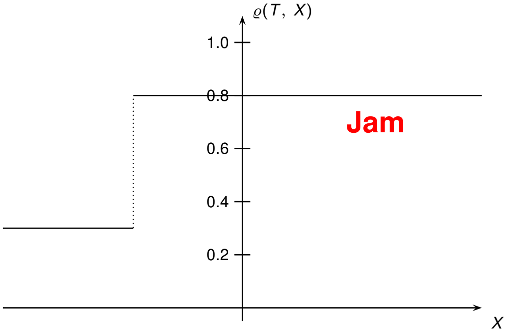

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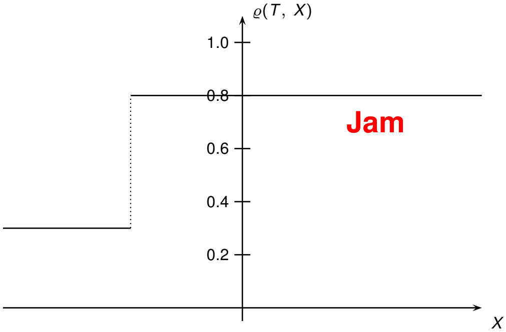

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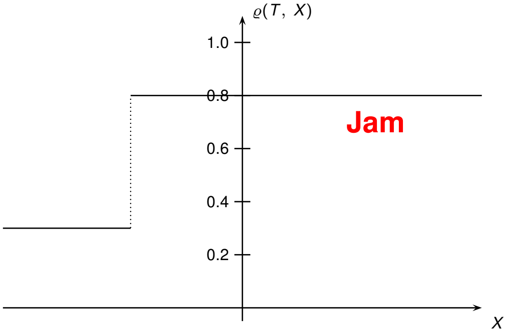

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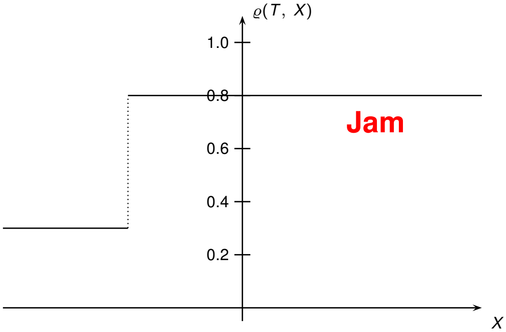

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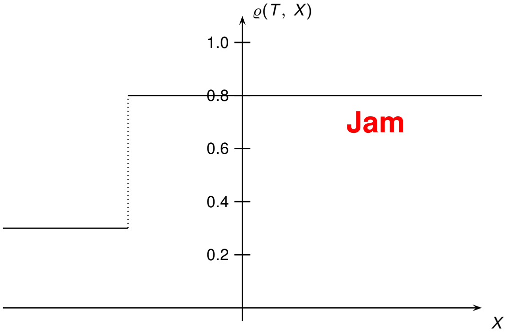

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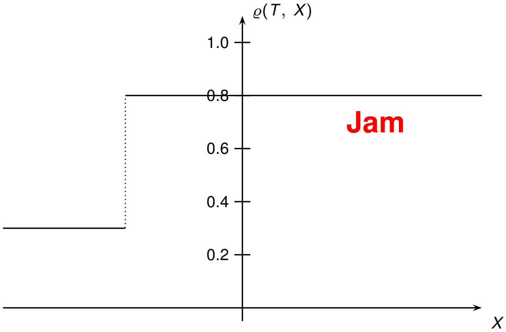

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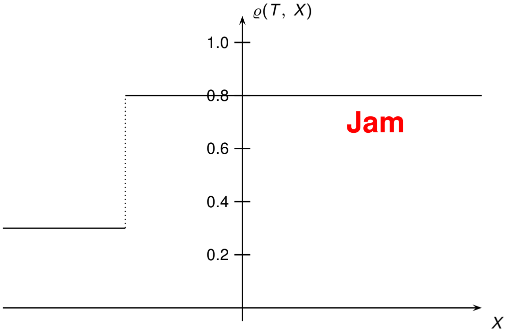

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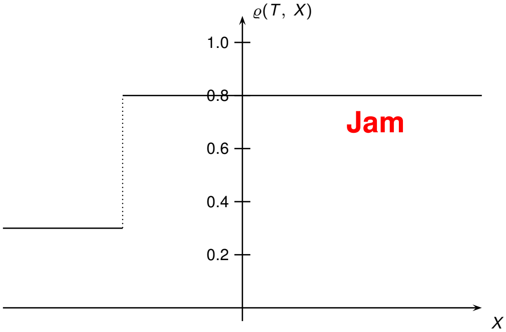

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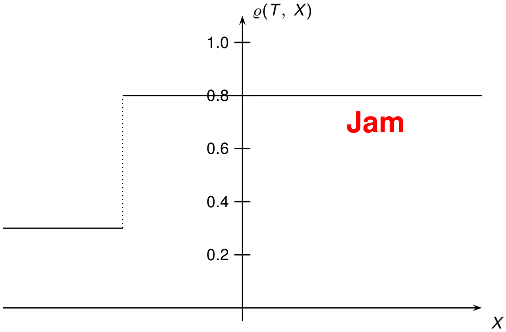

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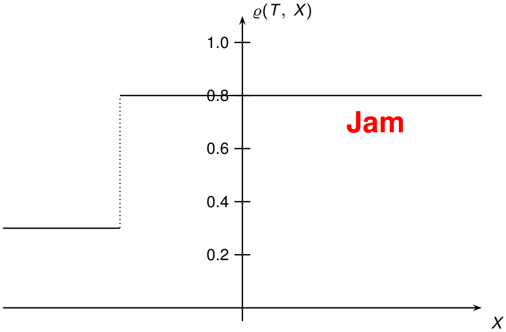

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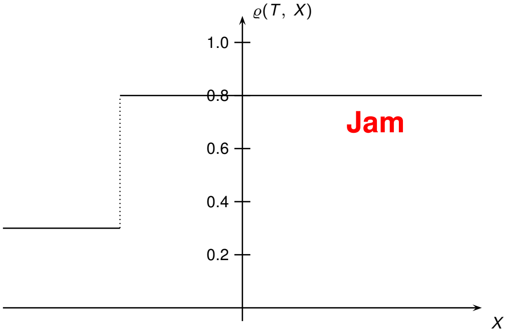

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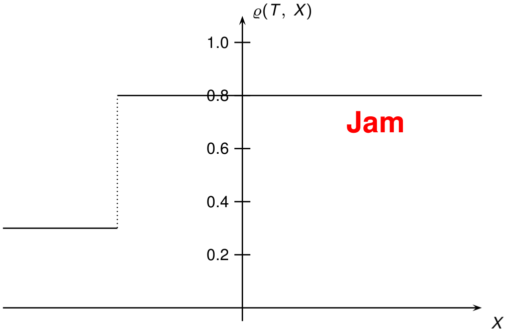

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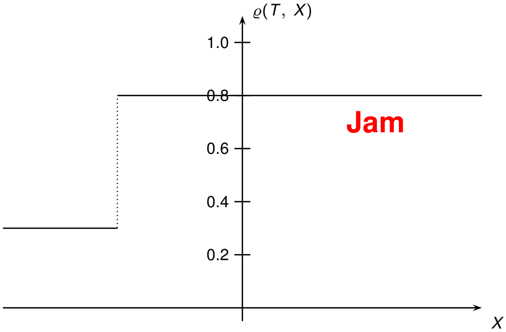

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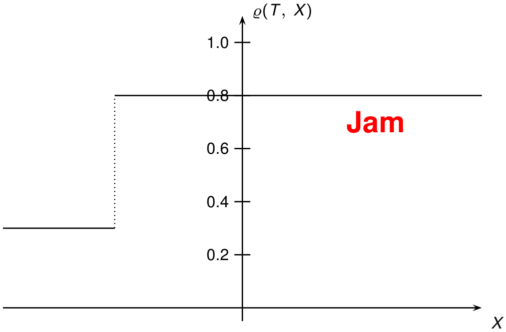

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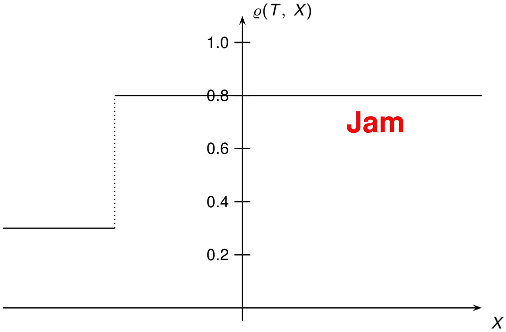

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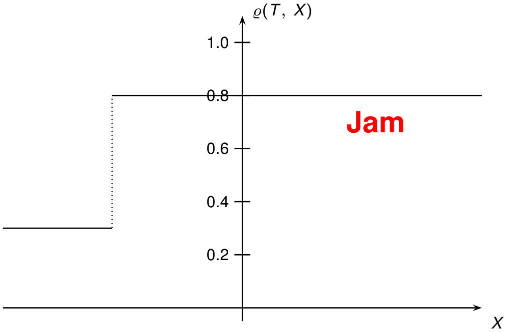

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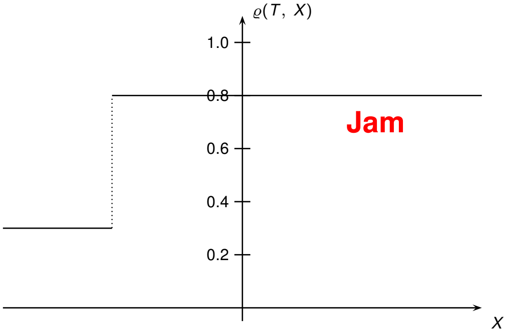

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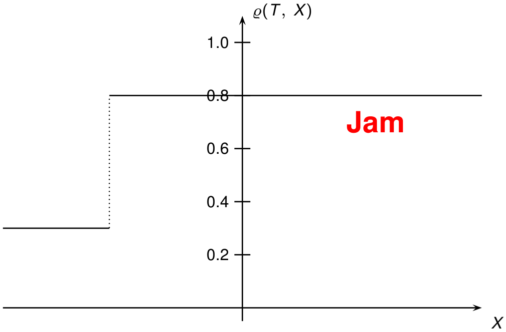

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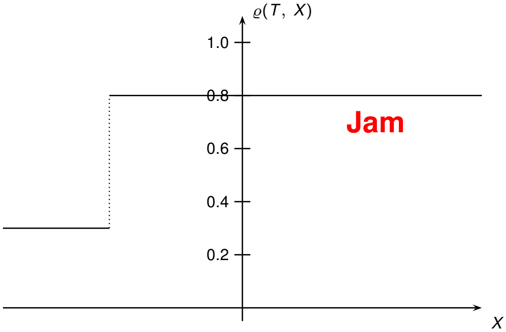
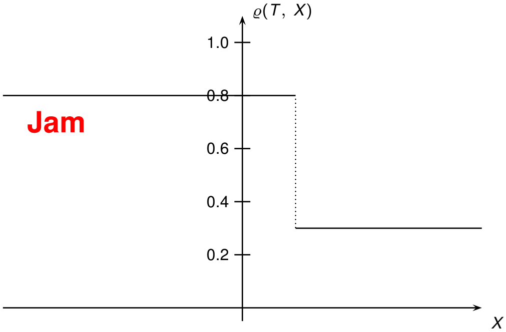


The start of the jam: sharpens.

On large scales



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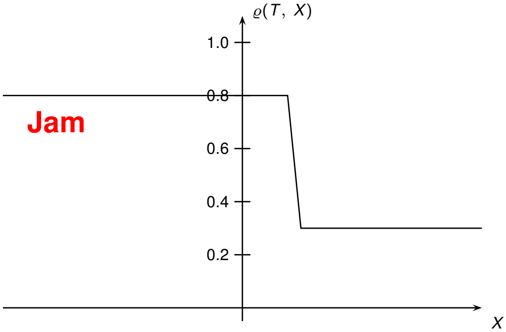

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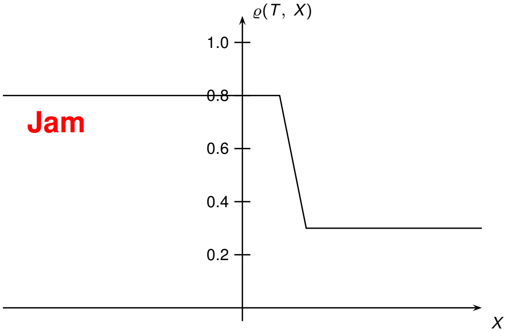

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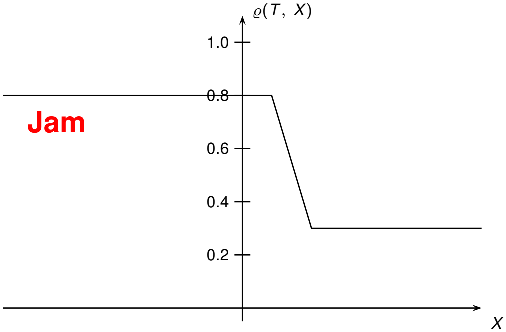

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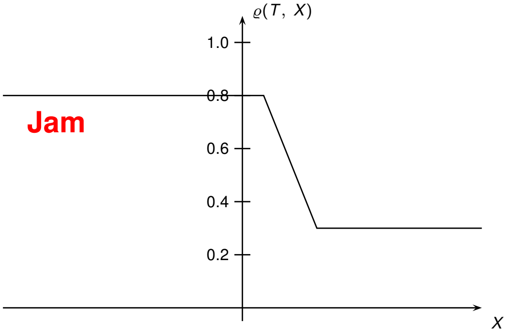

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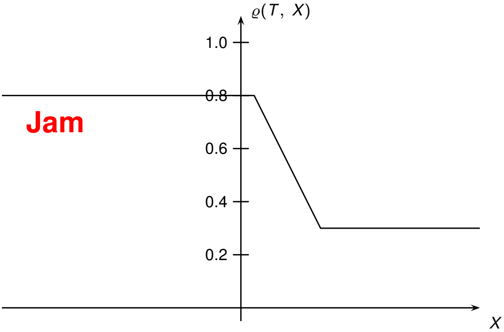

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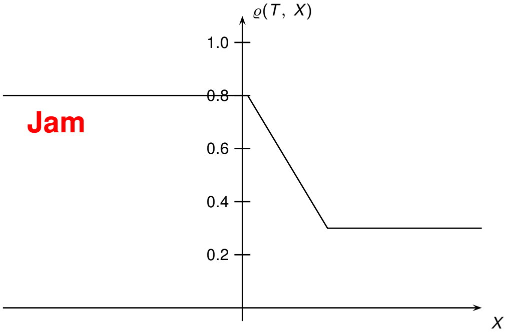

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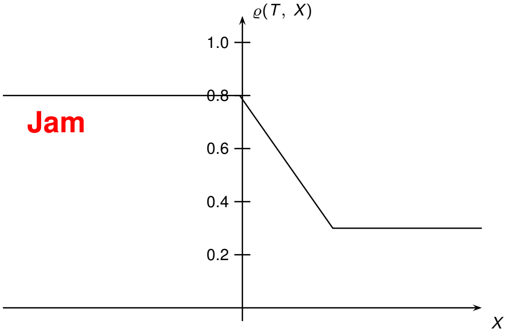

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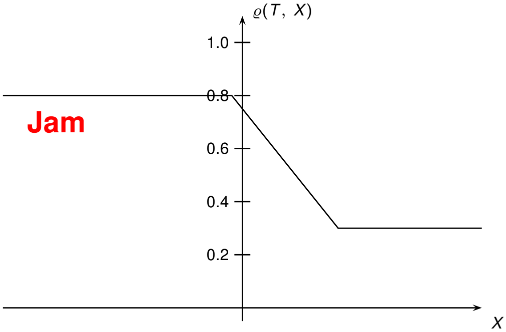

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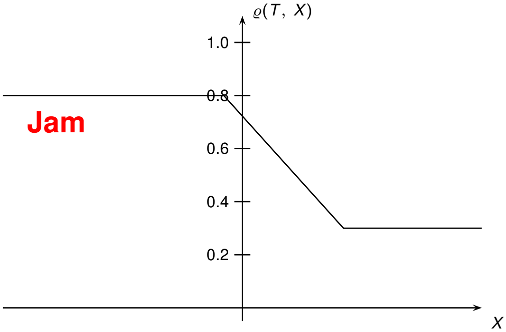

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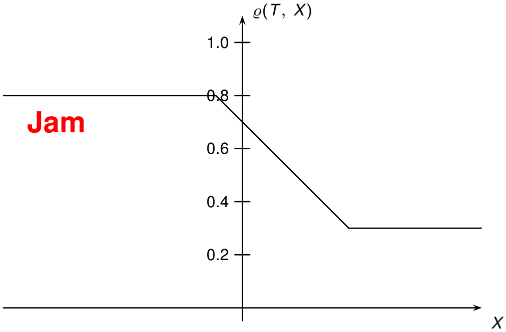

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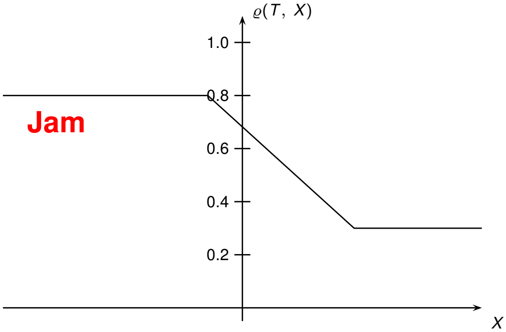

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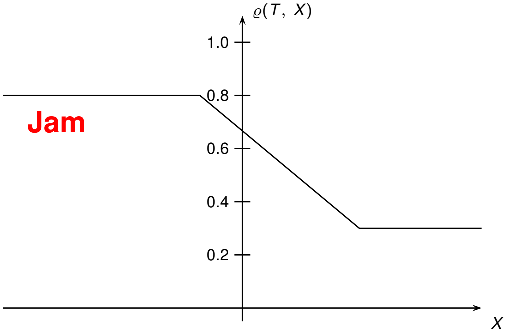

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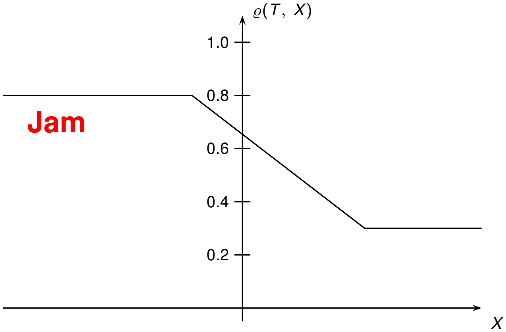

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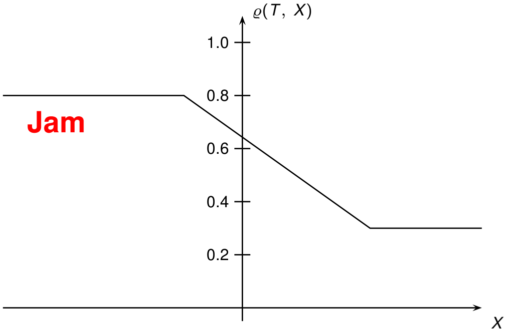

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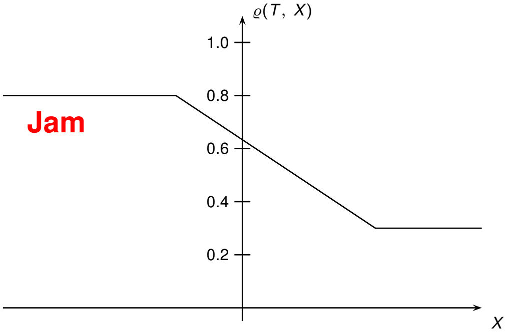

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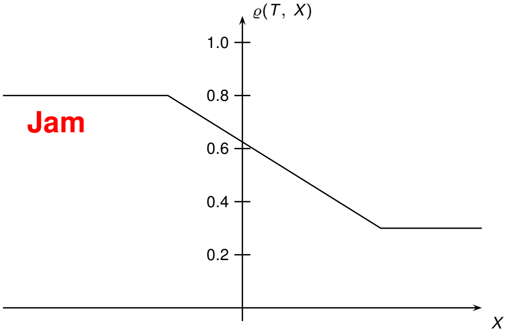

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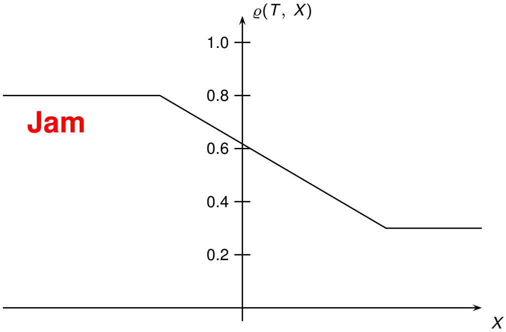

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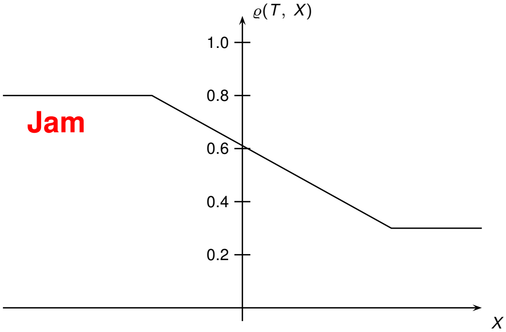

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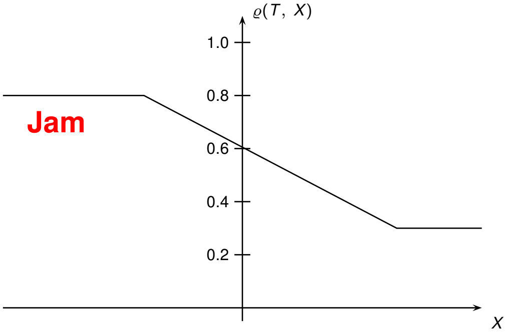

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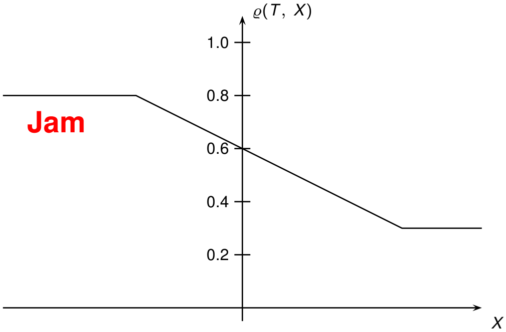

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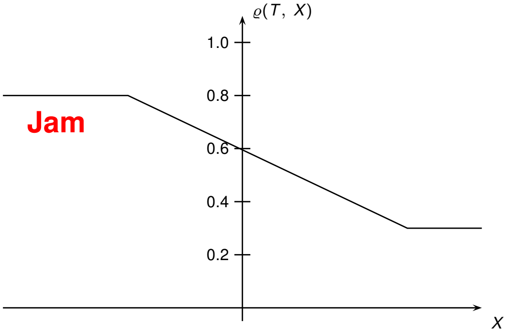

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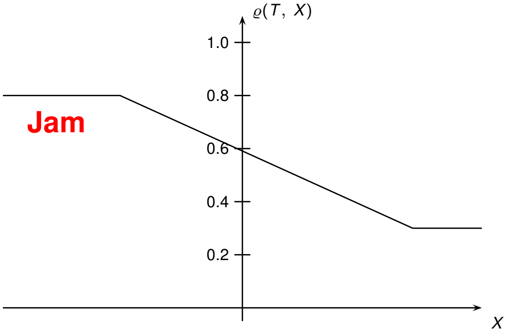

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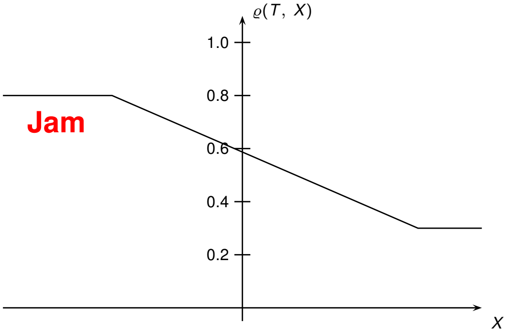

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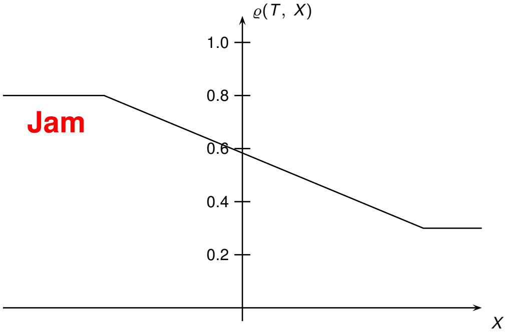

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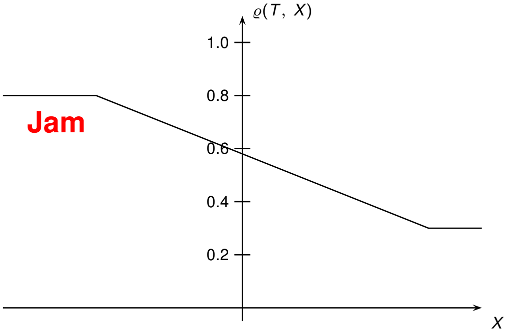

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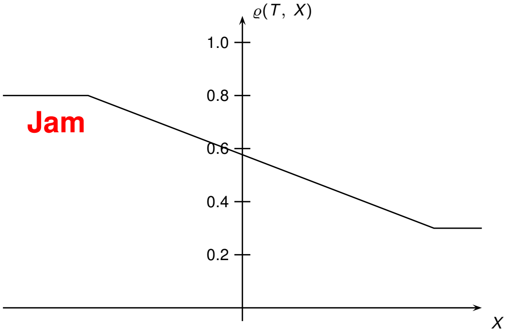

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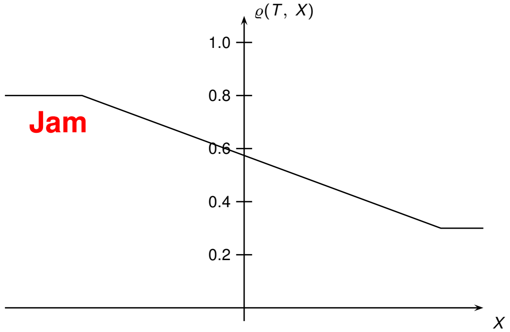

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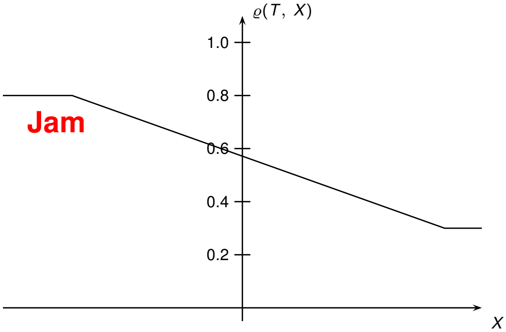

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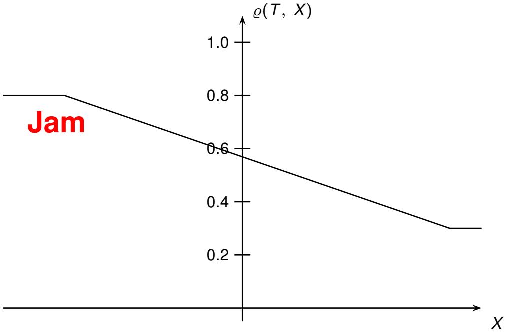

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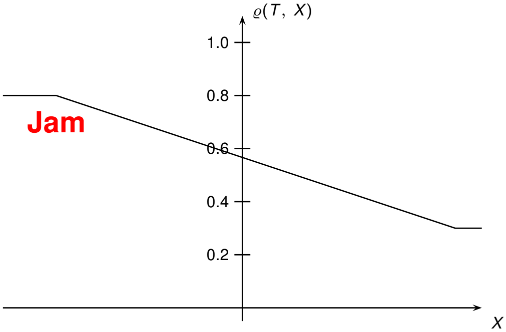

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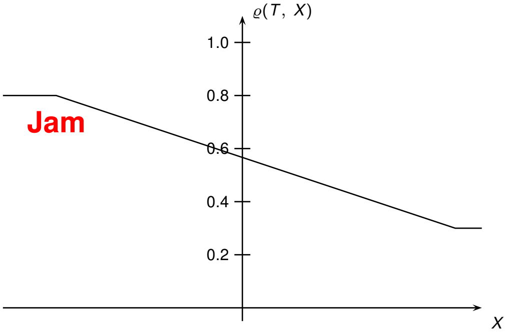
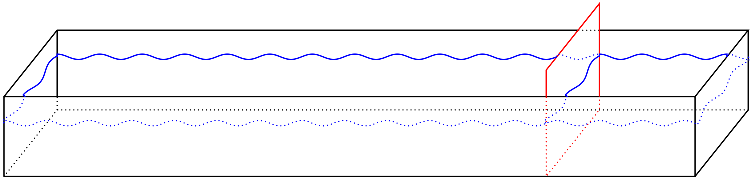


End of the jam: smoothens.

Remarks.

In general, non-linear differential equations are fun. (And difficult.)
E.g., solitary waves were discovered by John Scott Russell in 1834: he chased one along a canal for miles!



http://youtu.be/MADng1fqECY

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Thank you.