Homework set 5

Characteristic functions, CLT

Further Topics in Probability, 2nd teaching block, 2015

School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 16:00pm on Tuesday, 21st April. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Introducing: ◖ for half a mark. Random variables are defined on a common probability space unless otherwise stated.

5.1 Determine the characteristic functions of

a) ◖ the Bernoulli(p),

b) ◖ the Binomial(n, p),

c) ◖ the Poisson(λ),

d) ◖ the Optimistic and Pessimistic Geometric(p),

e) the Normal(μ, σ^2) if you haven't seen it on Tuesday,

f) ◖ the Exponential(λ),

g) ◖ the Uniform(a, b),

h) the Cauchy(b, a) if you haven't seen it on Tuesday

distributions.

5.2 Let φ be the characteristic function of a probability distribution. Are $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ characteristic functions?

5.3 Let $f(x) = 1 - |x-1|$ for $0 \le x \le 2$, and 0 otherwise. Determine the characteristic function of the distribution with density f.

5.4 Explain, using characteristic functions, the identities

$$\frac{\sin t}{t} = \frac{\sin(t/2)}{t/2} \cdot \cos(t/2) \qquad \text{and} \qquad \frac{\sin t}{t} = \prod_{k=1}^{\infty} \cos\Big(\frac{t}{2^k}\Big).$$

5.5 •••• Show that if φ is the characteristic function of an integer-valued distribution, then the mass function is of the form

$$p(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-ikt} \varphi(t)\,\mathrm{d}t, \qquad \forall k \in \mathbb{Z}.$$

5.6 ••• Let μ_n be the Normal(m_n, σ_n^2) probability distribution, $n = 1, 2, \ldots$. Prove that the sequence $\{\mu_n\}_n$ is tight if and only if the number sequences m_n and σ_n^2 are both bounded.

Below you will need the Continuity Lemma. It states

Theorem 1. *Let μ_n, $n = 1, 2, \ldots$ be a sequence of distributions, and φ_n the associated characteristic functions.*

1. If $\mu_n \xrightarrow{w} \mu$, then for all $t \in \mathbb{R}$ $\varphi_n(t) \to \varphi(t)$, where φ is the characteristic function of μ.

2. *If for all $t \in \mathbb{R}$ $\varphi(t) := \lim_{n\to\infty} \varphi_n(t)$ exists, and is continuous at $t = 0$, then φ is the characteristic function of a distribution μ, and $\mu_n \xrightarrow{w} \mu$.*

5.7 Let X_p be Pessimistic Geometric(p) distributed. Prove, via characteristic functions and the Continuity Lemma, that $p \cdot X_p$ converges to Exponential(1) in distribution as $p \searrow 0$.

5.8 •••• Let X be Poisson(λ) distributed. Prove, via characteristic functions and the Continuity Lemma, that

$$\frac{X - \lambda}{\sqrt{\lambda}} \xrightarrow{d} \text{Normal}(0,\ 1)$$

as $\lambda \to \infty$.

5.9 Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables, and $X_k = k \cdot Y_k$, $\quad S_n = X_1 + X_2 + \cdots + X_n$. Prove that

$$\frac{S_n}{\frac{n^2}{4}} \xrightarrow[n\to\infty]{\text{w}} 1, \qquad \text{and} \qquad \frac{S_n - \frac{n^2}{4}}{\frac{1}{6} n^{\frac{3}{2}}} \xrightarrow[n\to\infty]{\text{w}} \text{Normal}(0,\ 1).$$

Below you will need the Central Limit Theorem:

Theorem 2. *Let X_i be iid. random variables with finite mean m and variance σ^2. Then for every $a \in \mathbb{R}$,*

$$\mathbf{P}\Big\{\frac{X_1 + X_2 + \cdots + X_n - n \cdot m}{\sigma\sqrt{n}} \le a\Big\} \overset{n\to\infty}{\longrightarrow} \phi(a) \qquad \textit{(Standard Normal distribution).}$$

5.10 ••• We keep rolling a fair die until the sum of the numbers shown on it exceeds 300. Estimate the probability that at least 80 rolls are needed for this.

5.11 We round 50 real numbers to integers, then sum them up. Suppose that rounding makes a Uniform(-0.5, 0.5) error independently, for each of the 50 numbers. Estimate the probability that our result differs from the true sum by more than 3.