HOMEWORK SET 6

Renewal processes

Further Topics in Probability, 2nd teaching block, 2015

School of Mathematics, University of Bristol

Problems with $^\bullet$'s are to be handed in. These are due in class or in the blue locker with my name in the Main Maths Building before 16:00 on Friday, 8th May. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Random variables are defined on a common probability space unless otherwise stated.

6.1 $^{\bullet\bullet}$ A city allows for two-hour parking in all downtown spaces. Methodical parking officials patrol the downtown area, passing the same point every two hours. When an official encounters a car, he marks it with a chalk. If the car is still there two hours later, a ticket is written. Suppose that you park your car at a random moment for a random amount of time that is uniformly distributed in $(0, 4)$ hours. What is the probability that you will get a ticket?

6.2 $^{\bullet\bullet\bullet\bullet}$ (*Resnick.*) Find the renewal function $U(t) = \mathbf{E}N_t$ corresponding to

a) $F'(x) = \lambda \mathrm{e}^{-\lambda x}, \quad \lambda > 0, x > 0$. (This is the Poisson process.)

b) $F'(x) = \lambda^2 x \mathrm{e}^{-\lambda x}, \quad \lambda > 0, x > 0$. (This is a Gamma density.)

In both examples, verify $\lim_{t\to\infty} U(t)/t = 1/\mu$ directly. *HINT for b): What is the Taylor series of* $\mathrm{e}^y \pm \mathrm{e}^{-y}$*?*

6.3 (*Durrett.*) Let $A_t = t - T_{N(t)-1}$ (age) and $B_t = T_{N(t)} - t$ (residual time) in a renewal process.

a) $^{\bullet\bullet\bullet}$ Show that for every $x, y > 0$,

$$\mathbf{P}\{A_t > x,\ B_t > y\} \overset{t\to\infty}{\longrightarrow} \frac{1}{\mu}\int_{x+y}^{\infty} (1 - F(t))\,\mathrm{d}t.$$

b) $^\bullet$ Recall from class that A_t and B_t have the same limit distribution

$$\lim_{t\to\infty} \mathbf{P}\{A_t > x\} = \lim_{t\to\infty} \mathbf{P}\{B_t > x\} = \frac{1}{\mu}\int_{x}^{\infty} (1 - F(t))\,\mathrm{d}t.$$

In which case are A_t and B_t asymptotically independent? *HINT: For a non-increasing nonnegative function* g, $g(x + y) = g(x) \cdot g(y)\ \forall x, y$ *implies it's exponential.*

6.4 (*Durrett.*) Let $X_1, X_2, \ldots$ be iid. with distribution F_X, $Y_1, Y_2, \ldots$ be iid. and independent of the X_i's with distribution F_Y. Let $T_0 = 0$ and for $k \ge 1$ let $S_k = T_{k-1} + X_k$ and $T_k = S_k + Y_k$. In words, we have a machine that works for an amount of time X_k, breaks down, and then requires Y_k units of time to be repaired. Let $H(t)$ be the probability the machine is working at time t.

a) $^\bullet$ Show that if $F_X * F_Y$ is nonarithmetic, then as $t \to \infty$

$$H(t) \to \frac{\mu_X}{\mu_X + \mu_Y},$$

where $\mu_X = \mathbf{E}X_1$ and $\mu_Y = \mathbf{E}Y_1$.

b) • Show that in a renewal process

$$\mathbf{P}\{\text{the number of renewals in } [0,\, t] \text{ is odd}\} \to \frac{1}{2}.$$

HINT: This is a special case of a).

c) •• Let now X_i's and Y_i's each have the Exponential(λ) distribution. Compute $H(t)$ directly for finite $t > 0$. *HINT: From a previous problem we know that in a renewal process with Gamma(2, λ) interarrival distribution we have $U(t) = 3/4 + \lambda t/2 + \mathrm{e}^{-2\lambda t}/4$. Be very careful with $dU(0)$!*

6.5 ••• (*Resnick.*) In a renewal process, find the limit of

$$H(t) = \mathbf{P}\Big\{\bigcup_{n=0}^{\infty}\{T_n \in (t,\, t+x]\}\Big\}$$

as $t \to \infty$ with a fixed $x > 0$. Assume all regularity that you need. What is $H(t)$ in words?

6.6 Let $Y_1, Y_2, \ldots$ be iid. Uniform(0, 1) random variables. Define $S_n := \sum_{i=1}^{n} Y_i$ $(n \geq 1)$, and $M := \min\{n : S_n > 1\}$.

a) Fix $0 \leq z \leq 1$. Prove inductively that $\mathbf{P}(S_n < z) = z^n/n!$. (This is not true for $z > 1$!)

b) Rewrite $\mathbf{E}M$ in terms of the tail probabilities $\mathbf{P}(M > n)$. Then rewrite these in terms of S_n-probabilities and find $\mathbf{E}M$.

c) Since M is a stopping time, use Wald's identity: $\mathbf{E}S_M = \mathbf{E}M \cdot \mathbf{E}Y_1$ to find $\mathbf{E}(S_M - 1)$, the expected time to wait for the first renewal after 1.