Homework set 5

Optional stopping, martingale convergence

Martingale Theory with Applications, 1^{st} teaching block, 2025

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Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 13^{th} November. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Make sure you find all 10 $^\bullet$'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

5.1 *Bienaymé-Galton-Watson Branching Process.* Let $\xi_{n,k}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$ be iid. non-negative integer random variables with finite mean μ and variance σ^2. Define the Bienaymé-Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k},$$

and let $\mathcal{F}_n := \sigma(Z_j \,:\, 0 \le j \le n)$, $n \ge 0$ be the natural filtration.

(a) Prove that $M_n := Z_n/\mu^n$, $n \ge 0$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(b) Prove that $\mathbb{E}(Z_{n+1}^2 \,|\, \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$.

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}} \dfrac{\mu^n - 1}{\mu - 1} M_n, & \text{if } \mu \ne 1, \\ M_n^2 - n\sigma^2 M_n, & \text{if } \mu = 1 \end{cases}$$

is also an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then M_n is bounded in $\mathcal{L}^2$, while if $\mu \le 1$ then $\lim_{n\to\infty} \mathbb{E}\, M_n^2 = \infty$.

5.2 *Gambler's ruin.* Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbb{P}\{X_i = 1\} = p = 1 - q = 1 - \mathbb{P}\{X_i = -1\}$. Fix also $0 < a < b$ integers, and

$$S_n := a + \sum_{k=1}^{n} X_k, \qquad T := \inf\{n \,:\, S_n = 0 \text{ or } S_n = b\}.$$

(We think about S_n as a gambler's money at time n; the gambler starts at a, and is either ruined ($S_T = 0$) or wins it all ($S_T = b$).)

(a) Show that $\mathbb{E}\, T < \infty$. *Hint: we had a lemma for this…*

(b) Show that both

$$M_n := S_n - n(p - q) \qquad \text{and} \qquad N_n := \begin{cases} S_n^2 - n, & \text{if } p = q = \dfrac{1}{2}, \\ \left(\dfrac{q}{p}\right)^{S_n}, & \text{if } p \ne q \end{cases}$$

are martingales w.r.t. the natural filtration.

(c) Calculate the ruin probability $\mathbb{P}\{S_T = 0\}$ and the expected duration $\mathbb{E}\,T$ of the game.

5.3 ●●●● *Extending Doob's Optional Stopping Theorem, 1.* Let $\tau \geq 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Show $\{\tau \geq k\} \in \mathcal{F}_{k-1}$.

(b) Based on the identity

$$|X_{\tau\wedge n} - X_0| = \Big|\sum_{k=1}^{n}(X_k - X_{k-1}) \cdot \mathbf{1}\{\tau \geq k\}\Big| \leq \sum_{k=1}^{\infty}|X_k - X_{k-1}| \cdot \mathbf{1}\{\tau \geq k\}$$

and the proof of Doob's Optional Stopping Theorem, part (iii), show the following: If X is a supermartingale for which there exists a $C \in \mathbb{R}$ with

$$\mathbb{E}(|X_k - X_{k-1}| \,\big|\, \mathcal{F}_{k-1}) \leq C \qquad \forall k > 0, \text{ a.s.},$$

then $\mathbb{E}\,X_\tau \leq \mathbb{E}\,X_0$. Of course we have equality in case X is a martingale.

5.4 ●●● *Wald's identities, 1. (Notice: In contrary to this problem, Ábrahám Wald invented these without the use of martingales.)* Let Y_1, Y_2, … be iid. random variables in $\mathcal{L}^1$, $\mu := \mathbb{E}\,Y_i$, and $\tau \geq 1$ a stopping time (w.r.t. the natural filtration), $\mathbb{E}\,\tau < \infty$. Let $S_n = \sum\limits_{i=1}^{n} Y_i$. Show that $\mathbb{E}\,S_\tau = \mu \cdot \mathbb{E}\,\tau$. *Hint: use a martingale and the previous problem.*

5.5 ●●● *First mark after 1 in a Uniform renewal process.* Let Y_1, Y_2, … be iid. Uniform(0, 1) random variables, let $S_n := \sum\limits_{i=1}^{n} Y_i$, and $\tau := \min\{n : S_n > 1\}$.

(a) Show that for any fixed $0 \leq z \leq 1$, $\mathbb{P}\{S_n \leq z\} = z^n/n!$ holds. *Be careful, this is not true for $z > 1$!*

(b) Find $\mathbb{E}\,\tau$. *Hint: τ is non-negative, therefore we can sum tail probabilities. These latter are in close connection with part (a).*

(c) Since τ is a stopping time, use Wald's identity to calculate $\mathbb{E}(S_\tau - 1)$, the residual time at 1 until the next mark in a Uniform renewal process.

5.6 *Extending Doob's Optional Stopping Theorem, 2.* Let $\tau \geq 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Prove that for any process $(M_n)_{n=0}^{\infty}$ with $M_0 = 0$,

(1)
$$M_{\tau\wedge n}^2 = \sum_{k=1}^{n}\big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2\sum_{1\leq i<j\leq n}(M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \geq j\},$$

(2)
$$M_\tau^2 = \sum_{k=1}^{\infty}\big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2\sum_{1\leq i<j<\infty}(M_i - M_{i-1}) \cdot (M_j - M_{j-1}) \cdot \mathbf{1}\{\tau \geq j\},$$

holds. (Notice that a.s. finitely many terms are non-zero in each of these sums.).

(b) Show that the first sum on the right hand-side of (1) converges monotonically to that on the right hand-side of (2).

(c) Let M be a martingale with $M_0 = 0$ and a $C \in \mathbb{R}$ such that

$$\left|M_k - M_{k-1}\right| \le C \qquad \forall k > 0.$$

From now on, let us assume that the stopping time τ is in $\mathcal{L}^2$. Show that the second sums on the right hand-sides of both (1) and (2) are mean zero. *Hint: Fubini's Theorem.*

(d) With the condition as in the previous part, conclude $\lim\limits_{n\to\infty} \mathbb{E}\, M_{\tau\wedge n}^2 = \mathbb{E}\, M_\tau^2$.

5.7 *Wald's identities, 2.* If Y_i is bounded and, additionally, $\mathbb{E}\,\tau^2 < \infty$ also holds, then with $\sigma^2 := \mathbb{V}\mathrm{ar}\, Y_i$ show $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \cdot \mathbb{E}\,\tau$. (*This one is often used in case* $\mu = 0$.) *Hint: find a martingale for* S_n^2*, and use the previous problem on your martingale from 5.4. Do it for* $\mu = 0$ *first.*