HOMEWORK SET 2

Cauchy, Gamma, generating fuctions, branching processes

Further Topics in Probability, 1^{st} teaching block, 2013

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Problems with $^{\bullet}$'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 11:59pm on Wednesday, 23^{th} October. Please show your work leading to the result, not only the result. Each problem worth the number of $^{\bullet}$'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

2.1 Prove that if ξ is a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, then $X := 1/\xi$, $Y := 2\xi/(1-\xi^2)$, and $Z := (3\xi - \xi^3)/(1-3\xi^2)$ are also Cauchy distributed. *HINT: use the trigonometric identities: if* $\xi = \tan(\alpha)$, *then* $1/\xi = \tan(\frac{\pi}{2} - \alpha)$, $2\xi/(1-\xi^2) = \tan(2\alpha)$, *and* $(3\xi - \xi^3)/(1-3\xi^2) = \tan(3\alpha)$.

2.2 Let X be a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. We know that $\mathbf{E}|X| = \infty$, but $\mathbf{E}(|X|^{1-\varepsilon}) < \infty$ for any $\varepsilon > 0$. Determine the limit $\lim_{\varepsilon\searrow 0} \varepsilon\mathbf{E}(|X|^{1-\varepsilon})$. *HINT: Split the integral at a well chosen point.*

2.3 Let X and Y be iid. standard Normal random variables. Determine the joint density of the pair $U = X$, $V = X/Y$. Then use it to show that X/Y has the standard Cauchy distribution.

2.4 We are given a biased coin that shows Heads with probability p and Tails with probability $1-p$. The coin initially shows Heads, and at time moments given by a Poisson(λ) process we flip the coin again and again. What is the probability that the coin shows Heads at time t?

2.5 $^{\bullet\bullet\bullet}$ (*With a good understanding of stuff, you can solve this problem without any computation.*) Let X_1, X_2, ... be iid. random variables each with density xe^{-x} for $x > 0$ and 0 otherwise. Let $S_0 = 0$ and $S_n := X_1 + \cdots + X_n$, and $N(t) := \max\{n : S_n < t\}$.

a) Determine the density of the random variable S_2.

b) Find the mass function of the random variable $N(t)$.

2.6 $^{\bullet\bullet\bullet\bullet}$ Are the following functions generating functions of non-negative integer probability distributions?

$$\text{(a)}\quad \exp\left(\frac{s-1}{\lambda}\right),\quad \lambda > 0; \qquad \text{(b)}\quad \frac{(s+1)^6}{64};$$

$$\text{(c)}\quad \frac{2}{2-s}; \qquad \text{(d)}\quad \frac{2}{1+s}.$$

2.7 $^{\bullet\bullet\bullet}$ Let X be a non-negative integer-valued random variable with generating function $P(s)$. Find the generating function for the distribution of $Y := X + 1$ and the one for $Z := 2X$.

2.8 $^{\bullet\bullet\bullet}$ Let X_1, X_2, ... be iid. Optimistic Geometric(p_1) random variables, and Z an independent Optimistic Geometric(p_2) random variable. Prove, using generating functions, that

$$\sum_{i=1}^{Z} X_i \sim \text{Geometric}(p_1 p_2).$$

Give a probabilistic interpretation of this fact.

2.9 The distribution of a random variable X is called *infinitely divisible*, if for every $n \ge 1$ integer there exist $Y_1^{(n)}, \dots, Y_n^{(n)}$ iid. random variables such that

$$\sum_{i=1}^{n} Y_i^{(n)} \sim X.$$

a) Is the Poisson distribution infinitely divisible?

b) Is the Binomial distribution infinitely divisible?

c) Prove that for every $0 < p < 1$ there exists a probability mass function $p_1, p_2, \dots$ and $\lambda > 0$ with which $\sum_{i=1}^{Z} X_i \sim$ Pessimistic Geometric(p), where $X_1, X_2, \dots$ are iid., for $k \ge 1$ $\mathbf{P}\{X_i = k\} = p_k$, and Z is an independent Poisson(λ) random variable.

d) Show, with the help of c), that the Geometric distribution is infinitely divisible.

2.10 a) •• (*Resnick.*) In a branching process

$$P(s) = as^2 + bs + c$$

where $a > 0$, $b > 0$, $c > 0$, $P(1) = 1$. Compute π. Give a condition for sure extinction.

b) •• Little Johnny is salesman in a store where servicing a customer takes exactly 1 minute. Under this time, with probability 0.6 two new customers arrive in the queue, with probability 0.2 one new customer arrives, with probability 0.2 no new customers appear. Little Johnny can have his coffee if the queue empties out. What is the probability that this ever happens after the first customer steps in the store? Explain.

2.11 In a branching process let $X = \sum_{n=0}^{\infty} Z_n$ be the number of individuals that ever existed. You can use (and we'll see on Thursday why this is true) that its generating function is

$$Q(s) = \left(\frac{s}{P(s)}\right)^{-1} \quad \leftarrow\text{function inverse} \quad .$$

a) Find $Q(s)$ when the distribution of the number of children of an individual is Bernoulli(p).

b) Find $Q(s)$ when the distribution of the number of children of an individual is Pessimistic Geometric(p).

c) Find $\mathbf{E}X$ in both cases.