Homework set 8

Doob's submartingale inequality

Martingale Theory with Applications, 1^{st} teaching block, 2025

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Problems with $^\bullet$'s are to be handed in. These are due in Blackboard before noon on Thursday, 4^{th} December. Please show your work leading to the result, not only the result. Each problem worth the number of $^\bullet$'s you see right next to it. Make sure you find all 10 $^\bullet$'s!

Use of AI: Minimal - You may only use tools such as spelling and grammar checkers in this assignment, and their use should be limited to corrections of your own work rather than substantial re-writes or extended contributions.

8.1 (*Following Edward Crane's ideas; hard:*) Given are countably infinitely many gamblers, called $i = 1, 2, \ldots$, who respectively start with a fixed $X_0^{(i)} > 0$ (real) amount of money. After each day n, they have a non-negative $X_n^{(i)} \ge 0$ wealth. A gambler i goes *bankrupt* when they run out of money, $X_n^{(i)} = 0$, in which case they will never play again. Gamblers who are not bankrupt are called *active*. Each day $n > 0$ when there are at least two active gamblers, the first (in their index i) and second active gamblers pair up, as well as the third and fourth active gamblers pair up, also the fifth and sixth, etc. If there is an odd number of active gamblers then the one with the highest index will not play on day n.

Each paired up gamblers $i < j$ play the following game between each other: they bet the smaller of the two wealths they have, $\min(X_{n-1}^{(i)}, X_{n-1}^{(j)})$, on an independent and fair coinflip. If the coin comes up Head, gambler i pays this amount to gambler j, and if the coin comes up Tail then gambler j pays this amount to gambler i, this is how $X_n^{(i)}$ and $X_n^{(j)}$ are calculated. The game stops when there is at most one active gambler. Show that, a.s.:

- If the total amount in play is finite i.e., $\sum_i X_0^{(i)} < \infty$, then in the $n \to \infty$ time-limit exactly one gambler will have all the money.
- If the total amount in play is infinite i.e., $\sum_i X_0^{(i)} = \infty$, then in the time-limit either each gambler went bankrupt, or only one gambler has a finite amount of money. I.e., infinite amount of wealth is lost off to infinity.

8.2 N people queue for a concert the ticket for which costs £1. Each person, independently and with equal chance, has a £1 coin or a £2 coin so these customers need £1 change. The cashier starts selling tickets with a number m of £1 coins in reserve, and we are interested in how this number changes over time.

(a) Find a natural martingale M_n for the problem.

(b) Use M_n^2 to give a bound on the probability that the cashier ever runs out of coins.

(c) Use an exponential of M_n to bound the same. Make your bound as strong as possible.

8.3 $^{\bullet\bullet\bullet}$ $2N$ people queue for a concert the ticket for which costs £1. Exactly N of the queuing people have a £1 coin each and N of them have a £2 coin so these customers need £1 change. The problem is that the queue is in a uniformly random order, hence the cashier starts selling tickets with a number m of £1 coins in reserve. Find a natural martingale for the problem and use Doob's submartingale inequality on its square to give a bound on the probability that the cashier ever runs out of coins. *Hint: Use Problem 4.8. Deal with the first N customers only, then use the symmetry of the problem.*

8.4 *Azuma-Hoeffding concentration inequality.*

(a) Let $c > 0$, and $-c \le Y \le c$ a mean zero random variable. Then for any $\theta \in \mathbb{R}$ we have

$$\mathbf{E}e^{\theta Y} \le \cosh(\theta c) \le e^{\theta^2 c^2/2}.$$

Hint: for any convex function g and $-c \le y \le c$,

$$g(y) \le \frac{c-y}{2c} \cdot g(-c) + \frac{c+y}{2c} \cdot g(c).$$

$e^{\theta \cdot}$ *is a convex function.*

(b) Let M be a martingale with $M_0 = 0$, and assume $|M_n - M_{n-1}| \le c_n$, $\forall n$ with a deterministic sequence $\{c_n\}_{n \in \mathbb{N}}$. Then for any $x > 0$

$$\mathbf{P}\Big\{\sup_{k \le n} M_k \ge x\Big\} \le e^{-x^2/(2\sum_{k=1}^{n} c_k^2)}.$$

Hint: apply the above and Doob's submartingale inequality, then optimise in θ.

8.5 ••• Apply the Azuma-Hoeffding inequality to bound the probability that the cashier of Problem 8.2 ever runs out of coins.

8.6 Apply the Azuma-Hoeffding inequality to bound the probability that the cashier of Problem 8.3 ever runs out of coins.

8.7 Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra. Show that given $\mathcal{G}$, $\mathbb{E}(X \mid \mathcal{G})$ is the best predictor of X in the following sense: the minimum mean square error $\mathbb{E}(V - X)^2$ among $\mathcal{G}$-measurable random variables V is achieved for $V = \mathbb{E}(X \mid \mathcal{G})$. What is this minimal mean square error? *Hint: use a tower rule first, then minimise pointwise among $\mathcal{G}$-measurable functions.*

8.8 We are given n many intervals of i.i.d. Uniform$(0, 1)$ lengths that need to be packed into "boxes" that is, intervals, of length 1. Let B_n be the minimum number of boxes needed to do that. Apply the Azuma-Hoeffding inequality to bound the deviation between our best estimates after observing the first i Uniforms and the mean of B_n.

8.9 •••• Given are N balls and K, initially empty, urns. We place the balls, one by one, into the urns without removing them. Each ball independently goes to a uniformly chosen urn from 1 to K. These choices are denoted by $X_1, X_2, \ldots, X_N$, which are therefore i.i.d. discrete uniform on the set $\{1, 2, \ldots, K\}$. The generated filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for $n = 0, 1, \ldots, N$. Denote by Z the number of empty urns when all N balls have been placed, and Z_n the number of empty urns after the n^{th} step.

(a) Calculate the best prediction martingale (Problem 8.7) $M_n = \mathbb{E}(Z \mid \mathcal{F}_n)$, $(n = 0, 1, \ldots, N)$ explicitly, and show its martingale property via direct computation based on your explicit form. *Hint: use indicators for urns to stay empty.*

(b) What is M_0 and what is M_N?

(c) Find $\mathbb{E}\, Z_n$ $(0 \le n \le N)$ and $\mathbb{E}\, Z$.

(d) Apply the Azuma-Hoeffding inequality to bound the deviation between our best estimate for Z after observing the first n balls and the mean of Z.

8.10 The *Erdős-Rényi random graph* on n vertices is a random subset of $\binom{n}{2}$ possible edges between the vertices, where each edge is independently present with probability p. The *chromatic number* χ of a graph is the minimum number of colours for the vertices needed to avoid the same colour of any two vertices that are adjacent in the graph (i.e., connected by an edge). Let $\mathcal{F}_k$ be the sigma-algebra generated by the presence or absence of all edges among the first k vertices of the Erdős-Rényi graph, $k = 0 \ldots n$. Apply the Azuma-Hoeffding inequality with this filtration on the chromatic number of this graph.

8.11 A monkey repeatedly types any of the 26 letters of the English alphabet independently with equal chance, until a total of N letters are typed. Let X be the number of times the word "ABRACADABRA" appears. Overlaps are acceptable e.g., we have it three times in EABRACADABRACADABRABRACADABRAX. Show that for any $x > 0$,

$$\mathbb{P}\Big\{\Big|X - (N-10)\cdot\frac{1}{26^{11}}\Big| \geq x\Big\} \leq 2\mathrm{e}^{-x^2/8N}.$$