

HOMEWORK SET 3

Conditional expectation; probabilistic tools

Martingale Theory with Applications, 1st teaching block, 2026
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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 22nd October. Please show your work leading to the result, not only the result. Each problem is worth the number of •'s you see right next to it. Make sure you find all 10 •'s!

Use of AI: Category 2: Minimal - for example, using spell and grammar checkers to help identify mistakes but not rewrite chunks of text.

- 3.1 Fix $0 < p = 1 - q < 1$ reals, let $\Omega = \{0, 1, 2, \dots\}$ be the non-negative integers, $\mathcal{F} = \mathcal{P}(\Omega)$ the power set, and $\mathbb{P}$ the probability measure that assigns $\mathbb{P}\{n\} = q^n p$ to $n \geq 0$. Define the function $X : \Omega \rightarrow \{0, 1, 2, \dots\}$ to be the identity function. Notice that so far this is a way to describe a Pessimistic Geometric(p) random variable (counting the failures before the first success). Define also

$$Y := (X \bmod 2) = \mathbb{1}\{X \text{ is odd}\} = \begin{cases} 0, & \text{if } X \text{ is even,} \\ 1, & \text{if } X \text{ is odd.} \end{cases}$$

- (a) What is the σ -algebra generated by X ?
 - (b) What is the σ -algebra generated by Y ?
 - (c) Use Kolmogorov's theorem on conditional expectations to calculate $\mathbb{E}(X | Y)$.
- 3.2 ••• Let X and Y be two i.i.d. $\text{Exp}(1)$ random variables, and $Z = X + Y$.
- (a) Write down an actual probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to model this situation.
 - (b) Fix any $t \in \mathbb{R}$ with $0 < |t| < 1$ and show that the random variable $\frac{e^{tZ}-1}{tZ}$ satisfies Kolmogorov's theorem on conditional expectations for $\mathbb{E}(e^{tX} | Z)$, hence by uniqueness $\frac{e^{tZ}-1}{tZ}$ is a version of this conditional expectation.
Hint1: recall from earlier studies or accept the fact that the expectation of a function of X and Y is calculated as a double integral of that function with the joint density, which latter in our case is just the product of the marginal Exponential densities due to independence.
Hint2: $\sigma(Z)$ is generated by events of the form $G = \{Z \leq z\}$, hence it is enough to work with these.
Hint3: A substitution of $v = x + y$, then swapping integrals might prove useful.
 - (c) Conclude that $(X | Z) \sim \text{Uniform}(0, Z)$. In other words, X given the sum $Z = X + Y$ has the uniform distribution on the allowed range $(0, X + Y)$.
- 3.3 •• We call two random variables X and Y *jointly continuous with joint probability density function* $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, if for any borel set $B \in \mathcal{B}(\mathbb{R}^2)$ of $\mathbb{R}^2$,

$$\mathbb{P}\{(X, Y) \in B\} = \iint_B f(x, y) \, dx \, dy.$$

- (a) For such pair, show that Y is an absolutely continuous random variable with (marginal) probability density function

$$f_Y(y) := \int_{-\infty}^{\infty} f(x, y) \, dx.$$

- (b) Fix any y where $f_Y(y) > 0$. Use the Kolmogorov definition of conditional expectations to show that the conditional distribution of X given $Y = y$ is absolutely continuous with probability density function

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}.$$

- 3.4 Let $X_1, X_2, \dots, X_n$ be iid. random variables with finite mean, and S_n their sum. Calculate $\mathbb{E}(X_1 | S_n)$.
- 3.5 (*Hard:*) Let X and Y be random variables with finite mean on a probability space. Prove that if $\mathbb{E}(X | Y) = Y$ and $\mathbb{E}(Y | X) = X$, then $X = Y$ a.s.
- 3.6 (*Hard:*) Let X and Y be random variables with finite second moment on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub- σ algebra of $\mathcal{F}$. Suppose that $\mathbb{E}(X | \mathcal{G}) = Y$ and $\mathbb{E}X^2 = \mathbb{E}Y^2$. Prove that $X = Y$ a.s.
- 3.7 Let $\mathcal{G} \subset \mathcal{F}$ be a σ -algebra on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If X is $\mathcal{G}$ -measurable, then $\mathbb{E}(X | \mathcal{G}) = X$ which suggests that the map $X \mapsto \mathbb{E}(X | \mathcal{G})$ is a projection. Show that indeed: this map is an orthogonal projection in the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ (with inner product $\langle X, Y \rangle_{\mathbb{P}} = \mathbb{E}(XY)$) onto the subspace $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$.
- 3.8 •• We perform infinitely many independent experiments. The n^{th} one is successful with probability $n^{-\alpha}$ and fails with probability $1 - n^{-\alpha}$, $0 < \alpha$. Let $k \geq 1$. We are happy if we see k consecutive successes infinitely often. What is the probability of this?

- 3.9 (The longest run of heads, I.)

Let $X_1, X_2, \dots$ be iid. random variables with $\mathbf{P}\{X_k = 1\} = p$, $\mathbf{P}\{X_k = 0\} = q$, where $p + q = 1$. Fix a parameter $\lambda > 1$, and denote by $A_k^{(\lambda)}$ the following events for $k = 0, 1, 2, \dots$:

$$A_k^{(\lambda)} := \left\{ \exists r \in [\lfloor \lambda^k \rfloor, \lfloor \lambda^{k+1} \rfloor - k] \cap \mathbb{N} : X_r = X_{r+1} = \dots = X_{r+k-1} = 1 \right\}.$$

In plain words: $A_k^{(\lambda)}$ means that somewhere between $\lfloor \lambda^k \rfloor$ and $\lfloor \lambda^{k+1} \rfloor - 1$ there is a sequence of k consecutive 1's. Prove that

- If $\lambda < p^{-1}$, then a.s. only finitely many of the events $A_k^{(\lambda)}$ occur.
- If $\lambda > p^{-1}$, then a.s. infinitely many of the events $A_k^{(\lambda)}$ occur.
- What happens for $\lambda = p^{-1}$?

- 3.10 (The longest run of heads, II.)

Let

$$R_n := \sup\{k \geq 0 : X_n = X_{n+1} = \dots = X_{n+k-1} = 1\}.$$

That is: R_n is the length of the run of consecutive 1's that starts at n . (If $X_n = 0$, then set $R_n = 0$.) Prove that

$$\mathbf{P} \left\{ \limsup_{n \rightarrow \infty} \frac{R_n}{\log n} = |\log p|^{-1} \right\} = 1.$$

HINT: For a fixed parameter $\alpha > 0$, let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

If $\alpha > 1$, then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the $B_n^{(\alpha)}$'s occur a.s. If $\alpha \leq 1$, then from the previous exercise it follows that a.s. infinitely many of the $B_n^{(\alpha)}$'s occur.

- 3.11 Let $X_1, X_2, \dots$ be independent. Prove that $\sup_n X_n < \infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$ for some positive finite A .
- 3.12 Prove that for any sequence $X_1, X_2, \dots$ of random variables there exists a deterministic sequence $c_1, c_2, \dots$ of real numbers for which $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$.