HOMEWORK SET 1

Measure theory background

Martingale Theory with Applications, 1^{st} teaching block, 2025

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Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 9^{th} October. Please show your work leading to the result, not only the result. Each problem is worth the number of $^\bullet$'s you see right next to it. Hence, for example, Problem 1.1 is worth two marks. Make sure you find all 10 $^\bullet$'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

1.1 $^{\bullet\bullet}$ Let $(\Omega,\,\mathcal{F})$ be a measurable space. Prove that if $A,\,B\in\mathcal{F}$, then

$$A\cap B,\qquad A-B \text{ (set-difference,)}\qquad A\Delta B \text{ (symmetric set-difference)}$$

are also in $\mathcal{F}$.

1.2 $^{\bullet\bullet}$ Is the union of two σ-algebras (on the same set) also a σ-algebra? If yes, prove it, if no, give a counterexample.

1.3 Is the intersection of two σ-algebras (on the same set) also a σ-algebra? If yes, prove it, if no, give a counterexample.

1.4 $^{\bullet\bullet\bullet}$ Define the Borel σ-algebra on $\mathbb{R}$ as we did in class:

$$\begin{aligned}&\mathfrak{B}(\mathbb{R})\\ &:=\sigma\Big\{\bigcup_{i=1}^{n}(a_i,\,b_i]\,:\,n<\infty,\text{ and }a_1<b_1\le a_2<b_2\le\cdots\le a_n<b_n\text{ in }\mathbb{R}\cup\{-\infty\};\\ &\qquad\bigcup_{i=1}^{n}(a_i,\,b_i]\cup(c,\,\infty)\,:\,n<\infty,\text{ and }a_1<b_1\le\cdots\le a_n<b_n\le c\text{ in }\mathbb{R}\cup\{-\infty\}\Big\}.\end{aligned}$$

Show that each of

$$(a,\,b),\qquad [a,\,b),\qquad [a,\,b],\qquad \{a\},\qquad (a,\,\infty)$$

are in $\mathfrak{B}(\mathbb{R})$ for any $a<b$ in $\mathbb{R}$.

1.5 (*Shiryaev.*) Let Ω be a countable set and $\mathcal{F}$ the collection of all its subsets. Put $\mu(A)=0$ if A is finite and $\mu(A)=\infty$ if A is infinite. Show that the set function μ is finitely additive but not σ-additive.

1.6 $^{\bullet\bullet\bullet}$ (*Shiryaev.*) Let μ be the Lebesgue-Stieltjes measure generated by a continuous distribution function. Show that if the set A is at most countable, then $\mu(A)=0$.

1.7 (*Hard: Construction of the Vitali set – an example that cannot be Lebesgue measurable.*) Let $\Omega:=[0,\,1)$ and define on Ω the following equivalence relation:

$$x\sim y \text{ iff } x-y\in\mathbb{Q}\text{ (the rational numbers)}.$$

Let $V\subset[0,\,1)$ consist of *exactly one representative element from each equivalence class of* $\sim$. (Notice: this construction relies on the Axiom of Choice.) For $q\in\mathbb{Q}\cap[0,\,1)$, denote

$$V_q:=\{x+q\ (\mathrm{mod}\ 1)\,:\,x\in V\}.$$

Prove that

(a) The sets V_q are congruent: for any $q,\, q' \in \mathbb{Q} \cap [0,\, 1)$, $V_{q'} = (q' - q) + V_q \pmod{1}$.

(b) If $q \neq q'$ in $\mathbb{Q} \cap [0,\, 1)$, then $V_q \cap V_{q'} = \emptyset$.

(c) $\bigcup_{q \in \mathbb{Q} \cap [0,\, 1)} V_q = [0,\, 1)$.

Conclude that the Vitali set V cannot be Lebesgue measurable.