Homework set 2

Conditional expectation

Martingale Theory with Applications, 1^{st} teaching block, 2025

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Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 16^{th} October. Please show your work leading to the result, not only the result. Each problem is worth the number of $^\bullet$'s you see right next to it. Make sure you find all 10 $^\bullet$'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

2.1 $^{\bullet\bullet\bullet}$ Consider the example from class: $\Omega = \{1, 2, \dots, 12\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}$ is uniform on Ω, and the random variables X and Y are defined by $X(\omega) = \lceil \frac{\omega}{2} \rceil$, $Y(\omega) = \lceil \frac{\omega}{4} \rceil$, $\mathcal{G} = \sigma(Y)$. Show by explicit calculations in this example that $\mathbb{E}(XY \mid \mathcal{G}) = Y\,\mathbb{E}(X \mid \mathcal{G})$. This is referred to as '*taking out what's known*' or '*given Y, Y is not random*'.

2.2 $^{\bullet\bullet\bullet\bullet}$ In the example of Problem 2.1, let $\mathcal{H} := \sigma(X)$. Calculate each of

- $\mathbb{E}\big(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}\big)$,
- $\mathbb{E}\big(\mathbb{E}(X \mid \mathcal{H}) \mid \mathcal{G}\big)$,
- $\mathbb{E}\big(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}\big)$,
- $\mathbb{E}\big(\mathbb{E}(Y \mid \mathcal{H}) \mid \mathcal{G}\big)$.

Compare with $\mathbb{E}(X \mid \mathcal{G})$, $\mathbb{E}(X \mid \mathcal{H})$, $\mathbb{E}(Y \mid \mathcal{G})$, $\mathbb{E}(Y \mid \mathcal{H})$. *It is important here that one of the two σ-algebras contains the other!*

2.3 However, give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, sub-$\sigma$ algebras $\mathcal{F}_1 \subset \mathcal{F}, \mathcal{F}_2 \subset \mathcal{F}$, and a random variable X such that

$$\mathbb{E}\big(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2\big) \neq \mathbb{E}\big(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1\big).$$

Why is it not a contradiction with the previous problem?

2.4 (*Monty Hall problem with σ-algebras.*) The famous Monty Hall problem goes like this:

We have three doors. Behind one of them is a car, behind the others, goats.

1. You pick a door, let us assume it's door number 1.
2. Monty opens *another door with a goat behind it.*
3. Now you pick one of the two closed doors (repeat your choice, or switch to the other one).
4. Whatever is behind this door is yours.

Make the natural assumptions about the probabilities of the location of the car and the choice of door Monty opens (if he *has* a choice). Would you repeat your choice or switch?

(a) Write the full probability space of the experiment that involves the first two steps above.

(b) In this sample space, write the event $A = \{\text{door 3 has a goat}\}$, and its generated σ-algebra $\mathcal{F} = \sigma(A)$.

(c) Let $X = 1, 2, 3$ be the location of the car. Calculate $\mathbb{E}(X \mid \mathcal{F})$.

(d) Now write the event $B = \{\text{Monty opens door 3}\}$, and its generated σ-algebra $\mathcal{G} = \sigma(B)$.

(e) Calculate $\mathbb{E}(X \mid \mathcal{G})$.

(f) Conclude the optimal strategy for the player in this problem.

2.5 Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} = \sigma(Y)$. Show that X is independent of $\mathcal{G}$ if and only if for any bounded and measurable functions f and g, we have $\mathbb{E}\big(f(X) \cdot g(Y)\big) = \mathbb{E} f(X) \cdot \mathbb{E} g(Y)$ (the *Probability 1* definition of independence).

2.6 ••• Let A and B be two events in a probability space, B of positive probability. Derive the *Probability 1* definition of the conditional probability $\mathbb{P}\{A \mid B\}$ from our definition of conditional expectations.

2.7 Based on your definition above, show that for any fixed event B of positive probability in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set function $\mathbb{Q}(\cdot) := \mathbb{P}\{\cdot \mid B\}$ is a probability measure.

2.8 Continuing the previous problem, show that for any events B and C with a positive probability intersection,

$$\mathbb{Q}(\cdot \mid C) = \mathbb{P}(\cdot \mid B \cap C).$$