Inclusion-exclusion principle

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October 13, 2014

This little write-up is part of important foundations of probability that were left out of the unit Probability 1 due to lack of time and prerequisites. Here we prove the general (probabilistic) version of the inclusion-exclusion principle. Many other elementary statements about probability have been included in Probability 1. Notice that the inclusion-exclusion principle has various formulations including those for counting in combinatorics.

We start with the version for two events:

Proposition 1 (inclusion-exclusion principle for two events) *For any events* $E, F \in \mathcal{F}$

$$\mathbf{P}\{E \cup F\} = \mathbf{P}\{E\} + \mathbf{P}\{F\} - \mathbf{P}\{E \cap F\}.$$

Proof. We make use of the simple observation that E and $F - E$ are exclusive events, and their union is $E \cup F$:

$$\mathbf{P}\{E \cup F\} = \mathbf{P}\{E \cup (F - E)\} = \mathbf{P}\{E\} + \mathbf{P}\{F - E\}.$$

On the other hand, $F - E$ and $F \cap E$ are also exclusive events with union equal to F:

$$\mathbf{P}\{F\} = \mathbf{P}\{(F - E) \cup (F \cap E)\} = \mathbf{P}\{F - E\} + \mathbf{P}\{F \cap E\}.$$

The difference of the two equations gives the proof of the statement. □

Next, the general version for n events:

Theorem 2 (inclusion-exclusion principle) *Let* $E_1, E_2, \ldots, E_n$ *be any events. Then*

$$\begin{aligned}&\mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_n\}\\ &= \sum_{1\le i\le n} \mathbf{P}\{E_i\} - \sum_{1\le i_1<i_2\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2}\} + \sum_{1\le i_1<i_2<i_3\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap E_{i_3}\} - \cdots + (-1)^{n+1}\mathbf{P}\{E_1 \cap E_2 \cap \cdots \cap E_n\}.\end{aligned}$$

Intuitively, summing the probabilities we double-count all the two-intersections. Those we subtract with the second sum. (Observe that every two-intersection is contained exactly once in $\{E_{i_1} \cap E_{i_2} : 1 \le i_1 < i_2 \le n\}$.) Unfortunately, with this move we have now counted all three-intersections three times, then subtracted them three times, hence we have to add them back once. But then we run into trouble with four-intersections, etc.

When our state space is countable then counting arguments give a direct proof of the formula. This can also be extended to the general case. Here we give a different proof.

Proof. We argue inductively. The proof for $n = 2$ is seen above. Suppose that the formula is true for n, we show it for $n + 1$. First apply the $n = 2$ case, then distributivity of intersections:

$$\begin{aligned}&\mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_n \cup E_{n+1}\}\\ &\qquad= \mathbf{P}\{(E_1 \cup E_2 \cup \cdots \cup E_n) \cup E_{n+1}\}\\ &\qquad= \mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_n\} + \mathbf{P}\{E_{n+1}\} - \mathbf{P}\{(E_1 \cup E_2 \cup \cdots \cup E_n) \cap E_{n+1}\}\\ &\qquad= \mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_n\} + \mathbf{P}\{E_{n+1}\} - \mathbf{P}\{(E_1 \cap E_{n+1}) \cup (E_2 \cap E_{n+1}) \cup \cdots \cup (E_n \cap E_{n+1})\}.\end{aligned}$$

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The first and the last terms are n-unions, for which we assumed the formula to hold. Therefore

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\begin{aligned}
\mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_n \cup E_{n+1}\} = & \sum_{1\le i\le n} \mathbf{P}\{E_i\} && (1)\\
& - \sum_{1\le i_1<i_2\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2}\} && (2)\\
& + \sum_{1\le i_1<i_2<i_3\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap E_{i_3}\} && (3)\\
& - \cdots + (-1)^{n+1}\mathbf{P}\{E_1 \cap E_2 \cap \cdots \cap E_n\} && (4)\\
& + \mathbf{P}\{E_{n+1}\} && (5)\\
& - \sum_{1\le i\le n} \mathbf{P}\{E_i \cap E_{n+1}\} && (6)\\
& + \sum_{1\le i_1<i_2\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap E_{n+1}\} && (7)\\
& - \cdots - (-1)^n \sum_{1\le i_1<i_2<\cdots<i_{n-1}\le n} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap \cdots \cap Ei_{n-1} \cap E_{n+1}\} && (8)\\
& - (-1)^{n+1}\mathbf{P}\{E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1}\}
\end{aligned}
$$

Here (1) and (5) account for all the probabilities of single events from 1 to $n+1$. (2) includes all the two-intersection probabilities from 1 to n, and (6) all the two-intersection probabilities where the higher index equals $n+1$. These two sums thus account for all possible two-intersection probabilities from 1 to $n+1$. Similarly, (3) includes all three-intersection probabilities from 1 to n, and (7) those with highest index equal to $n+1$. Together they include all three-intersection probabilities from 1 to $n+1$. This continues until (4) and (8), which together give all n-intersection probabilities from 1 to $n+1$. Finally, we write down the last term, and

$$
\begin{aligned}
&\mathbf{P}\{E_1 \cup E_2 \cup \cdots \cup E_{n+1}\} = \\
&\quad = \sum_{1\le i\le n+1} \mathbf{P}\{E_i\} - \sum_{1\le i_1<i_2\le n+1} \mathbf{P}\{E_{i_1} \cap E_{i_2}\} + \sum_{1\le i_1<i_2<i_3\le n+1} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap E_{i_3}\} \\
&\quad\quad - \cdots + (-1)^{n+1} \sum_{1\le i_1<i_2<\cdots<i_n\le n+1} \mathbf{P}\{E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_n}\} + (-1)^{n+2}\mathbf{P}\{E_1 \cap E_2 \cap \cdots \cap E_{n+1}\},
\end{aligned}
$$

which justifies the formula for $n+1$. □

Corollary 3 *The right hand-side of the inclusion-exclusion formula alternates in the sense that the first sum is greater than or equal to the probability of the union on the left hand-side. The difference of the first two sums is smaller than or equal to the left hand-side. The first three sums together with their signs are larger than or equal, etc.*

Proof. This statement can be followed in an inductive fashion along the proof. □