HOMEWORK SET 3

Optional stopping, martingale convergence

Martingale Theory with Applications, 1^{st} teaching block, 2024

School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 14^{th} November. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Make sure you find all 20 $^\bullet$'s!

3.1 *Bellman's Optimality Principle.* We model a sequence of gamblings as follows. Let $\xi_1, \xi_2, \dots$ be iid. random variables with $\mathbb{P}\{\xi_n = +1\} = p$, $\mathbb{P}\{\xi_n = -1\} = q$, where $p = 1 - q > 1/2$. Define the *entropy* of this distribution by

$$\alpha = p \ln\Big(\frac{p}{1/2}\Big) + q \ln\Big(\frac{q}{1/2}\Big) = p \ln p + q \ln q + \ln 2.$$

A gambler starts playing with initial fortune $Y_0 > 0$. Her return at time n on a *unit bet* is the random variable ξ_n, and she plays C_n in round n. In other words, with probability p she doubles her bet and with probability q she looses it. Therefore her fortune after round n is

$$Y_n = Y_{n-1} + C_n \xi_n.$$

The bet C_n may depend on the values $\xi_1, \xi_2, \dots, \xi_{n-1}$, and has bounds $0 \le C_n < Y_{n-1}$. The expected rate of winnings up to time n is

$$r_n := \mathbb{E} \ln\Big(\frac{Y_n}{Y_0}\Big),$$

which the gambler wishes to maximise.

(a) Prove that no matter what strategy C the gambler chooses,

$$X_n := \ln Y_n - n\alpha$$

is a supermartingale, hence her expected average winning rate, $\frac{r_n}{n} \le \alpha$.

(b) However, there exists a gambling strategy that makes the above X a martingale, hence realises the average expected winning rate α. Find this strategy.

3.2 Let S_n be a simple symmetric random walk on the square lattice $\mathbb{Z}^2$ with $S_0 = (0, 0)$. That is, the walker starts from the origin and at each step independently, she steps one unit to East, North, West or South with equal chance. Denote by D_n the walker's Euclidean distance from the origin of $\mathbb{Z}^2$ at time n, and let $\nu_r = \inf\{n : D_n > r\}$.

(a) Show that $D_n^2 - n$ is a martingale.

(b) Show that $r^{-2}\,\mathbb{E}\,\nu_r \to 1$ as $r \to \infty$.

3.3 The problem is the same as the previous one, except that the walk is on $\mathbb{R}^2$ and steps are of length one in iid. Uniform$(0, 2\pi)$ directions.

3.4 Let S_n be a simple symmetric random walk on the cubic lattice $\mathbb{Z}^3$ with $S_0 = (0, 0, 0)$. That is, the walker starts from the origin and at each step independently, she steps one unit to up, down, left, right, forward or backward with equal chance. Denote by D_n the walker's Euclidean distance from the origin of $\mathbb{Z}^3$ at time n, and let $\nu_r = \inf\{n : D_n > r\}$.

(a) Show that $D_n^2 - n$ is a martingale.

(b) Show that $r^{-2}\,\mathbb{E}\,\nu_r \to 1$ as $r \to \infty$.

3.5 •••• We repeatedly toss a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HHHHHH for the first time?

(b) We stop when six consecutive tosses result in the same outcome, in other words when either the pattern HHHHHH or TTTTTT first appears. What is the expected number of tosses until this moment?

3.6 We repeatedly toss a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HTHT for the first time?

(b) What is the expected number of tosses until we have seen the pattern THTH for the first time?

(c) What is the expected number of tosses until we have seen the pattern HTTH for the first time?

(d) What is the expected number of tosses until we have seen the pattern THHT for the first time?

(e) Give an example of a four letter pattern of H-s and T-s that has the maximal expected number of tosses, of any four letter patterns, until it is seen.

3.7 The previous question with a biased coin. Explain your answer.

3.8 Let $m \geq 2$ be an integer. At time $n = 0$, an urn contains $2m$ balls of which m are red and m are blue. At each time $n = 1, 2, \ldots, 2m$ we draw a randomly chosen ball without replacement from the urn and record its colour. For $n = 0, 1, \ldots, 2m-1$ let N_n denote the number of red balls left in the urn after time n, and

$$P_n := \frac{N_n}{2m - n}$$

denote the fraction of them. Let $(\mathcal{F}_n)_{0\leq n\leq 2m}$ be the natural filtration generated by the process $(N_n)_{0\leq n\leq 2m}$.

(a) Show that $n \mapsto P_n$ is an $\mathcal{F}_n$-martingale.

(b) Let T be the first time at which the ball drawn is red. Show that the $(T+1)^{\text{st}}$ draw is equally likely to be red or blue.

3.9 *Galton-Watson Branching Process.* Let $\xi_{n,k}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$ be iid. non-negative integer random variables with finite mean μ and variance σ^2. Define the Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k},$$

and let $\mathcal{F}_n := \sigma(Z_j : 0 \leq j \leq n)$, $n \geq 0$ be the natural filtration.

(a) Prove that $M_n := Z_n/\mu^n$, $n \geq 0$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale.

(b) Prove that $\mathbb{E}(Z_{n+1}^2 \,|\, \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$.

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}} \dfrac{\mu^n - 1}{\mu - 1} M_n, & \text{if } \mu \neq 1, \\ M_n^2 - n\sigma^2 M_n, & \text{if } \mu = 1 \end{cases}$$

is also an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then M_n is bounded in $\mathcal{L}^2$, while if $\mu \le 1$ then $\lim_{n\to\infty} \mathbb{E}\, M_n^2 = \infty$.

3.10 *Gambler's ruin.* Let X_1, X_2, ... be iid. random variables with $\mathbb{P}\{X_i = 1\} = p = 1 - q = 1 - \mathbb{P}\{X_i = -1\}$. Fix also $0 < a < b$ integers, and

$$S_n := a + \sum_{k=1}^n X_k, \qquad T := \inf\{n \,:\, S_n = 0 \text{ or } S_n = b\}.$$

(We think about S_n as a gambler's money at time n; the gambler starts at a, and is either ruined ($S_T = 0$) or wins it all ($S_T = b$).)

(a) Show that $\mathbb{E}\, T < \infty$. *Hint: we had a lemma for this...*

(b) Show that both

$$M_n := S_n - n(p-q) \qquad \text{and} \qquad N_n := \begin{cases} S_n^2 - n, & \text{if } p = q = \dfrac{1}{2}, \\ \left(\dfrac{q}{p}\right)^{S_n}, & \text{if } p \neq q \end{cases}$$

are martingales w.r.t. the natural filtration.

(c) Calculate the ruin probability $\mathbb{P}\{S_T = 0\}$ and the expected duration $\mathbb{E}\, T$ of the game.

3.11 •••• *Extending Doob's Optional Stopping Theorem, 1.* Let $\tau \ge 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Show $\{\tau \ge k\} \in \mathcal{F}_{k-1}$.

(b) Based on the identity

$$|X_{\tau\wedge n} - X_0| = \left|\sum_{k=1}^n (X_k - X_{k-1}) \cdot \mathbf{1}\{\tau \ge k\}\right| \le \sum_{k=1}^\infty |X_k - X_{k-1}| \cdot \mathbf{1}\{\tau \ge k\}$$

and the proof of Doob's Optional Stopping Theorem, part (iii), show the following: If X is a supermartingale for which there exists a $C \in \mathbb{R}$ with

$$\mathbb{E}(|X_k - X_{k-1}| \,\big|\, \mathcal{F}_{k-1}) \le C \qquad \forall k > 0, \text{ a.s.},$$

then $\mathbb{E}\, X_\tau \le \mathbb{E}\, X_0$. Of course we have equality in case X is a martingale.

3.12 ••• *Wald's identities, 1. (Notice: In contrary to this problem, Ábrahám Wald invented these without the use of martingales.)* Let Y_1, Y_2, ... be iid. random variables in $\mathcal{L}^1$, $\mu := \mathbb{E}\, Y_i$, and $\tau \ge 1$ a stopping time (w.r.t. the natural filtration), $\mathbb{E}\,\tau < \infty$. Let $S_n = \sum_{i=1}^n Y_i$. Show that $\mathbb{E}\, S_\tau = \mu \cdot \mathbb{E}\,\tau$. *Hint: use a martingale and the previous problem.*

3.13 •••• *First mark after 1 in a Uniform renewal process.* Let Y_1, Y_2, ... be iid. Uniform(0, 1) random variables, let $S_n := \sum_{i=1}^{n} Y_i$, and $\tau := \min\{n : S_n > 1\}$.

(a) Show that for any fixed $0 \le z \le 1$, $\mathbb{P}\{S_n \le z\} = z^n/n!$ holds. *Be careful, this is not true for $z > 1$!*

(b) Find $\mathbb{E}\,\tau$. *Hint: τ is non-negative, therefore we can sum tail probabilities. These latter are in close connection with part (a).*

(c) Since τ is a stopping time, use Wald's identity to calculate $\mathbb{E}(S_\tau - 1)$, the residual time at 1 until the next mark in a Uniform renewal process.

3.14 *Extending Doob's Optional Stopping Theorem, 2.* Let $\tau \ge 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Prove that for any process $(M_n)_{n=0}^{\infty}$ with $M_0 = 0$,

(1)

$$M_{\tau\wedge n}^2 = \sum_{k=1}^{n} \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \ge k\} + 2 \sum_{1\le i<j\le n} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \ge j\},$$

(2)

$$M_{\tau}^2 = \sum_{k=1}^{\infty} \big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \ge k\} + 2 \sum_{1\le i<j<\infty} (M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \ge j\},$$

holds. (Notice that a.s. finitely many terms are non-zero in each of these sums.).

(b) Show that the first sum on the right hand-side of (1) converges monotonically to that on the right hand-side of (2).

(c) Let M be a martingale with $M_0 = 0$ and a $C \in \mathbb{R}$ such that

$$\big|M_k - M_{k-1}\big| \le C \qquad \forall k > 0.$$

From now on, let us assume that the stopping time τ is in $\mathcal{L}^2$. Show that the second sums on the right hand-sides of both (1) and (2) are mean zero. *Hint: Fubini's Theorem.*

(d) With the condition as in the previous part, conclude $\lim_{n\to\infty} \mathbb{E}\, M_{\tau\wedge n}^2 = \mathbb{E}\, M_\tau^2$.

3.15 *Wald's identities, 2.* If Y_i is bounded and, additionally, $\mathbb{E}\,\tau^2 < \infty$ also holds, then with $\sigma^2 := \mathbb{V}\mathrm{ar}\, Y_i$ show $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \cdot \mathbb{E}\,\tau$. (*This one is often used in case $\mu = 0$.*) *Hint: find a martingale for S_n^2, and use the previous problem on your martingale from 3.12. Do it for $\mu = 0$ first.*

3.16 ••••• *Pólya urn.* At time $n = 0$, an urn contains $B_0 = 1$ blue and $R_0 = 1$ red balls. At each time $n > 0$ a ball is chosen uniformly at random from the urn and returned to the urn, together with a new ball of the same colour. We denote by B_n and R_n the number of blue, respectively, red balls in the urn after the n^{th} turn of this procedure. Notice that $B_n + R_n = n + 2$. Let

$$M_n := \frac{B_n}{B_n + R_n}$$

be the proportion of blue balls in the urn just after turn n.

(a) Show that M_n is a martingale w.r.t. the natural filtration of the process.

(b) Show that B_n is discrete uniform: $\mathbb{P}\{B_n = k\} = \frac{1}{n+1}$ for $1 \le k \le n+1$.

(c) Show that $M_\infty := \lim_{n\to\infty} M_n$ exists a.s. What is its distribution?

(d) Let T be the time the first blue ball is drawn. Show that $T < \infty$ a.s. *Hint: show that the events $\{T > n\}$ are decreasing and find the limit of their probabilities.*

(e) Show that $\mathbb{E}\,\frac{1}{T+2} = \frac{1}{4}$.

3.17 Let $X_1,\, X_2\, \ldots$ be independent random variables with

$$\mathbb{P}\{X_n = i\} = \begin{cases} \mathrm{e}^{-n}, & \text{if } i = 0, \\ 1 - 2\mathrm{e}^{-n}, & \text{if } i = 1, \\ \mathrm{e}^{-n}, & \text{if } i = 2, \end{cases}$$

and $M_n = \prod_{k=1}^{n} X_k$.

(a) Show that M_n is a martingale w.r.t. the natural filtration.

(b) Show that M_n has an almost sure limit M_∞ that is almost surely finite.

(c) Construct a random variable Z that bounds M_n for each n, and is of finite mean.

(d) Show that $\mathbb{E}\,M_\infty = 1$.

3.18 Let $X_1,\, X_2,\, \ldots$ be strictly positive i.i.d. random variables such that $\mathbb{E}\,X_1 = 1$ and $\mathbb{P}\{X_1 = 1\} < 1$.

(a) Show that $M_n = \prod_{i=1}^{n} X_i$ is a martingale w.r.t. the natural filtration.

(b) Deduce that there exists a real valued random variable L such that $M_n \to L$ a.s. as $n \to \infty$.

(c) Show that $\mathbb{P}\{L = 0\} = 1$. *Hint: argue by contradiction and note that if $M_n,\, M_{n+1} \in (a - \varepsilon,\, a + \varepsilon)$ then $X_{n+1} \in \left(\frac{a-\varepsilon}{a+\varepsilon},\, \frac{a+\varepsilon}{a-\varepsilon}\right)$.*

(d) Use the Strong Law of Large Numbers to show that there exists $c \in \mathbb{R}$ such that $\frac{1}{n} \ln M_n \to c$ a.s. as $n \to \infty$. Use Jensen's inequality to show that $c < 0$.

3.19 Let $X_n \in [0,\, 1]$ be adapted to $\mathcal{F}_n$. Let $\alpha,\, \beta > 0$ such that $\alpha + \beta = 1$ and suppose

$$\mathbb{P}\{X_{n+1} = \alpha + \beta X_n \,|\, \mathcal{F}_n\} = X_n, \quad \mathbb{P}\{X_{n+1} = \beta X_n \,|\, \mathcal{F}_n\} = 1 - X_n.$$

Show:

(a) $\mathbb{P}\{\lim_n X_n = 0 \text{ or } 1\} = 1$. *Hint: Use martingale convergence, and try to find an independent sequence U_n that generates X_n.*

(b) If $X_0 = \theta$ then $\mathbb{P}\{\lim_n X_n = 1\} = \theta$.