Combinations with repetition

Márton Balázs* and Bálint Tóth*

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This little write-up is part of important foundations of probability that were left out of the unit Probability 1 due to lack of time and prerequisites. We show the sixth important basic case of counting scenarios, combinations with repetitions.

Our experiment consists of picking k objects from the multiset (that is, set of repeated objects)

$$H = \{h_1, \dots, h_1, h_2, \dots, h_2, \dots, h_r, \dots, h_r\},$$

with no respect to order. A simplifying assumption we make is that each type of objects is available of multiplicity at least k, that is, $n_1, n_2, \dots, n_r \ge k$. This makes it sure that we never run out of objects of any type. Notice that the outcome of the experiment is the number of times we choose the various types. Therefore,

$$\Omega = \{(e_1, e_2, \dots, e_r) : e_i \ge 0 \ (1 \le i \le r), \ \sum_{i=1}^{r} e_i = k\}.$$

We prove

$$|\Omega| = \binom{r+k-1}{k} = \binom{r+k-1}{r-1}$$

below. Notice first that replacing our choices by indistinguishable balls and the types we choose by urns, our experiment is equivalent to

Example 1 (Bose-Einstein distribution) *k indistinguishable balls can be placed into r urns in $\binom{r+k-1}{k}$ many ways.*

To see this number, imagine that the r urns are placed next to each other in a row, thus $r-1$ walls emerge that separate them. Any outcome of the experiment corresponds to exactly one ordering of the k identical balls and the $r-1$ identical separating walls. We have seen in Probability 1 that the number of such permutations with repetition is indeed $\binom{r+k-1}{k}$.

Example 2 *We can make $\binom{4+12-1}{12} = 455$ packages of a total of 12 cakes that can be chosen out of 4 different types. If we want at least one of each of the 4 types, then we first put those into the package, then we order 8 more cakes of any types, and we can do this in $\binom{4+8-1}{8} = 165$ many ways.*

*University of Bristol / Budapest University of Technology and Economics