

HOMEWORK SET 2  
*Convolutions, generating fcts.*  
Further Topics in Probability, 2<sup>nd</sup> teaching block, 2022  
School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 17<sup>th</sup> February. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

- 2.1 Prove that if  $\xi$  is a Cauchy random variable with density  $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ , then  $X := 1/\xi$ ,  $Y := 2\xi/(1-\xi^2)$ , and  $Z := (3\xi - \xi^3)/(1-3\xi^2)$  are also Cauchy distributed. *HINT: use the trigonometric identities: if  $\xi = \tan(\alpha)$ , then  $1/\xi = \tan(\frac{\pi}{2} - \alpha)$ ,  $2\xi/(1-\xi^2) = \tan(2\alpha)$ , and  $(3\xi - \xi^3)/(1-3\xi^2) = \tan(3\alpha)$ .*
- 2.2 Let  $X$  be a Cauchy random variable with density  $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ . We know that  $\mathbf{E}|X| = \infty$ , but  $\mathbf{E}(|X|^{1-\varepsilon}) < \infty$  for any  $\varepsilon > 0$ . Determine the limit  $\lim_{\varepsilon \searrow 0} \varepsilon \mathbf{E}(|X|^{1-\varepsilon})$ . *HINT: Split the integral at a well chosen point.*
- 2.3 Let  $X$  and  $Y$  be iid. standard Normal random variables. Determine the joint density of the pair  $U = X$ ,  $V = X/Y$ . Then use it to show that  $X/Y$  has the standard Cauchy distribution.
- 2.4 We are given a biased coin that shows Heads with probability  $p$  and Tails with probability  $1-p$ . The coin initially shows Heads, and at time moments given by a Poisson( $\lambda$ ) process we flip the coin again and again. What is the probability that the coin shows Heads at time  $t$ ?
- 2.5 ••• (With a good understanding of stuff, you can solve this problem without any computation.) Let  $X_1, X_2, \dots$  be iid. random variables each with density  $\frac{1}{2}x^2e^{-x}$  for  $x > 0$  and 0 otherwise. Let  $S_0 = 0$  and  $S_n := X_1 + \dots + X_n$ , and  $N(t) := \max\{n : S_n < t\}$ .
- a) Determine the density of the random variable  $S_2$ .
- b) Find the mass function of the random variable  $N(t)$ .
- 2.6 Show, using the convolution formula, that Gamma( $\alpha, \lambda$ ) and Gamma( $\beta, \lambda$ ) convolve to Gamma( $\alpha + \beta, \lambda$ ) for any positive reals  $\alpha, \beta, \lambda$ . Along the way there will be a tricky integral, which doesn't have to be explicitly calculated as long as you can scale out its dependence on the important variable. You can then use the fact that Gamma densities are normalized (thus proving a representation of Beta functions this way).
- 2.7 •• Find the density of the sum of an Exponential( $\lambda$ ) and an independent Exponential( $\mu$ ) random variable ( $\lambda \neq \mu$ ). What happens when  $\mu \rightarrow \lambda$ ?
- 2.8 ••• Are the following functions generating functions of non-negative integer probability distributions?
- (a)  $\exp\left(\frac{s-1}{\lambda}\right)$ ,  $\lambda > 0$ ; (b)  $\frac{(s+1)^5}{32}$ ;
- (c)  $\frac{2}{2-s}$ ; (d)  $\frac{2}{1+s}$ .
- 2.9 •• Let  $X$  be a non-negative integer-valued random variable with generating function  $P(s)$ . Find the generating function for the distribution of  $Y := X + 2$  and the one for  $Z := 3X$ .

2.10 ••• Let  $X_1, X_2, X_3, \dots$  be iid. random variables with common distribution function  $F$ . Let  $\nu \geq 1$  an independent, integer-valued variable with generating function  $P$ , and define  $Y := \max\{X_1, X_2, X_3, \dots, X_\nu\}$ . Show that  $Y$  has distribution function  $P(F(\cdot))$ .