Homework set 5
Borel-Cantelli Lemmas, Law of Large Numbers
Further Topics in Probability, 2$^{\text{nd}}$ teaching block, 2016
School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in class or in the blue locker with my name on the ground floor of the Main Maths Building before 16:00pm on Thursday, 21$^{\text{st}}$ April. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

5.1 $^{\bullet\bullet}$ (*Shiryaev.*) Let Ω be a countable set and $\mathcal{F}$ the collection of all its subsets. Put $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is infinite. Show that the set function μ is finitely additive but not σ-additive.

5.2 a) Prove that Markov's inequality is *sharp* in the following sense: fixing $0 < m \le \lambda$, there exists a non-negative random variable X with expectation $\mathbf{E}X = m$ and 'saturated' Markov's inequality: $\mathbf{P}\{X \ge \lambda\} = m/\lambda$.

b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable X with finite mean, $\lim\limits_{\lambda\to\infty} \lambda\mathbf{P}\{X \ge \lambda\}/\mathbf{E}X = 0$.

5.3 Show that $\mathbf{E}X^2 < \infty$ if and only if $\sum_{n=1}^{\infty} n \cdot \mathbf{P}\{|X| > n\} < \infty$.

5.4 $^{\bullet\bullet\bullet}$ Let the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, \ldots, X$ and Y be defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and suppose $X_n \xrightarrow{\mathbf{P}} X$ and $Y_n \xrightarrow{\mathbf{P}} Y$. Prove

a) $X_nY_n \xrightarrow{\mathbf{P}} XY$,

b) if $Y_n \ne 0$ and $Y \ne 0$ a.s., then $X_n/Y_n \xrightarrow{\mathbf{P}} X/Y$.

5.5 Let $X_1, X_2, \ldots$ be independent. Prove that $\sup_n X_n < \infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathbf{P}\{X_n > A\} < \infty$ for some positive finite A.

5.6 Prove that for any sequence $X_1, X_2, \ldots$ of random variables there exists a deterministic sequence $c_1, c_2, \ldots$ of real numbers for which $\frac{X_n}{c_n} \xrightarrow{\textbf{a.s.}} 0$.

5.7 $^{\bullet\bullet\bullet\bullet}$ Formulate necessary and sufficient conditions for independent (but not identically distributed) Exponential(λ_i) variables X_i to converge to 0

a) in distribution;

b) almost surely.

5.8 We perform infinitely many independent experiments. The n^{th} one is successful with probability $n^{-\alpha}$ and fails with probability $1 - n^{-\alpha}$, $0 < \alpha < 1$. Let $k \ge 1$. We are happy if we see k consecutive successes infinitely often. What is the probability of this?

5.9 (The longest run of heads, I.)
Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbf{P}\{X_k = 1\} = p$, $\mathbf{P}\{X_k = 0\} = q$, where $p + q = 1$. Fix a parameter $\lambda > 1$, and denote by $A_k^{(\lambda)}$ the following events for $k = 0, 1, 2, \ldots$:

$$A_k^{(\lambda)} := \Big\{\exists r \in \big[\lfloor\lambda^k\rfloor, \lfloor\lambda^{k+1}\rfloor - k\big] \cap \mathbb{N} : X_r = X_{r+1} = \cdots = X_{r+k-1} = 1\Big\}.$$

In plain words: $A_k^{(\lambda)}$ means that somewhere between $\lfloor\lambda^k\rfloor$ and $\lfloor\lambda^{k+1}\rfloor - 1$ there is a sequence of k consecutive 1's. Prove that

a) If $\lambda < p^{-1}$, then a.s. only finitely many of the events $A_k^{(\lambda)}$ occur.

b) If $\lambda > p^{-1}$, then a.s. infinitely many of the events $A_k^{(\lambda)}$ occur.

c) What happens for $\lambda = p^{-1}$?

5.10 (The longest run of heads, II.)
Let

$$R_n := \sup\{k \geq 0 \, : \, X_n = X_{n+1} = \cdots = X_{n+k-1} = 1\}.$$

That is: R_n is the length of the run of consecutive 1's that starts at n. (If $X_n = 0$, then set $R_n = 0$.) Prove that

$$\mathbf{P}\left\{\limsup_{n\to\infty} \frac{R_n}{\log n} = |\log p|^{-1}\right\} = 1.$$

HINT: For a fixed parameter $\alpha > 0$, let

$$B_n^{(\alpha)} := \{R_n > \alpha \log n / |\log p|\}.$$

If $\alpha > 1$, then by the first Borel-Cantelli Lemma and direct computation, only finitely many of the $B_n^{(\alpha)}$'s occur a.s. If $\alpha \leq 1$, then from the previous exercise it follows that a.s. infinitely many of the $B_n^{(\alpha)}$'s occur.

5.11 •• On the (simplified version of the) game Roulette, a player bets £ 1, and looses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Use the Weak Law of Large Numbers to find the probability that the casino looses money with this game on the (very) long run. Explain your answer.

5.12 Rolling a die 100 times, denote the outcome of roll i by X_i. Estimate the probability

$$\mathbf{P}\Big\{\prod_{i=1}^{100} X_i \leq a^{100}\Big\}$$

for real $1 < a < 6$.

5.13 ••• (The simplest form of the *McMillan Theorem.*)
Let $\mathbf{p} = (p_1, p_2, \ldots, p_r)$, where p_i, $i = 1, 2, \ldots, r$ are positive numbers with $p_1 + p_2 + \cdots + p_r = 1$. That is: given is a probability distribution on the set $\{1, 2, \ldots, r\}$. The *entropy* of the distribution $\mathbf{p}$ is defined by $H(\mathbf{p}) := -\sum_{j=1}^{r} p_j \log p_j$. Let $X_1, X_2, \ldots$ be iid. random variables from this distribution $\mathbf{p}$. Define the random variables $R_n := \prod_{k=1}^{n} p_{X_k}$: this is the a priori probability of the observed sequence $X_1, X_2, \ldots, X_n$ of outcomes. Prove that

$$\mathbf{P}\left\{\lim_{n\to\infty} n^{-1} \log R_n = -H(\mathbf{p})\right\} = 1.$$

5.14 ••• Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left((x_1 x_2 \cdots x_n)^{1/n}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

5.15 Prove

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n = \frac{2}{3}.$$

5.16 Let $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ be the surface of the n-dimensional Euclidean unit sphere. There is a unique probability measure $\nu^{(n-1)}$ on S^{n-1} that is invariant to orthogonal transformations of $\mathbb{R}^n$: for any Borel-measurable $A \subset S^{n-1}$ and H orthogonal transformation of $\mathbb{R}^n$, $\nu^{(n-1)}(HA) = \nu^{(n-1)}(A)$. (This measure is called the *Haar measure* on S^{n-1}, it is actually the uniform measure on S^{n-1}.)

a) Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be the vector of iid. Standard Normal components in $\mathbb{R}^n$. Prove that applying an arbitrary orthogonal transformation H on $\mathbf{X}$, the resulting vector $\mathbf{Y} := H\mathbf{X}$ again has iid. Standard Normal components $Y_1, Y_2, \ldots, Y_n$. From this and the uniqueness of the Haar measure prove that $\mathbf{X}/|\mathbf{X}| \in S^{n-1}$ has the uniform distribution $\nu^{(n-1)}$ on the surface of the sphere. (That is, for any Borel measurable $A \subset S^{n-1}$, we have $\mathbf{P}\big(\mathbf{X}/|\mathbf{X}| \in A\big) = \nu^{(n-1)}(A)$.)

b) Let $X_1, X_2, \ldots$ be iid. Standard Normal random variables, and

$$R_n := \left(X_1^2 + X_2^2 + \cdots + X_n^2\right)^{1/2}.$$

Prove $R_n/\sqrt{n} \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$.

c) Pick now a uniform random point P on the surface S^{n-1} of the unit sphere, and denote its coordinates in $\mathbb{R}^n$ by $(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)})$. Use the above a) and b) to prove the following limit theorems for P:

$$\lim_{n\to\infty} \mathbf{P}\Big(\sqrt{n}Y_1^{(n)} < y\Big) = \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, \mathrm{d}x,$$

$$\lim_{n\to\infty} \mathbf{P}\Big(\sqrt{n}Y_1^{(n)} < y_1;\ \sqrt{n}Y_2^{(n)} < y_2\Big) = \Phi(y_1)\Phi(y_2).$$

HINT: $\mathbf{P}\big(\sqrt{n}Y_1^{(n)} < y\big) = \mathbf{P}\big(\sqrt{n}X_1/R_n < y\big)$.