Homework set 5

DeMoivre-Laplace CLT and Law of large numbers

Further Topics in Probability, 2nd teaching block, 2022

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Problems with •'s are to be handed in. These are due in Blackboard before 12:00pm on Monday, 25th April. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

5.1 ••••

a) Prove that Markov's inequality is *sharp* in the following sense: fixing $0 < m \le \lambda$, there exists a non-negative random variable X with expectation $\mathbb{E}\,X = m$ and 'saturated' Markov's inequality: $\mathbb{P}\{X \ge \lambda\} = m/\lambda$.

b) Prove that Markov's inequality is *not sharp*, in the following sense: for any fixed non-negative random variable X with finite mean, $\lim\limits_{\lambda\to\infty} \lambda \mathbb{P}\{X \ge \lambda\}/\,\mathbb{E}\,X = 0$.

5.2 Show that $\mathbb{E}\,X^2 < \infty$ if and only if $\sum_{n=1}^{\infty} n \cdot \mathbb{P}\{|X| > n\} < \infty$.

5.3 (*Ross.*) Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

5.4 A fraction p of citizens in a city smoke. We are to determine the value of p by making a survey involving n citizens whom we select randomly. If k of these n people smoke, then $p' = k/n$ will be our result. How large should we choose n if we want our result p' to be closer to the real value p than 0.01 with probability at least 0.98? In other words: determine the smallest number n_0 such that $P(|p' - p| \le 0.01) \ge 0.98$ for any $p \in (0, 1)$ and $n \ge n_0$.

5.5 Given are two very similar insurance companies, each with 10 000 customers. In the beginning of the year, each customer pays £ 500, and during the year each customer independently claims, at most once, £ 1 500 with probability 1/3. The two companies start the year with capital £ 50 000 each. A company becomes bankrupt if it cannot pay the insurance claims during the year. Should the two companies unite to help avoiding bankruptcy? Explain.

5.6 ••• Use normal approximation to find the numerical value of

$$\binom{3600}{1224} \cdot 0.36^{1224} \cdot 0.64^{2376}.$$

5.7 •••• **Normal approximation of Poisson.** Let $X \sim \mathrm{Poi}(\lambda)$. Use asymptotics on the mass function directly to prove that, as $\lambda \to \infty$,

$$\sqrt{\lambda} \cdot \mathbb{P}\{X = \lfloor \lambda + x\sqrt{\lambda} \rfloor\} = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) + \mathcal{O}(\lambda^{-1/2}),$$

where the error term is uniformly small if x stays in a bounded range. *HINT: First argue that the floor is not important, then apply Stirling's formula. And be patient.*

As a corollary, prove

$$\mathbb{P}\Big\{a < \frac{X-\lambda}{\sqrt{\lambda}} \le b\Big\} \to \int_a^b \varphi(y)\,\mathrm{d}y = \phi(b) - \phi(a)$$

as $\lambda \to \infty$. (Notice $\mathbb{E}\,X = \lambda$ and $\mathbb{SD}\,X = \sqrt{\lambda}$.)

5.8 On the (simplified version of the) game Roulette, a player bets £1, and looses his bet with probability 19/37, but is given his bet and an extra pound back with probability 18/37. Use the Weak Law of Large Numbers to find the probability that the casino looses money with this game on the (very) long run. Explain your answer.

5.9 ••• Rolling a die 200 times, denote the outcome of roll i by X_i. Estimate the probability

$$\mathbb{P}\Big\{\prod_{i=1}^{200} X_i \le a^{200}\Big\}$$

for real $1 < a < 6$.

5.10 ••• Let $f : [0,\,1] \to \mathbb{R}$ be continuous. Prove

$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 f\left((x_1x_2\cdots x_n)^{1/n}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n = f\left(\frac{1}{e}\right).$$

5.11 Prove

$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 \frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1+x_2+\cdots+x_n}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n = \frac{2}{3}.$$

5.12 Let $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ be the surface of the n-dimensional Euclidean unit sphere. There is a unique probability measure $\nu^{(n-1)}$ on S^{n-1} that is invariant to orthogonal transformations of $\mathbb{R}^n$: for any Borel-measurable $A \subset S^{n-1}$ and H orthogonal transformation of $\mathbb{R}^n$, $\nu^{(n-1)}(HA) = \nu^{(n-1)}(A)$. (This measure is called the *Haar measure* on S^{n-1}, it is actually the uniform measure on S^{n-1}.)

a) Let $\mathbf{X} = (X_1,\,X_2,\,\ldots,\,X_n)$ be the vector of iid. Standard Normal components in $\mathbb{R}^n$. Prove that applying an arbitrary orthogonal transformation H on $\mathbf{X}$, the resulting vector $\mathbf{Y} := H\mathbf{X}$ again has iid. Standard Normal components $Y_1,\,Y_2,\,\ldots,\,Y_n$. From this and the uniqueness of the Haar measure prove that $\mathbf{X}/|\mathbf{X}| \in S^{n-1}$ has the uniform distribution $\nu^{(n-1)}$ on the surface of the sphere. (That is, for any Borel measurable $A \subset S^{n-1}$, we have $\mathbb{P}\big(\mathbf{X}/|\mathbf{X}| \in A\big) = \nu^{(n-1)}(A)$.)

b) Let $X_1,\,X_2,\,\ldots$ be iid. Standard Normal random variables, and

$$R_n := \left(X_1^2 + X_2^2 + \cdots + X_n^2\right)^{1/2}.$$

Prove $R_n/\sqrt{n} \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$.

c) Pick now a uniform random point P on the surface S^{n-1} of the unit sphere, and denote its coordinates in $\mathbb{R}^n$ by $(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)})$. Use the above a) and b) to prove the following limit theorems for P:

$$\lim_{n\to\infty} \mathbb{P}\Big(\sqrt{n}Y_1^{(n)} < y\Big) = \Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\,\mathrm{d}x,$$

$$\lim_{n\to\infty} \mathbb{P}\Big(\sqrt{n}Y_1^{(n)} < y_1;\ \sqrt{n}Y_2^{(n)} < y_2\Big) = \Phi(y_1)\Phi(y_2).$$

HINT: $\mathbb{P}\big(\sqrt{n}Y_1^{(n)} < y\big) = \mathbb{P}\big(\sqrt{n}X_1/R_n < y\big)$.