Homework set 5

Optional stopping, martingale convergence

Martingale Theory with Applications, 1st teaching block, 2026

School of Mathematics, University of Bristol

Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 12th November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Make sure you find all 10 •'s!

Use of AI: Category 2: Minimal - for example, using spell and grammar checkers to help identify mistakes but not rewrite chunks of text.

5.1 *Bienaymé-Galton-Watson Branching Process.* Let $\xi_{n,k}$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$ be iid. non-negative integer random variables with finite mean μ and variance σ^2. Define the Bienaymé-Galton-Watson branching process

$$Z_0 := 1, \qquad Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n+1,k},$$

and let $\mathcal{F}_n := \sigma(Z_j \,:\, 0 \le j \le n)$, $n \ge 0$ be the natural filtration.

(a) Prove that $M_n := Z_n/\mu^n$, $n \ge 0$ is an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(b) Prove that $\mathbb{E}(Z_{n+1}^2 \,|\, \mathcal{F}_n) = \mu^2 Z_n^2 + \sigma^2 Z_n$.

(c) Using the result from (b) prove that

$$N_n := \begin{cases} M_n^2 - \dfrac{\sigma^2}{\mu^{n+1}} \dfrac{\mu^n - 1}{\mu - 1} M_n, & \text{if } \mu \ne 1, \\ M_n^2 - n\sigma^2 M_n, & \text{if } \mu = 1 \end{cases}$$

is also an $(\mathcal{F}_n)_{n\ge 0}$-martingale.

(d) Using the result from (c) prove that if $\mu > 1$ then M_n is bounded in $\mathcal{L}^2$, while if $\mu \le 1$ then $\lim_{n\to\infty} \mathbb{E}\, M_n^2 = \infty$.

5.2 *Gambler's ruin.* Let $X_1, X_2, \ldots$ be iid. random variables with $\mathbb{P}\{X_i = 1\} = p = 1 - q = 1 - \mathbb{P}\{X_i = -1\}$. Fix also $0 < a < b$ integers, and

$$S_n := a + \sum_{k=1}^{n} X_k, \qquad T := \inf\{n \,:\, S_n = 0 \text{ or } S_n = b\}.$$

(We think about S_n as a gambler's money at time n; the gambler starts at a, and is either ruined ($S_T = 0$) or wins it all ($S_T = b$).)

(a) Show that $\mathbb{E}\, T < \infty$. *Hint: we had a lemma for this...*

(b) Show that both

$$M_n := S_n - n(p - q) \qquad \text{and} \qquad N_n := \begin{cases} S_n^2 - n, & \text{if } p = q = \dfrac{1}{2}, \\ \left(\dfrac{q}{p}\right)^{S_n}, & \text{if } p \ne q \end{cases}$$

are martingales w.r.t. the natural filtration.

(c) Calculate the ruin probability $\mathbb{P}\{S_T = 0\}$ and the expected duration $\mathbb{E}\,T$ of the game.

5.3 •••• *Extending Doob's Optional Stopping Theorem, 1.* Let $\tau \geq 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Show $\{\tau \geq k\} \in \mathcal{F}_{k-1}$.

(b) Based on the identity

$$|X_{\tau\wedge n} - X_0| = \Big|\sum_{k=1}^{n}(X_k - X_{k-1})\cdot \mathbf{1}\{\tau \geq k\}\Big| \leq \sum_{k=1}^{\infty}|X_k - X_{k-1}|\cdot \mathbf{1}\{\tau \geq k\}$$

and the proof of Doob's Optional Stopping Theorem, part (iii), show the following: If X is a supermartingale for which there exists a $C \in \mathbb{R}$ with

$$\mathbb{E}(|X_k - X_{k-1}|\,\big|\,\mathcal{F}_{k-1}) \leq C \qquad \forall k > 0, \text{ a.s.},$$

then $\mathbb{E}\,X_\tau \leq \mathbb{E}\,X_0$. Of course we have equality in case X is a martingale.

5.4 ••• *Wald's identities, 1. (Notice: In contrary to this problem, Ábrahám Wald invented these without the use of martingales.)* Let $Y_1, Y_2, \ldots$ be iid. random variables in $\mathcal{L}^1$, $\mu := \mathbb{E}\,Y_i$, and $\tau \geq 1$ a stopping time (w.r.t. the natural filtration), $\mathbb{E}\,\tau < \infty$. Let $S_n = \sum_{i=1}^{n} Y_i$. Show that $\mathbb{E}\,S_\tau = \mu \cdot \mathbb{E}\,\tau$. *Hint: use a martingale and the previous problem.*

5.5 ••• *First mark after 1 in a Uniform renewal process.* Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables, let $S_n := \sum_{i=1}^{n} Y_i$, and $\tau := \min\{n : S_n > 1\}$.

(a) Show that for any fixed $0 \leq z \leq 1$, $\mathbb{P}\{S_n \leq z\} = z^n/n!$ holds. *Be careful, this is not true for $z > 1$!*

(b) Find $\mathbb{E}\,\tau$. *Hint: τ is non-negative, therefore we can sum tail probabilities. These latter are in close connection with part (a).*

(c) Since τ is a stopping time, use Wald's identity to calculate $\mathbb{E}(S_\tau - 1)$, the residual time at 1 until the next mark in a Uniform renewal process.

5.6 *Extending Doob's Optional Stopping Theorem, 2.* Let $\tau \geq 0$ be a stopping time, $\mathbb{E}\,\tau < \infty$.

(a) Prove that for any process $(M_n)_{n=0}^{\infty}$ with $M_0 = 0$,

(1)

$$M_{\tau\wedge n}^2 = \sum_{k=1}^{n}\big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2\sum_{1\leq i<j\leq n}(M_i - M_{i-1})\cdot(M_j - M_{j-1})\cdot\mathbf{1}\{\tau \geq j\},$$

(2)

$$M_\tau^2 = \sum_{k=1}^{\infty}\big(M_k - M_{k-1}\big)^2 \cdot \mathbf{1}\{\tau \geq k\} + 2\sum_{1\leq i<j<\infty}(M_i - M_{i-1})\cdot(M_j - M_{j-1})\cdot\mathbf{1}\{\tau \geq j\},$$

holds. (Notice that a.s. finitely many terms are non-zero in each of these sums.).

(b) Show that the first sum on the right hand-side of (1) converges monotonically to that on the right hand-side of (2).

(c) Let M be a martingale with $M_0 = 0$ and a $C \in \mathbb{R}$ such that

$$\left|M_k - M_{k-1}\right| \leq C \qquad \forall k > 0.$$

From now on, let us assume that the stopping time τ is in $\mathcal{L}^2$. Show that the second sums on the right hand-sides of both (1) and (2) are mean zero. *Hint: Fubini's Theorem.*

(d) With the condition as in the previous part, conclude $\lim_{n\to\infty} \mathbb{E}\, M_{\tau\wedge n}^2 = \mathbb{E}\, M_\tau^2$.

5.7 *Wald's identities, 2.* If Y_i is bounded and, additionally, $\mathbb{E}\,\tau^2 < \infty$ also holds, then with $\sigma^2 := \mathbb{V}\mathrm{ar}\, Y_i$ show $\mathbb{E}(S_\tau - \mu\tau)^2 = \sigma^2 \cdot \mathbb{E}\,\tau$. (*This one is often used in case* $\mu = 0$.) *Hint: find a martingale for* S_n^2*, and use the previous problem on your martingale from 5.4. Do it for* $\mu = 0$ *first.*