Homework set 4

*Modes of convergence, optional stopping*

Martingale Theory with Applications, 1st teaching block, 2025

School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 6th November. Please show your work leading to the result, not only the result. Each problem is worth the number of $^\bullet$'s you see right next to it. Make sure you find all 10 $^\bullet$'s!

*Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.*

4.1 Formulate necessary and sufficient conditions for $\alpha_i < \beta_i$ such that independent (but not identically distributed) Uniform$(\alpha_i,\, \beta_i)$ variables $X_i$ converge to 0

a) in distribution;

b) almost surely.

4.2 $^{\bullet\bullet\bullet}$ Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential$(\lambda_i)$ variables $X_i$ to converge to 0

a) in distribution;

b) almost surely.

4.3 Let $\xi_1,\, \xi_2,\, \ldots$ be i.i.d. Poisson(1) random variables. (Recall their moment generating function: $\mathbb{E}(\mathrm{e}^{t\xi_i}) = \mathrm{e}^{\mathrm{e}^t-1}$.) Let $a,\, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = \mathrm{e}^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > a,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow b > \frac{\mathrm{e}^{ra}-1}{r}.$$

4.4 $^{\bullet\bullet}$ Let $\xi_1,\, \xi_2,\, \ldots$ be i.i.d. standard normal random variables. (Recall their moment generating function: $\mathbb{E}(\mathrm{e}^{\lambda\xi_i}) = \mathrm{e}^{\lambda^2/2}$.) Let $a,\, b \in \mathbb{R}$,

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad \text{and} \qquad X_n = \mathrm{e}^{aS_n - bn}.$$

Show that

$$X_n \to 0 \text{ a.s.} \Leftrightarrow b > 0,$$

but for any $r \geq 1$

$$X_n \to 0 \text{ in } \mathcal{L}^r \Leftrightarrow r < \frac{2b}{a^2}.$$

4.5 Let $S$ and $T$ be stopping times w.r.t. the filtration $\mathcal{F}_n$. Which of these are stopping times? Explain.

$$S \wedge T := \min(S,\, T), \qquad S \vee T := \max(S,\, T), \qquad T+S, \qquad T-S \text{ (assume } T \geq S \text{ here.)}$$

4.6 Let $X_1, X_2, \ldots$ be i.i.d. Exponential(1) random variables, $S_n = X_1 + \cdots + X_n$, and $\{\mathcal{F}_n\}$ the natural filtration. Show that

$$\frac{n!}{(1+S_n)^{n+1}} \mathrm{e}^{S_n}$$

is a martingale w.r.t. $\{\mathcal{F}_n\}$.

4.7 An urn contains $n$ white and $n$ black balls. We draw them one by one without replacement. We receive £1 for any white ball, while nothing happens upon drawing a black one. Denote by $X_i$ our money after the $i^{\text{th}}$ draw ($X_0 = 0$). Let

$$\begin{aligned} Y_i &= \frac{2X_i - i}{2n - i} \quad (1 \le i \le 2n-1), \qquad \text{and} \\ Z_i &= \frac{2n-i}{2n-i-1} Y_i^2 - \frac{1}{2n-i-1} \quad (1 \le i \le 2n-2). \end{aligned}$$

(a) Show that both $Y_i$ and $Z_i$ are martingales.

(b) Calculate the mean and variance of $X_i$.

4.8 ••• An urn contains $n$ white and $n$ black balls. We draw them one by one without replacement. We pay £1 for any black ball drawn but receive £1 for any white one. Denote by $X_i$ our money after the $i^{\text{th}}$ draw ($X_0 = 0$). Let

$$Y_i = \frac{X_i}{2n-i} \quad (1 \le i \le 2n-1), \qquad \text{and} \qquad Z_i = \frac{X_i^2 - (2n-i)}{(2n-i)(2n-i-1)} \quad (1 \le i \le 2n-2).$$

(a) Show that both $Y_i$ and $Z_i$ are martingales.

(b) Calculate the variance of $X_i$.

4.9 Let $X_j$, $j \ge 1$, be absolutely integrable random variables, and $\mathcal{F}_n := \sigma(X_j,\ 1 \le j \le n)$, $n \ge 0$, their natural filtration. Define the new random variables

$$Z_0 := 0, \qquad Z_n := \sum_{j=0}^{n-1} \big( X_{j+1} - \mathbb{E}(X_{j+1} \,|\, \mathcal{F}_j) \big).$$

Prove that the process $n \mapsto Z_n$ is an $(\mathcal{F}_n)_{n \ge 0}$-martingale.

4.10 A biased coin shows HEAD with probability $\theta \in (0, 1)$, and TAIL with probability $1-\theta$. The value $\theta$ of the bias in *not known*. For $t \in [0, 1]$ and $n \in \mathbb{N}$ we define $p_{n,t} : \{0, 1\}^n \to [0, 1]$ by

$$p_{n,t}(x_1, x_2, \ldots, x_n) = t^{\sum_{j=1}^n x_j} \cdot (1-t)^{n - \sum_{j=1}^n x_j}.$$

We make two hypotheses about the possible value of $\theta$: either $\theta = a$, or $\theta = b$, where $a, b \in [0, 1]$ and $a \ne b$. We toss the coin repeatedly and form the sequence of random variables

$$Z_n := \frac{p_{n,a}(\xi_1, \xi_2, \ldots, \xi_n)}{p_{n,b}(\xi_1, \xi_2, \ldots, \xi_n)},$$

where we write $\xi_j = 1$ if the $j^{\text{th}}$ flip is HEAD and $\xi_j = 0$ if it is TAIL. Show that the process $n \mapsto Z_n$ is a martingale (w.r.t. the natural filtration generated by the coin tosses) if and only if the true bias of the coin is $\theta = b$.

4.11 Let $\eta_n$ be a homogeneous Markov chain on the countable state space $S := \{0,\, 1,\, 2,\, \dots\}$ and $\mathcal{F}_n := \sigma(\eta_j,\ 0 \le j \le n)$, $n \ge 0$ its natural filtration. For $i \in S$ denote by $Q(i)$ the probability that the Markov chain starting from site $i$ *ever reaches* the point $0 \in S$:

$$Q(i) := \mathbb{P}\{\exists m < \infty \,:\, \eta_m = 0 \,|\, \eta_0 = i\}.$$

Prove that $Z_n := Q(\eta_n)$ is an $(\mathcal{F}_n)_{n \ge 0}$-martingale.

4.12 *Bellman's Optimality Principle.* We model a sequence of gamblings as follows. Let $\xi_1,\, \xi_2,\, \dots$ be i.i.d. random variables with $\mathbb{P}\{\xi_n = +1\} = p$, $\mathbb{P}\{\xi_n = -1\} = q$, where $p = 1 - q > 1/2$. Define the *entropy* of this distribution by

$$\alpha = p \ln\Big(\frac{p}{1/2}\Big) + q \ln\Big(\frac{q}{1/2}\Big) = p \ln p + q \ln q + \ln 2.$$

A gambler starts playing with initial fortune $Y_0 > 0$. Her return at time $n$ on a *unit bet* is the random variable $\xi_n$, and she plays $C_n$ in round $n$. In other words, with probability $p$ she doubles her bet and with probability $q$ she looses it. Therefore her fortune after round $n$ is

$$Y_n = Y_{n-1} + C_n \xi_n.$$

The bet $C_n$ may depend on the values $\xi_1,\, \xi_2,\, \dots,\, \xi_{n-1}$, and has bounds $0 \le C_n < Y_{n-1}$. The expected rate of winnings up to time $n$ is

$$r_n := \mathbb{E} \ln\Big(\frac{Y_n}{Y_0}\Big),$$

which the gambler wishes to maximise.

(a) Prove that no matter what strategy $C$ the gambler chooses,

$$X_n := \ln Y_n - n\alpha$$

is a supermartingale, hence her expected average winning rate, $\frac{r_n}{n} \le \alpha$.

(b) However, there exists a gambling strategy that makes the above $X$ a martingale, hence realises the average expected winning rate $\alpha$. Find this strategy.

4.13 •• Let $S_n$ be a simple symmetric random walk on the square lattice $\mathbb{Z}^2$ with $S_0 = (0,\, 0)$. That is, the walker starts from the origin and at each step independently, she steps one unit to East, North, West or South with equal chance. Denote by $D_n$ the walker's Euclidean distance from the origin of $\mathbb{Z}^2$ at time $n$, and let $\nu_r = \inf\{n \,:\, D_n > r\}$.

(a) Show that $D_n^2 - n$ is a martingale.

(b) Show that $r^{-2}\, \mathbb{E}\, \nu_r \to 1$ as $r \to \infty$.

4.14 The problem is the same as the previous one, except that the walk is on $\mathbb{R}^2$ and steps are of length one in i.i.d. Uniform$(0,\, 2\pi)$ directions.

4.15 Let $S_n$ be a simple symmetric random walk on the cubic lattice $\mathbb{Z}^3$ with $S_0 = (0,\, 0,\, 0)$. That is, the walker starts from the origin and at each step independently, she steps one unit to up, down, left, right, forward or backward with equal chance. Denote by $D_n$ the walker's Euclidean distance from the origin of $\mathbb{Z}^3$ at time $n$, and let $\nu_r = \inf\{n \,:\, D_n > r\}$.

(a) Show that $D_n^2 - n$ is a martingale.

(b) Show that $r^{-2}\, \mathbb{E}\, \nu_r \to 1$ as $r \to \infty$.

4.16 We repeatedly toss a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HHHHHH for the first time?

(b) We stop when six consecutive tosses result in the same outcome, in other words when either the pattern HHHHHH or TTTTTT first appears. What is the expected number of tosses until this moment?

4.17 We repeatedly toss a fair coin.

(a) What is the expected number of tosses until we have seen the pattern HTHT for the first time?

(b) What is the expected number of tosses until we have seen the pattern THTH for the first time?

(c) What is the expected number of tosses until we have seen the pattern HTTH for the first time?

(d) What is the expected number of tosses until we have seen the pattern THHT for the first time?

(e) Give an example of a four letter pattern of H-s and T-s that has the maximal expected number of tosses, of any four letter patterns, until it is seen.

4.18 The previous question with a biased coin. Explain your answer.

4.19 Let $m \geq 2$ be an integer. At time $n = 0$, an urn contains $2m$ balls of which $m$ are red and $m$ are blue. At each time $n = 1, 2, \ldots, 2m$ we draw a randomly chosen ball without replacement from the urn and record its colour. For $n = 0, 1, \ldots, 2m-1$ let $N_n$ denote the number of red balls left in the urn after time $n$, and

$$P_n := \frac{N_n}{2m-n}$$

denote the fraction of them. Let $(\mathcal{F}_n)_{0 \leq n \leq 2m}$ be the natural filtration generated by the process $(N_n)_{0 \leq n \leq 2m}$.

(a) Show that $n \mapsto P_n$ is an $\mathcal{F}_n$-martingale.

(b) Let $T$ be the first time at which the ball drawn is red. Show that the $(T+1)^{\text{st}}$ draw is equally likely to be red or blue.