

HOMEWORK SET 5  
*Borel-Cantelli Lemmas, Law of Large Numbers*  
 Further Topics in Probability, 1<sup>st</sup> teaching block, 2013  
 School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 11:59pm on Wednesday, 4<sup>th</sup> December. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

- 5.1 a) Prove that Markov's inequality is *sharp* in the following sense: fixing  $0 < m \leq \lambda$ , there exists a non-negative random variable  $X$  with expectation  $\mathbf{E}X = m$  and 'saturated' Markov's inequality:  $\mathbf{P}\{X \geq \lambda\} = m/\lambda$ .
- b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable  $X$  with finite mean,  $\lim_{\lambda \rightarrow \infty} \lambda \mathbf{P}\{X \geq \lambda\} / \mathbf{E}X = 0$ .

5.2 Show that  $\mathbf{E}X^2 < \infty$  if and only if  $\sum_{n=1}^{\infty} n \cdot \mathbf{P}\{|X| > n\} < \infty$ .

5.3 •• Let the random variables  $X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n, \dots, X$  and  $Y$  be defined on a common probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and suppose  $X_n \xrightarrow{\mathbf{P}} X$  and  $Y_n \xrightarrow{\mathbf{P}} Y$ . Prove

a)  $X_n Y_n \xrightarrow{\mathbf{P}} XY$ ,

b) if  $Y_n \neq 0$  and  $Y \neq 0$  a.s., then  $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$ .

5.4 Let  $X_1, X_2, \dots$  be independent. Prove that  $\sup_n X_n < \infty$  a.s. if and only if  $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$  for some positive finite  $A$ .

5.5 Prove that for any sequence  $X_1, X_2, \dots$  of random variables there exists a deterministic sequence  $c_1, c_2, \dots$  of real numbers for which  $\frac{X_n}{c_n} \xrightarrow{\text{a.s.}} 0$ .

5.6 ••• Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential( $\lambda_i$ ) variables  $X_i$  to converge to 0

a) in distribution;

b) almost surely.

5.7 We perform infinitely many independent experiments. The  $n^{\text{th}}$  one is successful with probability  $n^{-\alpha}$  and fails with probability  $1 - n^{-\alpha}$ ,  $0 < \alpha < 1$ . Let  $k \geq 1$ . We are happy if we see  $k$  consecutive successes infinitely often. What is the probability of this?

5.8 (The longest run of heads, I.)

Let  $X_1, X_2, \dots$  be iid. random variables with  $\mathbf{P}\{X_k = 1\} = p$ ,  $\mathbf{P}\{X_k = 0\} = q$ , where  $p + q = 1$ . Fix a parameter  $\lambda > 1$ , and denote by  $A_k^{(\lambda)}$  the following events for  $k = 0, 1, 2, \dots$ :

$$A_k^{(\lambda)} := \left\{ \exists r \in \left[ \lfloor \lambda^k \rfloor, \lfloor \lambda^{k+1} \rfloor - k \right] \cap \mathbb{N} : X_r = X_{r+1} = \dots = X_{r+k-1} = 1 \right\}.$$

In plain words:  $A_k^{(\lambda)}$  means that somewhere between  $\lfloor \lambda^k \rfloor$  and  $\lfloor \lambda^{k+1} \rfloor - 1$  there is a sequence of  $k$  consecutive 1's. Prove that

- a) If  $\lambda < p^{-1}$ , then a.s. only finitely many of the events  $A_k^{(\lambda)}$  occur.  
 b) If  $\lambda > p^{-1}$ , then a.s. infinitely many of the events  $A_k^{(\lambda)}$  occur.  
 c) What happens for  $\lambda = p^{-1}$ ?

5.9 (The longest run of heads, II.)

Let

$$R_n := \sup\{k \geq 0 : X_n = X_{n+1} = \cdots = X_{n+k-1} = 1\}.$$

That is:  $R_n$  is the length of the run of consecutive 1's that starts at  $n$ . (If  $X_n = 0$ , then set  $R_n = 0$ .) Prove that

$$\mathbf{P} \left\{ \limsup_{n \rightarrow \infty} \frac{R_n}{\log n} = |\log p|^{-1} \right\} = 1.$$

*HINT: For a fixed parameter  $\alpha > 0$ , let*

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

*If  $\alpha > 1$ , then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the  $B_n^{(\alpha)}$ 's occur a.s. If  $\alpha \leq 1$ , then from the previous exercise it follows that a.s. infinitely many of the  $B_n^{(\alpha)}$ 's occur.*

- 5.10 ••• On the (simplified version of the) game Roulette, a player bets £1, and loses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Use the Weak Law of Large Numbers to find the probability that the casino loses money with this game on the (very) long run. Explain your answer.

- 5.11 Rolling a die 100 times, denote the outcome of roll  $i$  by  $X_i$ . Estimate the probability

$$\mathbf{P} \left\{ \prod_{i=1}^{100} X_i \leq a^{100} \right\}$$

for real  $1 < a < 6$ .

- 5.12 ••• (The simplest form of the *McMillan Theorem*.)

Let  $\mathbf{p} = (p_1, p_2, \dots, p_r)$ , where  $p_i, i = 1, 2, \dots, r$  are positive numbers with  $p_1 + p_2 + \cdots + p_r = 1$ . That is: given is a probability distribution on the set  $\{1, 2, \dots, r\}$ . The *entropy* of the distribution  $\mathbf{p}$  is defined by  $H(\mathbf{p}) := -\sum_{j=1}^r p_j \log p_j$ . Let  $X_1, X_2, \dots$  be iid. random variables from this distribution  $\mathbf{p}$ . Define the random variables  $R_n := \prod_{k=1}^n p_{X_k}$ : this is the a priori probability of the observed sequence  $X_1, X_2, \dots, X_n$  of outcomes. Prove that

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} n^{-1} \log R_n = -H(\mathbf{p}) \right\} = 1.$$

- 5.13 ••• Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Prove

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 f \left( \frac{x_1 + x_2 + \cdots + x_n}{n} \right) dx_1 dx_2 \cdots dx_n &= f \left( \frac{1}{2} \right). \\ \lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 f \left( (x_1 x_2 \cdots x_n)^{1/n} \right) dx_1 dx_2 \cdots dx_n &= f \left( \frac{1}{e} \right). \end{aligned}$$

- 5.14 Prove

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n = \frac{2}{3}.$$

5.15 Let  $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$  be the surface of the  $n$ -dimensional Euclidean unit sphere. There is a unique probability measure  $\nu^{(n-1)}$  on  $S^{n-1}$  that is invariant to orthogonal transformations of  $\mathbb{R}^n$ : for any Borel-measurable  $A \subset S^{n-1}$  and  $H$  orthogonal transformation of  $\mathbb{R}^n$ ,  $\nu^{(n-1)}(HA) = \nu^{(n-1)}(A)$ . (This measure is called the *Haar measure* on  $S^{n-1}$ , it is actually the uniform measure on  $S^{n-1}$ .)

- a) Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  be the vector of iid. Standard Normal components in  $\mathbb{R}^n$ . Prove that applying an arbitrary orthogonal transformation  $H$  on  $\mathbf{X}$ , the resulting vector  $\mathbf{Y} := H\mathbf{X}$  again has iid. Standard Normal components  $Y_1, Y_2, \dots, Y_n$ . From this and the uniqueness of the Haar measure prove that  $\mathbf{X}/|\mathbf{X}| \in S^{n-1}$  has the uniform distribution  $\nu^{(n-1)}$  on the surface of the sphere. (That is, for any Borel measurable  $A \subset S^{n-1}$ , we have  $\mathbf{P}(\mathbf{X}/|\mathbf{X}| \in A) = \nu^{(n-1)}(A)$ .)
- b) Let  $X_1, X_2, \dots$  be iid. Standard Normal random variables, and

$$R_n := (X_1^2 + X_2^2 + \dots + X_n^2)^{1/2}.$$

Prove  $R_n/\sqrt{n} \xrightarrow{\mathbf{P}} 1$  as  $n \rightarrow \infty$ .

- c) Pick now a uniform random point  $P$  on the surface  $S^{n-1}$  of the unit sphere, and denote its coordinates in  $\mathbb{R}^n$  by  $(Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)})$ . Use the above a) and b) to prove the following limit theorems for  $P$ :

$$\lim_{n \rightarrow \infty} \mathbf{P}(\sqrt{n}Y_1^{(n)} < y) = \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx,$$

$$\lim_{n \rightarrow \infty} \mathbf{P}(\sqrt{n}Y_1^{(n)} < y_1; \sqrt{n}Y_2^{(n)} < y_2) = \Phi(y_1)\Phi(y_2).$$

*HINT:*  $\mathbf{P}(\sqrt{n}Y_1^{(n)} < y) = \mathbf{P}(\sqrt{n}X_1/R_n < y)$ .

- 5.16 a) Let  $X_1, X_2, \dots, X_n, \dots$  be iid. random variables with  $\mathbf{P}\{X_i = 0\} = \mathbf{P}\{X_i = 1\} = \frac{1}{2}$ . Let  $Y := \sum_{n=1}^{\infty} 2^{-n}X_n$ . Prove that  $Y$  is uniformly distributed on the real interval  $[0, 1]$ .
- b) Let  $\mathbf{P}_1$  and  $\mathbf{P}_2$  be probability measures on the common measurable space  $(\Omega, \mathcal{F})$ . Let  $\mathcal{A}$  be an algebra such that  $\sigma(\mathcal{A}) = \mathcal{F}$ . We say that  $\mathbf{P}_1$  and  $\mathbf{P}_2$  are *mutually singular*, if there is an  $H \in \mathcal{F}$ , for which  $\mathbf{P}_1(H) = 1$  and  $\mathbf{P}_2(H) = 0$ . Prove that this happens if and only if

$$\forall \varepsilon > 0 \quad \exists A \in \mathcal{A} : \quad \mathbf{P}_1(A) \geq 1 - \varepsilon, \quad \mathbf{P}_2(A) \leq \varepsilon.$$

- c) Let  $X_1, X_2, \dots, X_n, \dots$  be iid. random variables with  $\mathbf{P}\{X_i = 0\} = 1 - \mathbf{P}\{X_i = 1\} = p \neq \frac{1}{2}$ . Let  $Y := \sum_{n=1}^{\infty} 2^{-n}X_n$ . Prove that  $Y$  has a continuous, but not *absolutely continuous* distribution: it is singular with respect to the Lebesgue measure, cf. part a).