Homework set 2

Probabilistic tools, convergence of random variables
(and a bit of martingales)
Martingale Theory with Applications, 1^{st} teaching block, 2023
School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 2^{nd} November. Please show your work leading to the result, not only the result. Each problem is worth the number of $^\bullet$'s you see right next to it. Make sure you find all 20 $^\bullet$'s!

2.1 $^{\bullet\bullet\bullet}$ We perform infinitely many independent experiments. The n^{th} one is successful with probability $n^{-\alpha}$ and fails with probability $1-n^{-\alpha}$, $0<\alpha$. Let $k\ge 1$. We are happy if we see k consecutive successes infinitely often. What is the probability of this?

2.2 (The longest run of heads, I.)
Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbf{P}\{X_k=1\}=p$, $\mathbf{P}\{X_k=0\}=q$, where $p+q=1$. Fix a parameter $\lambda>1$, and denote by $A_k^{(\lambda)}$ the following events for $k=0, 1, 2, \ldots$:

$$A_k^{(\lambda)} := \Big\{\exists r\in\big[\lfloor\lambda^k\rfloor\,,\,\lfloor\lambda^{k+1}\rfloor-k\big]\cap\mathbb{N}\,:\,X_r=X_{r+1}=\cdots=X_{r+k-1}=1\Big\}.$$

In plain words: $A_k^{(\lambda)}$ means that somewhere between $\lfloor\lambda^k\rfloor$ and $\lfloor\lambda^{k+1}\rfloor-1$ there is a sequence of k consecutive 1's. Prove that

a) If $\lambda<p^{-1}$, then a.s. only finitely many of the events $A_k^{(\lambda)}$ occur.
b) If $\lambda>p^{-1}$, then a.s. infinitely many of the events $A_k^{(\lambda)}$ occur.
c) What happens for $\lambda=p^{-1}$?

2.3 (The longest run of heads, II.)
Let

$$R_n := \sup\{k\ge 0\,:\,X_n=X_{n+1}=\cdots=X_{n+k-1}=1\}.$$

That is: R_n is the length of the run of consecutive 1's that starts at n. (If $X_n=0$, then set $R_n=0$.) Prove that

$$\mathbf{P}\left\{\limsup_{n\to\infty}\frac{R_n}{\log n}=|\log p|^{-1}\right\}=1.$$

HINT: For a fixed parameter $\alpha>0$, let

$$B_n^{(\alpha)} := \{R_n>\alpha\log n/|\log p|\}.$$

If $\alpha>1$, then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the $B_n^{(\alpha)}$'s occur a.s. If $\alpha\le 1$, then from the previous exercise it follows that a.s. infinitely many of the $B_n^{(\alpha)}$'s occur.

2.4 Let $X_1, X_2, \ldots$ be independent. Prove that $\sup_n X_n<\infty$ a.s. if and only if $\sum_{n=1}^\infty\mathbf{P}\{X_n>A\}<\infty$ for some positive finite A.

2.5 Prove that for any sequence $X_1, X_2, \ldots$ of random variables there exists a deterministic sequence $c_1, c_2, \ldots$ of real numbers for which $\frac{X_n}{c_n}\xrightarrow{\textbf{a.s.}}0$.

2.6 Let the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n + Y_n \xrightarrow{\mathbf{P}} X + Y$,

b) $X_n - Y_n \xrightarrow{\mathbf{P}} X - Y$.

2.7 Let the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n Y_n \xrightarrow{\mathbf{P}} XY$,

b) if $Y_n \neq 0$ and $Y \neq 0$ a.s., then $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$.

2.8 Formulate necessary and sufficient conditions for $\alpha_i < \beta_i$ such that independent (but not identically distributed) Uniform(α_i, β_i) variables X_i converge to 0

a) in distribution;

b) almost surely.

2.9 •••••• Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential(λ_i) variables X_i to converge to 0

a) in distribution;

b) almost surely.

2.10 Let $\xi_1, \xi_2, \ldots$ be iid. Poisson(1) random variables. (Recall their moment generating function: $\mathbb{E}(e^{t\xi_i}) = e^{e^t - 1}$.) Let $a, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = e^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > a,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow b > \frac{e^{ra} - 1}{r}.$$

2.11 ••••• Let $\xi_1, \xi_2, \ldots$ be iid. standard normal random variables. (Recall their moment generating function: $\mathbb{E}(e^{\lambda\xi_i}) = e^{\lambda^2/2}$.) Let $a, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = e^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > 0,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow r < \frac{2b}{a^2}.$$

2.12 Let S and T be stopping times w.r.t. the filtration $\mathcal{F}_n$. Which of these are stopping times? Explain.

$S \wedge T := \min(S, T)$, $\quad S \vee T := \max(S, T)$, $\quad T + S$, $\quad T - S$ (assume $T \geq S$ here.)

2.13 Let $X_1, X_2, \ldots$ be iid. Exponential(1) random variables, $S_n = X_1 + \cdots + X_n$, and $\{\mathcal{F}_n\}$ the natural filtration. Show that

$$\frac{n!}{(1+S_n)^{n+1}} \mathrm{e}^{S_n}$$

is a martingale w.r.t. $\{\mathcal{F}_n\}$.

2.14 An urn contains n white and n black balls. We draw them one by one without replacement. We receive £1 for any white ball, while nothing happens upon drawing a black one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{2X_i - i}{2n - i} \quad (1 \le i \le 2n-1), \qquad \text{and}$$
$$Z_i = \frac{2n-i}{2n-i-1} Y_i^2 - \frac{1}{2n-i-1} \quad (1 \le i \le 2n-2).$$

(a) Show that both Y_i and Z_i are martingales.

(b) Calculate the mean and variance of X_i.

2.15 •••••• An urn contains n white and n black balls. We draw them one by one without replacement. We pay £1 for any black ball drawn but receive £1 for any white one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{X_i}{2n-i} \quad (1 \le i \le 2n-1), \qquad \text{and} \qquad Z_i = \frac{X_i^2 - (2n-i)}{(2n-i)(2n-i-1)} \quad (1 \le i \le 2n-2).$$

(a) Show that both Y_i and Z_i are martingales.

(b) Calculate the variance of X_i.

2.16 Let X_j, $j \ge 1$, be absolutely integrable random variables, and $\mathcal{F}_n := \sigma(X_j,\ 1 \le j \le n)$, $n \ge 0$, their natural filtration. Define the new random variables

$$Z_0 := 0, \qquad Z_n := \sum_{j=0}^{n-1} \big(X_{j+1} - \mathbb{E}(X_{j+1} \,|\, \mathcal{F}_j)\big).$$

Prove that the process $n \mapsto Z_n$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

2.17 A biased coin shows HEAD with probability $\theta \in (0,\, 1)$, and TAIL with probability $1-\theta$. The value θ of the bias in *not known*. For $t \in [0,\, 1]$ and $n \in \mathbb{N}$ we define $p_{n,t} : \{0,\, 1\}^n \to [0,\, 1]$ by

$$p_{n,t}(x_1,\, x_2,\, \ldots,\, x_n) = t^{\sum_{j=1}^n x_j} \cdot (1-t)^{n - \sum_{j=1}^n x_j}.$$

We make two hypotheses about the possible value of θ: either $\theta = a$, or $\theta = b$, where $a,\, b \in [0,\, 1]$ and $a \ne b$. We toss the coin repeatedly and form the sequence of random variables

$$Z_n := \frac{p_{n,a}(\xi_1,\, \xi_2,\, \ldots,\, \xi_n)}{p_{n,b}(\xi_1,\, \xi_2,\, \ldots,\, \xi_n)},$$

where we write $\xi_j = 1$ if the j^{th} flip is HEAD and $\xi_j = 0$ if it is TAIL. Show that the process $n \mapsto Z_n$ is a martingale (w.r.t. the natural filtration generated by the coin tosses) if and only if the true bias of the coin is $\theta = b$.

2.18 Let η_n be a homogeneous Markov chain on the countable state space $S := \{0,\, 1,\, 2,\, \ldots\}$ and $\mathcal{F}_n := \sigma(\eta_j,\ 0 \le j \le n)$, $n \ge 0$ its natural filtration. For $i \in S$ denote by $Q(i)$ the probability that the Markov chain starting from site i *ever reaches* the point $0 \in S$:

$$Q(i) := \mathbb{P}\{\exists m < \infty \,:\, \eta_m = 0 \,|\, \eta_0 = i\}.$$

Prove that $Z_n := Q(\eta_n)$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.