

HOMEWORK SET 1
Measure theory background
 Martingale Theory with Applications, 1st teaching block, 2026
 School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 8th October. Please show your work leading to the result, not only the result. Each problem is worth the number of •'s you see right next to it. Hence, for example, Problem 1.1 is worth two marks. Make sure you find all 10 •'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

- 1.1 •• Let $(\Omega, \mathcal{F})$ be a measurable space. Prove that if $A, B \in \mathcal{F}$, then

$$A \cap B, \quad A - B \text{ (set-difference),} \quad A \Delta B \text{ (symmetric set-difference)}$$

are also in $\mathcal{F}$.

- 1.2 Is the union of two σ -algebras (on the same set) also a σ -algebra? If yes, prove it, if no, give a counterexample.
- 1.3 •• Is the intersection of two σ -algebras (on the same set) also a σ -algebra? If yes, prove it, if no, give a counterexample.
- 1.4 ••• Define the Borel σ -algebra on $\mathbb{R}$ as we did in class:

$$\begin{aligned} \mathfrak{B}(\mathbb{R}) \\ := \sigma \left\{ \bigcup_{i=1}^n (a_i, b_i] : n < \infty, \text{ and } a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \text{ in } \mathbb{R} \cup \{-\infty\}; \right. \\ \left. \bigcup_{i=1}^n (a_i, b_i] \cup (c, \infty) : n < \infty, \text{ and } a_1 < b_1 \leq \dots \leq a_n < b_n \leq c \text{ in } \mathbb{R} \cup \{-\infty\} \right\}. \end{aligned}$$

Show that each of

$$(a, b), \quad [a, b), \quad [a, b], \quad \{a\}, \quad (a, \infty)$$

are in $\mathfrak{B}(\mathbb{R})$ for any $a < b$ in $\mathbb{R}$.

- 1.5 ••• (*Shiryaev.*) Let Ω be a countable set and $\mathcal{F}$ the collection of all its subsets. Put $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is infinite. Show that the set function μ is finitely additive but not σ -additive.
- 1.6 (*Shiryaev.*) Let μ be the Lebesgue-Stieltjes measure generated by a continuous distribution function. Show that if the set A is at most countable, then $\mu(A) = 0$.
- 1.7 (*Hard: Construction of the Vitali set – an example that cannot be Lebesgue measurable.*) Let $\Omega := [0, 1)$ and define on Ω the following equivalence relation:

$$x \sim y \text{ iff } x - y \in \mathbb{Q} \text{ (the rational numbers).}$$

Let $V \subset [0, 1)$ consist of *exactly one representative element from each equivalence class of* $\sim$. (Notice: this construction relies on the Axiom of Choice.) For $q \in \mathbb{Q} \cap [0, 1)$, denote

$$V_q := \{x + q \pmod{1} : x \in V\}.$$

Prove that

- (a) The sets V_q are congruent: for any $q, q' \in \mathbb{Q} \cap [0, 1)$, $V_{q'} = (q' - q) + V_q \pmod{1}$.
- (b) If $q \neq q'$ in $\mathbb{Q} \cap [0, 1)$, then $V_q \cap V_{q'} = \emptyset$.
- (c) $\bigcup_{q \in \mathbb{Q} \cap [0, 1)} V_q = [0, 1)$.

Conclude that the Vitali set V cannot be Lebesgue measurable.