

HOMEWORK SET 3

*Convergence of random variables
(and a bit of martingales)*

Martingale Theory with Applications, 1st teaching block, 2022
School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 3rd November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it.

3.1 Let X_1, X_2, \dots be independent. Prove that $\sup_n X_n < \infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$ for some positive finite A .

3.2 Prove that for any sequence X_1, X_2, \dots of random variables there exists a deterministic sequence c_1, c_2, \dots of real numbers for which $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$.

3.3 Let the random variables $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, \dots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n + Y_n \xrightarrow{\mathbf{P}} X + Y$,

b) $X_n - Y_n \xrightarrow{\mathbf{P}} X - Y$.

3.4 Let the random variables $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, \dots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_n Y_n \xrightarrow{\mathbf{P}} XY$,

b) if $Y_n \neq 0$ and $Y \neq 0$ a.s., then $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$.

3.5 ••••• Formulate necessary and sufficient conditions for $\alpha_i < \beta_i$ such that independent (but not identically distributed) Uniform(α_i, β_i) variables X_i converge to 0

a) in distribution;

b) almost surely.

3.6 Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential(λ_i) variables X_i to converge to 0

a) in distribution;

b) almost surely.

3.7 •••• Let ξ_1, ξ_2, \dots be iid. Poisson(1) random variables. (Recall their moment generating function: $\mathbb{E}(e^{t\xi_i}) = e^{e^t-1}$.) Let $a, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^n \xi_k, \quad \text{and} \quad X_n = e^{aS_n - bn}.$$

Show that

$$X_n \rightarrow 0 \text{ a.s.} \Leftrightarrow b > a,$$

but for any $r \geq 1$

$$X_n \rightarrow 0 \text{ in } \mathcal{L}^r \Leftrightarrow b > \frac{e^{ra} - 1}{r}.$$

3.8 Let ξ_1, ξ_2, \dots be iid. standard normal random variables. (Recall their moment generating function: $\mathbb{E}(e^{\lambda\xi_i}) = e^{\lambda^2/2}$.) Let $a, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^n \xi_k, \quad \text{and} \quad X_n = e^{aS_n - bn}.$$

Show that

$$X_n \rightarrow 0 \text{ a.s.} \Leftrightarrow b > 0,$$

but for any $r \geq 1$

$$X_n \rightarrow 0 \text{ in } \mathcal{L}^r \Leftrightarrow r < \frac{2b}{a^2}.$$

3.9 Let S and T be stopping times w.r.t. the filtration \mathcal{F}_n . Which of these are stopping times? Explain.

$$S \wedge T := \min(S, T), \quad S \vee T := \max(S, T), \quad T + S, \quad T - S \text{ (assume } T \geq S \text{ here.)}$$

3.10 Let X_1, X_2, \dots be iid. Exponential(1) random variables, $S_n = X_1 + \dots + X_n$, and $\{\mathcal{F}_n\}$ the natural filtration. Show that

$$\frac{n!}{(1 + S_n)^{n+1}} e^{S_n}$$

is a martingale w.r.t. $\{\mathcal{F}_n\}$.

3.11 ••••• An urn contains n white and n black balls. We draw them one by one without replacement. We receive £1 for any white ball, while nothing happens upon drawing a black one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{2X_i - i}{2n - i} \quad (1 \leq i \leq 2n - 1), \quad \text{and}$$

$$Z_i = \frac{2n - i}{2n - i - 1} Y_i^2 - \frac{1}{2n - i - 1} \quad (1 \leq i \leq 2n - 2).$$

- (a) Show that both Y_i and Z_i are martingales.
- (b) Calculate the mean and variance of X_i .

3.12 An urn contains n white and n black balls. We draw them one by one without replacement. We pay £1 for any black ball drawn but receive £1 for any white one. Denote by X_i our money after the i^{th} draw ($X_0 = 0$). Let

$$Y_i = \frac{X_i}{2n - i} \quad (1 \leq i \leq 2n - 1), \quad \text{and} \quad Z_i = \frac{X_i^2 - (2n - i)}{(2n - i)(2n - i - 1)} \quad (1 \leq i \leq 2n - 2).$$

- (a) Show that both Y_i and Z_i are martingales.
- (b) Calculate the variance of X_i .

3.13 Let $X_j, j \geq 1$, be absolutely integrable random variables, and $\mathcal{F}_n := \sigma(X_j, 1 \leq j \leq n)$, $n \geq 0$, their natural filtration. Define the new random variables

$$Z_0 := 0, \quad Z_n := \sum_{j=0}^{n-1} (X_{j+1} - \mathbb{E}(X_{j+1} | \mathcal{F}_j)).$$

Prove that the process $n \mapsto Z_n$ is an $(\mathcal{F}_n)_{n \geq 0}$ -martingale.

- 3.14 A biased coin shows HEAD with probability $\theta \in (0, 1)$, and TAIL with probability $1-\theta$. The value θ of the bias is *not known*. For $t \in [0, 1]$ and $n \in \mathbb{N}$ we define $p_{n,t} : \{0, 1\}^n \rightarrow [0, 1]$ by

$$p_{n,t}(x_1, x_2, \dots, x_n) = t^{\sum_{j=1}^n x_j} \cdot (1-t)^{n-\sum_{j=1}^n x_j}.$$

We make two hypotheses about the possible value of θ : either $\theta = a$, or $\theta = b$, where $a, b \in [0, 1]$ and $a \neq b$. We toss the coin repeatedly and form the sequence of random variables

$$Z_n := \frac{p_{n,a}(\xi_1, \xi_2, \dots, \xi_n)}{p_{n,b}(\xi_1, \xi_2, \dots, \xi_n)},$$

where we write $\xi_j = 1$ if the j^{th} flip is HEAD and $\xi_j = 0$ if it is TAIL. Show that the process $n \mapsto Z_n$ is a martingale (w.r.t. the natural filtration generated by the coin tosses) if and only if the true bias of the coin is $\theta = b$.

- 3.15 Let η_n be a homogeneous Markov chain on the countable state space $S := \{0, 1, 2, \dots\}$ and $\mathcal{F}_n := \sigma(\eta_j, 0 \leq j \leq n)$, $n \geq 0$ its natural filtration. For $i \in S$ denote by $Q(i)$ the probability that the Markov chain starting from site i ever reaches the point $0 \in S$:

$$Q(i) := \mathbb{P}\{\exists m < \infty : \eta_m = 0 \mid \eta_0 = i\}.$$

Prove that $Z_n := Q(\eta_n)$ is an $(\mathcal{F}_n)_{n \geq 0}$ -martingale.