Homework set 3

Martingales, stopping times, convergence

Martingale Theory with Applications, 1^{st} teaching block, 2018

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Problems with $^\bullet$'s are to be handed in. These are due in the blue locker with "Martingale theory" on it on the ground floor of the Main Maths Building before 12:00pm on Monday, 12^{th} November. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Hence, for example, Problem 3.4 worth four marks.

3.1 *Bellman's Optimality Principle.* We model a sequence of gamblings as follows. Let $\xi_1, \xi_2, \dots$ be iid. random variables with $\mathbb{P}\{\xi_n = +1\} = p$, $\mathbb{P}\{\xi_n = -1\} = q$, where $p = 1 - q > 1/2$. Define the *entropy* of this distribution by

$$\alpha = p \ln\Big(\frac{p}{1/2}\Big) + q \ln\Big(\frac{q}{1/2}\Big) = p \ln p + q \ln q + \ln 2.$$

A gambler starts playing with initial fortune $Y_0 > 0$. Her return at time n on a *unit bet* is the random variable ξ_n, and she plays C_n in round n. In other words, with probability p she doubles her bet and with probability q she looses it. Therefore her fortune after round n is

$$Y_n = Y_{n-1} + C_n \xi_n.$$

The bet C_n may depend on the values $\xi_1, \xi_2, \dots, \xi_{n-1}$, and has bounds $0 \le C_n \le Y_{n-1}$. The expected rate of winnings up to time n is

$$r_n := \mathbb{E} \ln\Big(\frac{Y_n}{Y_0}\Big),$$

which the gambler wishes to maximise.

(a) Prove that no matter what strategy C the gambler chooses,

$$X_n := \ln Y_n - n\alpha$$

is a supermartingale, hence her expected average winning rate, $\frac{r_n}{n} \le \alpha$.

(b) However, there exists a gambling strategy that makes the above X a martingale, hence realises the average expected winning rate α. Find this strategy.

3.2 *Galton-Watson Branching Process.* Let $\xi_{n,k}$, $n = 1, 2, \dots$, $k = 1, 2, \dots$ be iid. non-negative integer random variables with finite mean μ and variance σ^2. Define the Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k},$$

and let $\mathcal{F}_n := \sigma(Z_j : 0 \le j \le n)$, $n \ge 0$ be the natural filtration.

(a) Prove that $M_n := Z_n / \mu^n$, $n \ge 0$ is an $(\mathcal{F}_n)_{n\ge0}$-martingale.

(b) Prove that $\mathbb{E}(Z_{n+1}^2 \,|\, \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$.

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}} \dfrac{\mu^n - 1}{\mu - 1} M_n, & \text{if } \mu \neq 1, \\ M_n^2 - n\sigma^2 M_n, & \text{if } \mu = 1 \end{cases}$$

is also an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then M_n is bounded in $\mathcal{L}^2$, while if $\mu \leq 1$ then $\lim_{n\to\infty} \mathbb{E}M_n^2 = \infty$.

3.3 *Gambler's ruin.* Let X_1, X_2, ... be iid. random variables with $\mathbb{P}\{X_i = 1\} = p = 1 - q = 1 - \mathbb{P}\{X_i = -1\}$. Fix also $0 < a < b$ integers, and

$$S_n := a + \sum_{k=1}^n X_k, \qquad T := \inf\{n : S_n = 0 \text{ or } S_n = b\}.$$

(We think about S_n as a gambler's money at time n; the gambler starts at a, and is either ruined ($S_T = 0$) or wins it all ($S_T = b$).)

(a) Show that $\mathbb{E}T < \infty$. *Hint: we had a lemma for this...*

(b) Show that both

$$M_n := S_n - n(p-q) \qquad \text{and} \qquad N_n := \begin{cases} S_n^2 - n, & \text{if } p = q = \dfrac{1}{2}, \\ \left(\dfrac{q}{p}\right)^{S_n}, & \text{if } p \neq q \end{cases}$$

are martingales w.r.t. the natural filtration.

(c) Calculate the ruin probability $\mathbb{P}\{S_T = 0\}$ and the expected duration $\mathbb{E}T$ of the game.

3.4 •••• *Extending Doob's Optional Stopping Theorem.* Let $\tau \geq 0$ be a stopping time, $\mathbb{E}\tau < \infty$.

(a) Show $\{\tau \geq k\} \in \mathcal{F}_{k-1}$.

(b) Based on the identity

$$|X_{\tau\wedge n} - X_0| = \Big|\sum_{k=1}^n (X_k - X_{k-1}) \cdot \mathbf{1}\{\tau \geq k\}\Big| \leq \sum_{k=1}^\infty |X_k - X_{k-1}| \cdot \mathbf{1}\{\tau \geq k\}$$

and the proof of Doob's Optional Stopping Theorem, part (iii), show the following: If X a supermartingale for which there exists a $C \in \mathbb{R}$ with

$$\mathbb{E}(|X_k - X_{k-1}| \,\big|\, \mathcal{F}_{k-1}) \leq C \qquad \forall k > 0, \text{ a.s.},$$

then $\mathbb{E}X_\tau \leq \mathbb{E}X_0$. Of course we have equality in case X is a martingale.

(c) Prove that for any process $(M_n)_{n=0}^\infty$ with $M_0 = 0$,

(1)

$$M_{\tau\wedge n}^2 = \sum_{k=1}^n \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2 \sum_{1\leq i<j\leq n} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \geq j\},$$

(2)

$$M_\tau^2 = \sum_{k=1}^\infty \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2 \sum_{1\leq i<j<\infty} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \geq j\},$$

holds. (Notice that a.s. finitely many terms are non-zero in each of these sums.).

(d) Show that the first sum on the right hand-side of (1) converges monotonically to that on the right hand-side of (2).

(e) Let M be a martingale with $M_0 = 0$ and a $C \in \mathbb{R}$ such that

$$\left|M_k - M_{k-1}\right| \le C \qquad \forall k > 0.$$

From now on, let us assume that the stopping time τ is in $\mathcal{L}^2$. Show that the second sums on the right hand-sides of both (1) and (2) are mean zero. *Hint: Fubini's Theorem.*

(f) With the condition as in the previous part, conclude $\lim\limits_{n\to\infty} \mathbb{E}M^2_{\tau\wedge n} = \mathbb{E}M^2_\tau$.

3.5 •••• *Wald's identities. (Notice: In contrary to this problem, Ábrahám Wald invented these without the use of martingales.)* Let $Y_1, Y_2, \ldots$ be iid. random variables in $\mathcal{L}^1$, $\mu := \mathbb{E}Y_i$, and $\tau \ge 1$ a stopping time (w.r.t. the natural filtration), $\mathbb{E}\tau < \infty$. Let $S_n = \sum\limits_{i=1}^{n} Y_i$. Show that

(a) $\mathbb{E}S_\tau = \mu \cdot \mathbb{E}\tau$. *Hint: use a martingale and the previous problem.*

(b) If Y_i is bounded and, additionally, $\mathbb{E}\tau^2 < \infty$ also holds, then with $\sigma^2 := \mathbf{Var}Y_i$ we have $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \cdot \mathbb{E}\tau$. (*This one is often used in case $\mu = 0$.*) *Hint: find a martingale for S^2_τ, and use the previous problem on your martingale from part (a). Do it for $\mu = 0$ first.*

3.6 ••• *First mark after 1 in a Uniform renewal process.* Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables, let $S_n := \sum\limits_{i=1}^{n} Y_i$, and $\tau := \min\{n : S_n > 1\}$.

(a) Show that for any fixed $0 \le z \le 1$, $\mathbb{P}\{S_n \le z\} = z^n/n!$ holds. *Be careful, this is not true for $z > 1$!*

(b) Find $\mathbb{E}\tau$. *Hint: τ is non-negative, therefore we can sum tail probabilities. These latter are in close connection with part (a).*

(c) Since τ is a stopping time, use Wald's identity to calculate $\mathbb{E}(S_\tau - 1)$, the residual time at 1 until the next mark in a Uniform renewal process.

3.7 *Pólya urn.* At time $n = 0$, an urn contains $B_0 = 1$ blue and $R_0 = 1$ red balls. At each time $n > 0$ a ball is chosen uniformly at random from the urn and returned to the urn, together with a new ball of the same colour. We denote by B_n and R_n the number of blue, respectively, red balls in the urn after the n^{th} turn of this procedure. Notice that $B_n + R_n = n + 2$. Let

$$M_n := \frac{B_n}{B_n + R_n}$$

be the proportion of blue balls in the urn just after turn n.

(a) Show that M_n is a martingale w.r.t. the natural filtration of the process.

(b) Show that B_n is discrete uniform: $\mathbb{P}\{B_n = k\} = \frac{1}{n+1}$ for $1 \le k \le n+1$.

(c) Show that $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. What is its distribution?

(d) Let T be the time the first blue ball is drawn. Show that $T < \infty$ a.s. *Hint: show that the events $\{T > n\}$ are decreasing and find the limit of their probabilities.*

(e) Show that $\mathbb{E}\frac{1}{T+2} = \frac{1}{4}$.

3.8 *Beta function.* Prove that for any $a, b \geq 0$ integers,

$$I_{a,b} := \int_0^1 \theta^a (1-\theta)^b \,\mathrm{d}\theta = \frac{a! \cdot b!}{(a+b+1)!}.$$

Hint: show via integration by parts that for $b \geq 1$, $I_{a,b} = \frac{b}{a+1} \cdot I_{a+1,b-1}$, while the case $b = 0$ is easy. From here, a recursive argument does the trick.

3.9 *Bayes urn.* Assume we have a *randomly biased* coin that shows HEAD with probability θ and TAIL with probability $1-\theta$. This parameter θ is random and has the Uniform(0, 1) distribution. We flip this coin repeatedly and record

$$B_0 := 1, \qquad B_n := 1 + \text{no. of HEADs in the first } n \text{ trials},$$
$$R_0 := 1, \qquad R_n := 1 + \text{no. of TAILs in the first } n \text{ trials}.$$

Notice that $B_n + R_n = n + 2$. Define the filtration generated by the first n flips, $\mathcal{F}_n = \sigma(B_1, B_2, \ldots, B_n)$, and mind that θ is *not* included in here.

(a) Determine the probability of a given sequence of flips,

$$\mathbb{P}\{B_1 = b_1, B_2 = b_2, \ldots, B_n = b_n\}.$$

Hint: Condition on θ and use Problem 3.8.

(b) Based on the previous part, find the distribution of B_{n+1}, given $\mathcal{F}_n$. Compare with the Pólya urn. Remember: θ is not included in $\mathcal{F}_n$.

(c) Show that, modulo zero measure sets, θ is $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^\infty \mathcal{F}_n\right)$-measurable.

(d) What is the conditional expectation of θ, given the first n flips? Explain. *Hint: the Pólya urn, and our theorem on uniformly integrable martingales...*

(e) Use the Bayes urn to find the conditional density of M_∞, given $\mathcal{F}_n$ in the Pólya urn.

3.10 Let $X_1, X_2, \ldots$ be strictly positive iid. random variables such that $\mathbb{E}X_1 = 1$ and $\mathbb{P}\{X_1 = 1\} < 1$.

(a) Show that $M_n = \prod_{i=1}^n X_i$ is a martingale w.r.t. the natural filtration.

(b) Deduce that there exists a real valued random variable L such that $M_n \to L$ a.s. as $n \to \infty$.

(c) Show that $\mathbb{P}\{L = 0\} = 1$. *Hint: argue by contradiction and note that if $M_n, M_{n+1} \in (a-\varepsilon, a+\varepsilon)$ then $X_{n+1} \in \left(\frac{a-\varepsilon}{a+\varepsilon}, \frac{a+\varepsilon}{a-\varepsilon}\right)$.*

(d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n}\ln M_n \to c$ a.s. as $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

3.11 •• Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra. Show that given $\mathcal{G}$, $\mathbb{E}(X \mid \mathcal{G})$ is the best predictor of X in the following sense: the minimum mean square error $\mathbb{E}(V - X)^2$ among $\mathcal{G}$-measurable random variables V is achieved for $V = \mathbb{E}(X \mid \mathcal{G})$. What is this minimal mean square error? *Hint: use a tower rule first, then minimise pointwise among $\mathcal{G}$-measurable functions.*

3.12 •••• Given are N balls and K, initially empty, urns. We place the balls, one by one, into the urns without removing them. Each ball independently goes to a uniformly chosen urn from 1 to K. These choices are denoted by $X_1, X_2, \ldots, X_N$, which are therefore iid. discrete uniform on the set $\{1, 2, \ldots, K\}$. The generated filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for $n = 0, 1, \ldots, N$. Denote by Z the number of empty urns when all N balls have been placed, and Z_n the number of empty urns after the n^{th} step.

(a) Calculate the best prediction martingale at time n (see the previous problem) $M_n = \mathbb{E}(Z \,|\, \mathcal{F}_n)$, $(n = 0,\, 1,\, \ldots,\, N)$ explicitly, and show its martingale property via direct computation. *Hint: use indicators for urns to stay empty.*

(b) What is M_0 and what is M_N?

(c) Find $\mathbb{E}Z_n$ $(0 \le n \le N)$ and $\mathbb{E}Z$.

3.13 Let M be a uniformly integrable martingale in the filtration $(\mathcal{F}_n)_{n\ge 0}$ in the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$. Let $S \le T$ a.s. be finite stopping times. We denote by $\mathcal{F}_T$ the collection of all events $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all n, which can be thought of as the set of events whose occurrence or non-occurrence is known by time T.

(a) Prove that $\mathcal{F}_T$ is a σ-algebra.

(b) Prove that $M_T = \mathbb{E}(M_\infty \,|\, \mathcal{F}_T)$ and that $M_S = \mathbb{E}(M_T \,|\, \mathcal{F}_S)$. *Hint: observe that $\mathcal{F}_T$ is generated by sets $A \cap \{T = n\}$ where $A \in \mathcal{F}$ and $n \in \mathbb{Z}^+$.*

3.14 Let $X_n \in [0,\, 1]$ be adapted to $\mathcal{F}_n$. Let $\alpha,\, \beta > 0$ such that $\alpha + \beta = 1$ and suppose

$$\mathbb{P}\{X_{n+1} = \alpha + \beta X_n \,|\, \mathcal{F}_n\} = X_n, \quad \mathbb{P}\{X_{n+1} = \beta X_n \,|\, \mathcal{F}_n\} = 1 - X_n.$$

Show:

(a) $\mathbb{P}\{\lim_n X_n = 0 \text{ or } 1\} = 1$. *Hint: Use martingale convergence, and try to find an independent sequence U_n that generates X_n.*

(b) If $X_0 = \theta$ then $\mathbb{P}\{\lim_n X_n = 1\} = \theta$.