Homework set 4

Doob's decomposition, uniformly integrable martingales, Doob's submartingale inequality

Martingale Theory with Applications, 1^{st} teaching block, 2024

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Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 28^{th} November. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Make sure you find all 20 $^\bullet$'s!

4.1 Recall the ABRACADABRA problem, and the variable X_n being the total wealth of all gamblers in play after the n^{th} letter has been typed. Perform Doob's decomposition on X_n.

4.2 Let $\Gamma_0 = 0$, and $\Gamma_1, \Gamma_2, \ldots$ be the marks of a rate λ homogeneous Poisson process on $\mathbb{R}^+$.

(a) Perform Doob's decomposition on Γ_n.

(b) Show that the martingale M you found above is $\mathcal{L}^2$. Find its brackets process $\langle M\rangle$ and the Doob decomposition of M^2.

4.3 Fix $\beta < \lambda$, and for the Poisson process of Problem 4.2 define $X_n := \mathrm{e}^{\beta\Gamma_n}$. Perform Doob's decomposition on this process. (*It is not going to be very nice.*)

4.4 $^{\bullet\bullet\bullet}$ *Decomposing in a product sense.* Let X_n be an adapted process, and assume that it is bounded: $\sup_n X_n < K < \infty$ and bounded away from zero: $\inf_n X_n > \delta > 0$ for some non-random K and δ. Show that

(a) there is

- a martingale $(M_n)_{n\ge 0}$ with $M_0 = 1$,
- a predictable process $(B_n)_{n\ge 0} > 0$ with $B_0 = 1$

such that $X_n = X_0 \cdot M_n \cdot B_n$. This decomposition is almost everywhere unique in the sense that for any other pair $(\widehat{M}_n,\ \widehat{B}_n)_{n\ge 0}$ with the above properties we have $\mathbb{P}\{M_n = \widehat{M}_n \text{ and } B_n = \widehat{B}_n \text{ for all } n \ge 0\} = 1$.

(b) $(X_n)_{n\ge 0}$ is a submartingale if and only if $\mathbb{P}\{B_n \le B_{n+1} \text{ for all } n \ge 0\} = 1$ in the above decomposition.

4.5 $^{\bullet\bullet\bullet\bullet}$ Fix $\beta < \lambda$, and for the Poisson process of Problem 4.2 define $X_n := \mathrm{e}^{\beta\Gamma_n}$.

(a) Find a predictable process $B_n > 0$ for which $X_n = M_n \cdot B_n$ where M_n is a martingale. (*Hint: while X_n does not have the bounds assumed in 4.4, the calculation still works. Check that the martingale is in $\mathcal{L}^1$ separately.*)

(b) Fix $t > 0$ and $T := \inf\{n : \Gamma_n > t\}$. Assume $\beta > 0$ and use the memoryless property of the Exponential distribution to bound the stopped martingale M^T from above by a random variable of finite mean. Hence show that optional stopping applies.

(c) Show from the above that the number of marks before t is Poisson$(\lambda \cdot t)$ distributed (just by assuming the i.i.d. Exponential(λ) interarrival times in the Poisson(λ) process). (*Hint: use its generating function.*)

4.6 Let Y_i, $i = 1, 2, \ldots$ be i.i.d. variables in $\mathcal{L}^2$ with mean μ and variance σ^2. Define $X_0 = 0$, $X_k := \frac{1}{k}\sum_{i=1}^{k} Y_i$ $(k \ge 1)$.

(a) Perform Doob's decomposition on X.

(b) Show that the martingale M you found above is $\mathcal{L}^2$. Find its brackets process $\langle M \rangle$ and the Doob decomposition of M^2.

(c) Show that the martingale M converges a.s.

4.7 *Beta function.* Prove that for any a, $b \geq 0$ integers,

$$I_{a,b} := \int_0^1 \theta^a (1-\theta)^b \, \mathrm{d}\theta = \frac{a! \cdot b!}{(a+b+1)!}.$$

Hint: show via integration by parts that for $b \geq 1$, $I_{a,b} = \frac{b}{a+1} \cdot I_{a+1,b-1}$, *while the case* $b = 0$ *is easy. From here, a recursive argument does the trick.*

4.8 *Bayes urn.* Assume we have a *randomly biased* coin that shows HEAD with probability θ and TAIL with probability $1 - \theta$. This parameter θ is random and has the Uniform(0, 1) distribution. We flip this coin repeatedly and record

$$B_0 := 1, \qquad B_n := 1 + \text{no. of HEADs in the first } n \text{ trials},$$
$$R_0 := 1, \qquad R_n := 1 + \text{no. of TAILs in the first } n \text{ trials}.$$

Notice that $B_n + R_n = n + 2$. Define the filtration generated by the first n flips, $\mathcal{F}_n = \sigma(B_1, B_2, \ldots, B_n)$, and mind that θ is *not* included in here.

(a) •• Determine the probability of a given sequence of flips,

$$\mathbb{P}\{B_1 = b_1, B_2 = b_2, \ldots, B_n = b_n\}.$$

Hint: Condition on θ *and use Problem 4.7.*

(b) •• Based on the previous part, find the distribution of B_{n+1}, given $\mathcal{F}_n$. Compare with the Pólya urn. Remember: θ is not included in $\mathcal{F}_n$.

(c) •• Show that, modulo zero measure sets, θ is $\mathcal{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$-measurable.

(d) •• What is the conditional expectation of θ, given the first n flips? Explain. *Hint: the Pólya urn, and our theorem on uniformly integrable martingales...*

(e) *No marks for this, only for pride, as you might not have met conditional densities before.* Use the Bayes urn to find the conditional density of M_∞, given $\mathcal{F}_n$ in the Pólya urn.

4.9 Let M be a uniformly integrable martingale in the filtration $(\mathcal{F}_n)_{n \geq 0}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $S \leq T$ a.s. be finite stopping times. We denote by $\mathcal{F}_T$ the collection of all events $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all n, which can be thought of as the set of events whose occurrence or non-occurrence is known by time T.

(a) Prove that $\mathcal{F}_T$ is a σ-algebra.

(b) Prove that $M_T = \mathbb{E}(M_\infty \,|\, \mathcal{F}_T)$ and that $M_S = \mathbb{E}(M_T \,|\, \mathcal{F}_S)$. *Hint: observe that* $\mathcal{F}_T$ *is generated by sets* $A \cap \{T = n\}$ *where* $A \in \mathcal{F}$ *and* $n \in \mathbb{Z}^+$.

4.10 Let $Y_0, Y_1, Y_2, \ldots$ be independent random variables with

$$\mathbb{P}\{Y_n = 1\} = \mathbb{P}\{Y_n = -1\} = \frac{1}{2}, \qquad \forall n.$$

For $n \geq 1$, define

$$X_n := Y_0 Y_1 \cdots Y_n.$$

Prove that the variables $X_1, X_2, \ldots$ are independent. Define

$$\mathcal{Y} := \sigma(Y_1, Y_2, \ldots), \qquad \mathcal{T}_n := \sigma(X_{n+1}, X_{n+2}, \ldots).$$

Prove that

$$\mathcal{L} := \bigcap_n \sigma(\mathcal{Y}, \mathcal{T}_n) \neq \sigma\Big(\mathcal{Y}, \bigcap_n \mathcal{T}_n\Big) =: \mathcal{R}.$$

Hint: Prove that Y_0 is $\mathcal{L}$-measurable but independent of $\mathcal{R}$.

4.11 N people queue for a concert the ticket for which costs £1. Each person, independently and with equal chance, has a £1 coin or a £2 coin so these customers need £1 change. The cashier starts selling tickets with a number m of £1 coins in reserve, and we are interested in how this number changes over time.

(a) Find a natural martingale M_n for the problem.

(b) Use M_n^2 to give a bound on the probability that the cashier ever runs out of coins.

(c) Use an exponential of M_n to bound the same. Make your bound as strong as possible.

4.12 ••• $2N$ people queue for a concert the ticket for which costs £1. Exactly N of the queuing people have a £1 coin each and N of them have a £2 coin so these customers need £1 change. The problem is that the queue is in a uniformly random order, hence the cashier starts selling tickets with a number m of £1 coins in reserve. Find a natural martingale for the problem and use Doob's submartingale inequality on its square to give a bound on the probability that the cashier ever runs out of coins. *Hint: Use Problem 2.15. Deal with the first N customers only, then use the symmetry of the problem.*

4.13 *Azuma-Hoeffding concentration inequality.*

(a) Let $c > 0$, and $-c \le Y \le c$ a mean zero random variable. Then for any $\theta \in \mathbb{R}$ we have

$$\mathbf{E}e^{\theta Y} \le \cosh(\theta c) \le e^{\theta^2 c^2/2}.$$

Hint: for any convex function g and $-c \le y \le c$,

$$g(y) \le \frac{c-y}{2c} \cdot g(-c) + \frac{c+y}{2c} \cdot g(c).$$

$e^{\theta \cdot}$ *is a convex function.*

(b) Let M be a martingale with $M_0 = 0$, and assume $|M_n - M_{n-1}| \le c_n$, $\forall n$ with a deterministic sequence $\{c_n\}_{n\in\mathbb{N}}$. Then for any $x > 0$

$$\mathbf{P}\Big\{\sup_{k\le n} M_k \ge x\Big\} \le e^{-x^2/(2\sum_{k=1}^{n} c_k^2)}.$$

Hint: apply the above and Doob's submartingale inequality, then optimise in θ.

4.14 Apply the Azuma-Hoeffding inequality to bound the probability that the cahsier of Problem 4.11 ever runs out of coins.

4.15 •• Apply the Azuma-Hoeffding inequality to bound the probability that the cahsier of Problem 4.12 ever runs out of coins.

4.16 Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra. Show that given $\mathcal{G}$, $\mathbb{E}(X \mid \mathcal{G})$ is the best predictor of X in the following sense: the minimum mean square error $\mathbb{E}(V - X)^2$ among $\mathcal{G}$-measurable random variables V is achieved for $V = \mathbb{E}(X \mid \mathcal{G})$. What is this minimal mean square error? *Hint: use a tower rule first, then minimise pointwise among $\mathcal{G}$-measurable functions.*

4.17 We are given n many intervals of i.i.d. Uniform$(0, 1)$ lengths that need to be packed into "boxes" that is, intervals, of length 1. Let B_n be the minimum number of boxes needed to do that. Apply the Azuma-Hoeffding inequality to bound the deviation between our best estimates after observing the first i Uniforms and the mean of B_n.

4.18 Given are N balls and K, initially empty, urns. We place the balls, one by one, into the urns without removing them. Each ball independently goes to a uniformly chosen urn from 1 to K. These choices are denoted by $X_1, X_2, \ldots, X_N$, which are therefore i.i.d. discrete uniform on the set $\{1, 2, \ldots, K\}$. The generated filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for $n = 0, 1, \ldots, N$. Denote by Z the number of empty urns when all N balls have been placed, and Z_n the number of empty urns after the n^{th} step.

(a) Calculate the best prediction martingale (Problem 4.16) $M_n = \mathbb{E}(Z \mid \mathcal{F}_n)$, $(n = 0, 1, \ldots, N)$ explicitly, and show its martingale property via direct computation based on your explicit form. *Hint: use indicators for urns to stay empty.*

(b) What is M_0 and what is M_N?

(c) Find $\mathbb{E}\, Z_n$ $(0 \le n \le N)$ and $\mathbb{E}\, Z$.

(d) Apply the Azuma-Hoeffding inequality to bound the deviation between our best estimate for Z after observing the first n balls and the mean of Z.

4.19 The *Erdős-Rényi random graph* on n vertices is a random subset of $\binom{n}{2}$ possible edges between the vertices, where each edge is independently present with probability p. The *chromatic number* χ of a graph is the minimum number of colours for the vertices needed to avoid the same colour of any two vertices that are adjacent in the graph (i.e., connected by an edge). Let $\mathcal{F}_k$ be the sigma-algebra generated by the presence or absence of all edges among the first k vertices of the Erdős-Rényi graph, $k = 0 \ldots n$. Apply the Azuma-Hoeffding inequality with this filtration on the chromatic number of this graph.

4.20 A monkey repeatedly types any of the 26 letters of the English alphabet independently with equal chance, until a total of N letters are typed. Let X be the number of times the word "ABRACADABRA" appears. Overlaps are acceptable e.g., we have it three times in EABRACADABRACADABRABRACADABRAX. Show that for any $x > 0$,

$$\mathbb{P}\Big\{\Big|X - (N - 10) \cdot \frac{1}{26^{11}}\Big| \ge x\Big\} \le 2\mathrm{e}^{-x^2/8N}.$$