Homework set 7

Doob's decomposition, uniformly integrable martingales

Martingale Theory with Applications, 1st teaching block, 2025

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Problems with •'s are to be handed in. These are due in Blackboard before noon on Thursday, 27th November. Please show your work leading to the result, not only the result. Each problem worth the number of •'s you see right next to it. Make sure you find all 10 •'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

7.1 ••• Let $\Gamma_0 = 0$, and $\Gamma_1, \Gamma_2, \dots$ be the marks of a rate λ homogeneous Poisson process on $\mathbb{R}^+$. Fix $\beta < \lambda$ and define $X_n := \mathrm{e}^{\beta\Gamma_n}$.

(a) Find a predictable process $B_n > 0$ for which $X_n = M_n \cdot B_n$ where M_n is a martingale. (*Hint: while X_n does not have the bounds assumed in Problem 6.8, the calculation still works. Check that the martingale is in $\mathcal{L}^1$ separately.*)

(b) Fix $t > 0$ and $T := \inf\{n : \Gamma_n > t\}$. Assume $\beta > 0$ and use the memoryless property of the Exponential distribution to bound the stopped martingale M^T from above by a random variable of finite mean. Hence show that optional stopping applies.

(c) Show from the above that the number of marks before t is Poisson($\lambda \cdot t$) distributed (just by assuming the i.i.d. Exponential(λ) interarrival times in the Poisson(λ) process). (*Hint: use its generating function.*)

7.2 Let Y_i, $i = 1, 2, \dots$ be i.i.d. variables in $\mathcal{L}^2$ with mean μ and variance σ^2. Define $X_0 = 0$, $X_k := \frac{1}{k}\sum_{i=1}^{k} Y_i$ $(k \geq 1)$.

(a) Perform Doob's decomposition on X.

(b) Show that the martingale M you found above is $\mathcal{L}^2$. Find its brackets process $\langle M \rangle$ and the Doob decomposition of M^2.

(c) Show that the martingale M converges a.s.

7.3 *Beta function.* Prove that for any $a, b \geq 0$ integers,

$$I_{a,b} := \int_0^1 \theta^a (1-\theta)^b \,\mathrm{d}\theta = \frac{a! \cdot b!}{(a+b+1)!}.$$

Hint: show via integration by parts that for $b \geq 1$, $I_{a,b} = \frac{b}{a+1} \cdot I_{a+1,b-1}$, while the case $b = 0$ is easy. From here, a recursive argument does the trick.

7.4 *Bayes urn.* Assume we have a *randomly biased* coin that shows HEAD with probability θ and TAIL with probability $1 - \theta$. This parameter θ is random and has the Uniform(0, 1) distribution. We flip this coin repeatedly and record

$$B_0 := 1, \qquad B_n := 1 + \text{no. of HEADs in the first } n \text{ trials},$$
$$R_0 := 1, \qquad R_n := 1 + \text{no. of TAILs in the first } n \text{ trials}.$$

Notice that $B_n + R_n = n + 2$. Define the filtration generated by the first n flips, $\mathcal{F}_n = \sigma(B_1, B_2, \dots, B_n)$, and mind that θ is *not* included in here.

(a) •• Determine the probability of a given sequence of flips,

$$\mathbb{P}\{B_1 = b_1,\, B_2 = b_2,\, \dots,\, B_n = b_n\}.$$

Hint: Condition on θ and use Problem 7.3.

(b) •• Based on the previous part, find the distribution of B_{n+1}, given $\mathcal{F}_n$. Compare with the Pólya urn. Remember: θ is not included in $\mathcal{F}_n$.

(c) • Show that, modulo zero measure sets, θ is $\mathcal{F}_\infty = \sigma\Big(\bigcup_{n=1}^\infty \mathcal{F}_n\Big)$-measurable.

(d) •• What is the conditional expectation of θ, given the first n flips? Explain. *Hint: the Pólya urn, and our theorem on uniformly integrable martingales...*

(e) *No marks for this, only for pride, as you might not have met conditional densities before.* Use the Bayes urn to find the conditional density of M_∞, given $\mathcal{F}_n$ in the Pólya urn.

7.5 Let M be a uniformly integrable martingale in the filtration $(\mathcal{F}_n)_{n\geq 0}$ in the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$. Let $S \leq T$ a.s. be finite stopping times. We denote by $\mathcal{F}_T$ the collection of all events $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all n, which can be thought of as the set of events whose occurrence or non-occurrence is known by time T.

(a) Prove that $\mathcal{F}_T$ is a σ-algebra.

(b) Prove that $M_T = \mathbb{E}(M_\infty \,|\, \mathcal{F}_T)$ and that $M_S = \mathbb{E}(M_T \,|\, \mathcal{F}_S)$. *Hint: observe that $\mathcal{F}_T$ is generated by sets $A \cap \{T = n\}$ where $A \in \mathcal{F}$ and $n \in \mathbb{Z}^+$.*

7.6 Let $Y_0,\, Y_1,\, Y_2,\, \dots$ be independent random variables with

$$\mathbb{P}\{Y_n = 1\} = \mathbb{P}\{Y_n = -1\} = \frac{1}{2}, \qquad \forall n.$$

For $n \geq 1$, define

$$X_n := Y_0 Y_1 \cdots Y_n.$$

Prove that the variables $X_1,\, X_2,\, \dots$ are independent. Define

$$\mathcal{Y} := \sigma(Y_1,\, Y_2,\, \dots), \qquad \mathcal{T}_n := \sigma(X_{n+1},\, X_{n+2},\, \dots).$$

Prove that

$$\mathcal{L} := \bigcap_n \sigma(\mathcal{Y},\, \mathcal{T}_n) \neq \sigma\Big(\mathcal{Y},\, \bigcap_n \mathcal{T}_n\Big) =: \mathcal{R}.$$

Hint: Prove that Y_0 is $\mathcal{L}$-measurable but independent of $\mathcal{R}$.