Sequences of random variables - examples
Further Topics in Probability
School of Mathematics, University of Bristol

Example 1. Let

$$X_n = \begin{cases} n^2 - 1, & \text{with probability } \dfrac{1}{n^2}, \\ -1, & \text{with probability } 1 - \dfrac{1}{n^2}, \end{cases}$$

be independent. One easily checks $\mathbf{E}X_n = 0\ \forall n$, hence $\lim_{n\to\infty} \mathbf{E}X_n = 0$. However, the probabilities in the first line are summable, hence Borel-Cantelli implies that a.s. $X_n \neq -1$ only happens for finitely many n. It follows that $X_n \to -1$ a.s., the limit does not swap with the expectation. Conditions of both Monotone and Dominated convergence fail.

Example 2. Let $U \sim \text{Uniform}(0, 1)$, and

$$X_1 = \mathbf{1}_{[0,\,1]}(U), \quad X_2 = \mathbf{1}_{[0,\,\frac{1}{2}]}(U), \quad X_3 = \mathbf{1}_{[\frac{1}{2},\,1]}(U), \quad X_4 = \mathbf{1}_{[0,\,\frac{1}{3}]}(U), \quad X_5 = \mathbf{1}_{[\frac{1}{3},\,\frac{2}{3}]}(U),$$
$$X_6 = \mathbf{1}_{[\frac{2}{3},\,1]}(U), \quad X_7 = \mathbf{1}_{[0,\,\frac{1}{4}]}(U), \quad X_8 = \mathbf{1}_{[\frac{1}{4},\,\frac{2}{4}]}(U), \quad X_9 = \mathbf{1}_{[\frac{2}{4},\,\frac{3}{4}]}(U), \quad X_{10} = \mathbf{1}_{[\frac{3}{4},\,1]}(U),$$
$$X_{11} = \mathbf{1}_{[0,\,\frac{1}{5}]}(U), \quad \ldots$$

This sequence converges to 0 in L^p, therefore in probability and weakly as well. (Just check the probability that $X_n \neq 0$.) However, there is always a later X_n with value 1, hence the sequence does not converge a.s.

Example 3. Let $U \sim \text{Uniform}(0, 1)$, and

- $X_n := U^n$. This converges to 0 in all senses.
- $X_n := nU^n$. This converges to 0 a.s., hence in probability and weakly as well. However,

  $$\mathbf{E}|X_n - 0| = n\mathbf{E}U^n = n \cdot \frac{1}{n+1} \to 1 \neq 0,$$

  therefore L^1 convergence does not hold. Notice how both Monotone and Dominated convergence fail for X_n.
- We can take this to more extreme by $X_n := \mathrm{e}^n \mathbf{1}_{[0,\,\frac{1}{n}]}(U)$. Again, this converges to 0 a.s. However, $||0 - X_n||_p = \frac{1}{n^{1/p}} \cdot \mathrm{e}^n \to \infty$ for any $p > 0$.

Example 4. Given a sequence $0 \leq p_n \leq 1$, let $X_n \sim \text{Bernoulli}(p_n)$ and independent. Then

- by the definitions, $p_n \to 0$ is equivalent to each of L^p and in probability convergence to 0,
- by the two Borel-Cantelli lemmas, $\sum_n p_n < \infty$ is equivalent to a.s. convergence to 0.

The choice $p_n = \frac{1}{n}$ therefore gives L^p but not a.s. convergence.