Overview of universality phenomena for random matrices
Workhorse: Multi-resolvent local laws

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The question that mathematicians failed to ask

What can be said about the statistical properties of the eigenvalues of a large random matrix?

Do some universal patterns emerge?

Eugene Wigner (1954)

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \implies (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

N = size of the matrix, will go to infinity.

WHY? Matrices are the Hamilton operators of quantum systems!

Wigner's vision: energy levels of large quantum systems exhibit a universal behavior, i.e. all the details are irrelevant, only basic physical symmetries matter.

Main questions we look at, first in a toy setup

Consider independent identically distributed (i.i.d.) random variables $X_1, \dots, X_N$, such that $\mathbf{E}X_1 = 0, \mathbf{E}X_1^2 = 1.$

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- Extremal statistics: There exist sequences a_N, b_N such that

$$b_N \left[\mathrm{Max}(X_1, \ldots, X_N) - a_N\right] \Rightarrow G,$$

where G is a Gumbel random variable with distribution function $F_G(x) = e^{-e^{-x}}$.

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For example, if $X_i \sim \mathcal{N}(0,1)$, then
$$a_N = \sqrt{2\log N} - \frac{\log\log N + \log 4\pi}{2\sqrt{2\log N}}, \qquad b_N = \sqrt{2\log N}.$$

Now we look at these questions in the eigenvalues/vectors of random matrices.
First, for the much more studied Hermitian case, then a bit non-Hermitian theory.

A brief history of classical results

Definition [Wigner matrix]: $N \times N$ Hermitian random matrix $H = H^*$

- Independent identically distributed (iid) entries up to Hermitian symmetry $h_{ab} = \overline{h}_{ba}$
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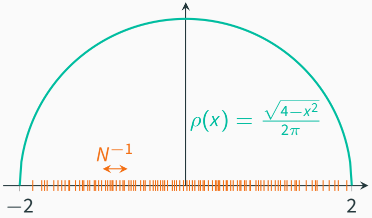

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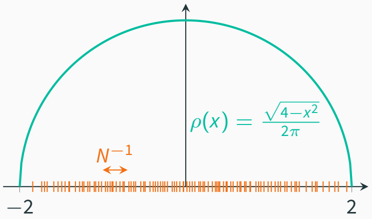

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"Law of Large Numbers"-type result on macro scale – insensitive to individual eigenvalues.

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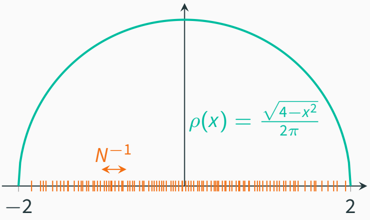
Wigner's revolutionary observation: the eigenvalue gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric), independent of the distribution of h_{ab}.



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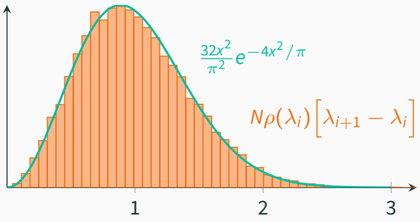

Histogram of rescaled bulk gaps and Wigner surmise

Formulated as the Wigner-Dyson-Mehta conjecture in 60's, proven around **2010** [**Erdős-Schlein-Yau, Tao-Vu**].

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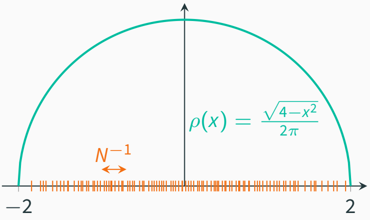

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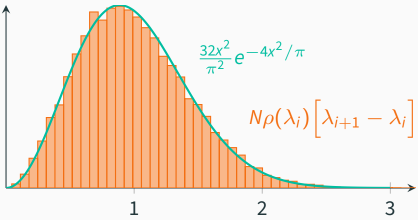

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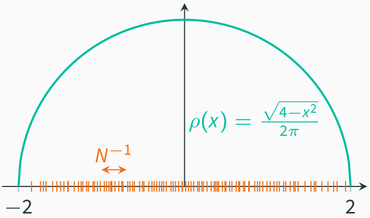

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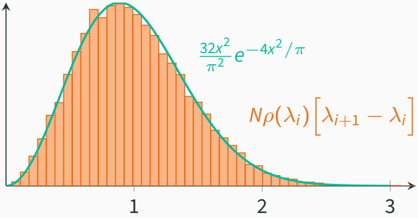

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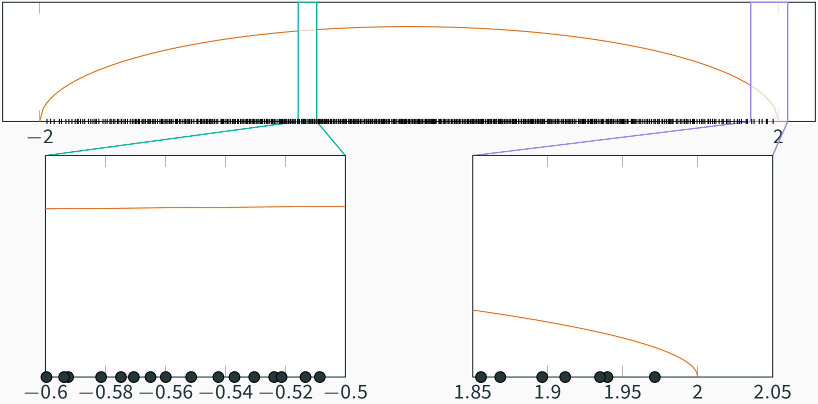
The distributions both in the bulk and at the edge are novel, the underlying point processes are strongly correlated (but determinantal) **[Dyson-Mehta, 1960's] [Tracy-Widom, 1993]**



Wigner-Dyson-Mehta on scale N^{-1}

Tracy-Widom on scale $N^{-2/3}$

Third universal statistics: Pearcey-statistic at cubic cusp singularities of $\rho(x) \sim |x|^{1/3}$.

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General belief: universality of local eigenvalue statistics holds for ”any sufficiently random” matrix (or even operator) in the delocalization regime in the sense of Anderson’s metal-insulator transition. In particular, it holds in mean-field systems.

Global density of eigenvalues is model specific, but cannot be “arbitrary”.

Matrix Dyson Equation and universality of singularities of the density

Most general (Hermitian) random matrix model: correlated (non-centred) entries.
Characterized by a data pair: expectation (matrix) A and correlation (tensor) $\mathcal{S}$:

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$$\rho(x) := \lim_{\eta\to 0+0} \frac{1}{\pi} \Im \langle M(x+i\eta)\rangle, \qquad \langle\cdot\rangle := \frac{1}{N}\operatorname{Tr}$$

where $M : \mathbf{C}_+ \to \mathbf{C}^{N\times N}$ is the unique solution to the *Matrix Dyson Equation (MDE)*

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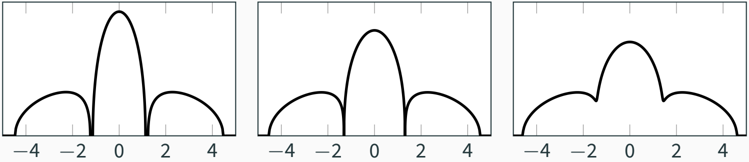
Theorem (Ajanki, Alt, E., Krüger 2016–2018)

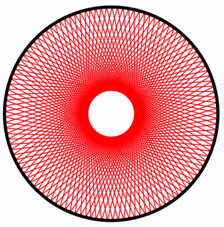
If $c\langle\cdot\rangle \le \mathcal{S} \le C\langle\cdot\rangle$ ("mean field"), then suppρ consists of finitely many intervals, ρ is real analytic in its interior and ρ has either square root singularity at the edges or cubic cusp singularity if two intervals touch. No other singularity can occur.



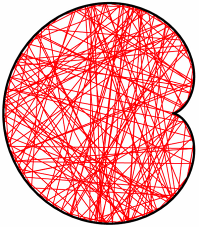
Eigenvectors? Quantum Unique Ergodicity (QUE)

Motto:

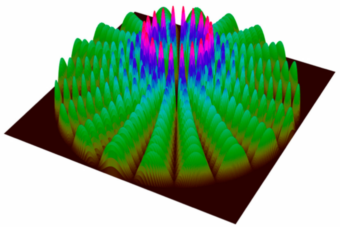
Eigenfunctions of the quantization of a chaotic classical dynamics are uniformly distributed.



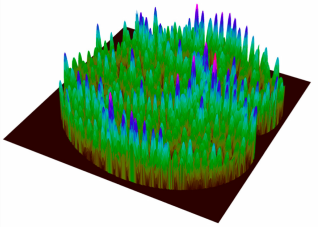
Regular (integrable) billiard



Chaotic billiard



Wavefunctions with symmetries



Chaotic wavefunctions

Universality for eigenvectors (Hermitian case)

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It also holds quite nontrivially for any Wigner matrix (universality), in particular:

- Asymptotic Gaussianity of entries: Finitely many $\sqrt{N}u_i(a_i)$ are jointly Gaussian; **[Bourgade-Yau-Yin, 2018], [Marcinek-Yau 2020]**

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  Random matrix version of Snirelman’s theorem with optimal fluctuation error. ETH also proven for Deformed Wigner and Wigner type matrices with a modified $\langle A \rangle$ **[Deutsch 1991][Feingold-Perez 1986] [Cipolloni-E-Henheik-Kolupaiev-Schröder, 2020–2023], [Benigni-Lopatto, 2021], [Benigni-Cipolloni, 2022], [E-Riabov, 2024]**
- Extremal statistics: $\max_i \langle \mathbf{u}_i, A\mathbf{u}_i \rangle$ (after rescaling) has Gumbel distribution if $\operatorname{rank}(A) \ll N^{1/2}$ **[E-McKenna 2023]**

Non-Hermitian case: i.i.d. random matrices

$$X = \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NN} \end{pmatrix}$$

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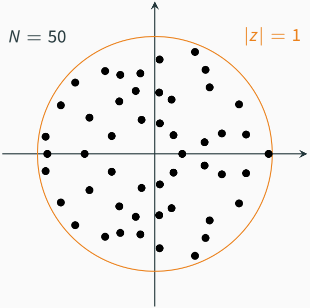

Figure 4: Real entries

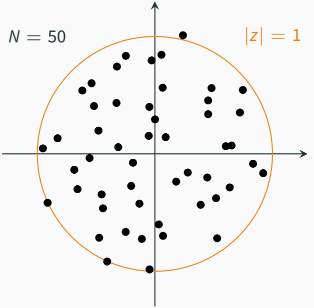

Figure 5: Complex entries with $\mathbf{E}\, x_{ab}^2 = 0$

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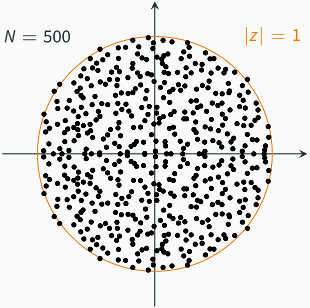

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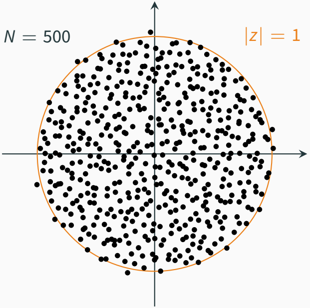

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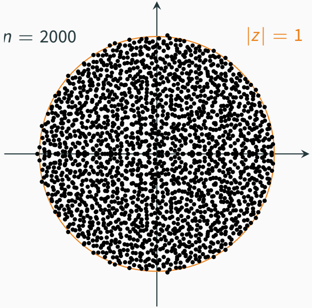

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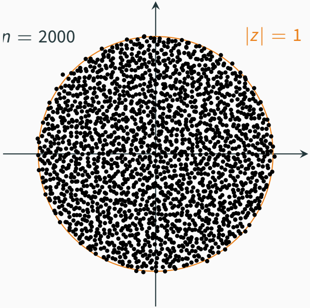

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Non-Hermitian case: i.i.d. random matrices

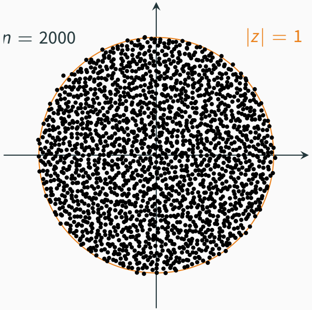
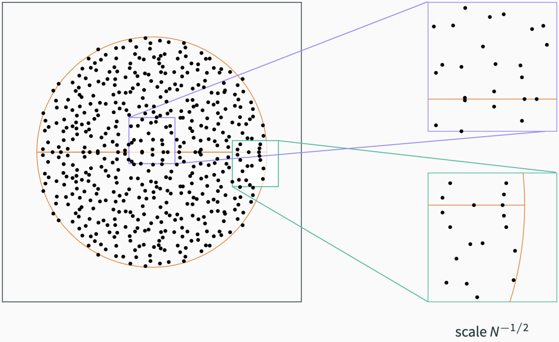


Figure 6: Complex entries with $\mathbf{E}\, x_{ab}^2 = 0$

- Circular law: Convergence to the uniform distribution on the unit disk: non-Hermitian analogue of Wigner's semicircle law.
- Eigenvalue spacing$\sim N^{-1/2}$.
- Accumulation of $\sim \sqrt{N}$ eigenvalues on the real axis for real matrices.

Universal phenomena similar to Hermitian matrices. After rescaling:



Spectral universality for i.i.d. matrices

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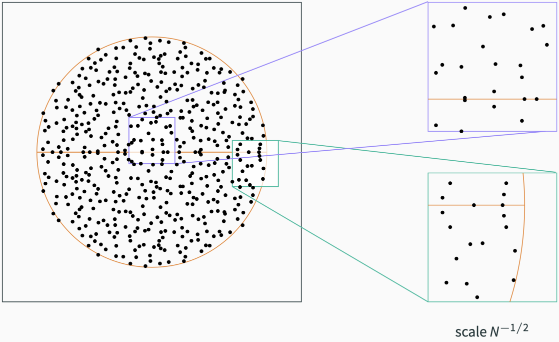

Edge universality: **[Cipolloni-E-Schröder (2019)]** (complex and real)
Similar result for deformed iid, $A + X$ (even in the “cusp” case when two domains touch)
[Liu-Zhang], [Campbell, Cipolloni, E., Ji] (2023-2024)

Bulk universality: **[Maltsev-Osman, 2023]** (complex), **[Osman, 2023]** (real)

State of the art for Hermitian vs. non-Hermitian random matrices

Hermitian prototype: GUE/GOE, more generally Wigner (h_{ab} are iid for $a \geq b$, $h_{ba} = \bar{h}_{ab}$).

- Spectrum is real, density of states is given by the semicircle law ($\surd$);
- Wigner-Dyson (bulk) and Tracy-Widom (edge) universality for eigenvalues ($\surd$);
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Non-Hermitian prototype: Ginibre matrix (Gaussian), more generally iid matrix (non-Gaussian): h_{ab} are i.i.d., no symmetry.

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General Pattern: Gaussian cases (GOE/GUE and Ginibre) are explicitly computable, then sophisticated techniques needed to show that the answer does not change if we change the distribution of h_{ab}. Well beyond any perturbation theory !! [Hermitian ev's live on scale $1/N$, changing only one h_{ab} is already a change of order $1/\sqrt{N}$ and there are N^2 of them!]

Basic scales and tools

Mesoscopic scale: Cumulative effect of $\gg 1$ eigenvalues.

Resolvents $G(z) = (H - z)^{-1}$ become deterministic if $\Im z \gg 1/N$ ("Local law"). Used as

- A priori bounds for bulk universality proofs;
- Directly for edge universality proofs;
- Green function comparison theorems: two ensembles with sufficiently many matching moments have the same local ev. statistics (originally **[Tao-Vu, 2009]**).

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Microscopic scale: Sensitive to individual eigenvalues

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- Supersymmetric formalism: Major reduction of variables **[Disertori-Pinson-Spencer 2002], [M. and T. Shcherbina 2011–]**
- Partial Schur decomposition : Explicit Haar/Gaussian calculations **[Edelman-Kostlan-Schub, 1994], [Fyodorov-Khoruzhenko, 2007], [Fyodorov, 2018] [Maltsev-Osman, 2023], [Osman 2023]**

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This week: Multi-resolvent local laws with a new zigzag strategy. This will have several applications beyond standard universality questions.

Applications 1: ETH

Let H be a Wigner matrix, $H\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ and A be a deterministic matrix.

$$\left|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle\right| \leq \frac{N^\xi}{\sqrt{N}} \qquad \text{w.v.h.p.} \tag{1}$$

Set $\mathring{A} := A - \langle A\rangle I$ to be the traceless part of A, we need to show to

$$\left|\langle \boldsymbol{u}_i, \mathring{A}\boldsymbol{u}_j\rangle\right|^2 \leq \frac{N^\xi}{N} \qquad \text{w.v.h.p.} \tag{2}$$

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Look at

$$\langle \Im G(z_1)\mathring{A}\Im G(z_2)\mathring{A}\rangle = \frac{1}{N}\sum_{a,b=1}^{N}\left|\langle \boldsymbol{u}_a, \mathring{A}\boldsymbol{u}_b\rangle\right|^2 \frac{\eta}{(\lambda_a-\gamma_i)^2+\eta^2}\frac{\eta}{(\lambda_b-\gamma_j)^2+\eta^2}, \qquad \eta \sim N^{-1+\xi}, \tag{3}$$

with $z_1 = \gamma_i + i\eta, \quad z_2 = \gamma_j + i\eta$.

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with $z_1 = \gamma_i + i\eta, \quad z_2 = \gamma_j + i\eta$.

Two-resolvent local law (with the $\sqrt{\eta}$ improvement due to the tracelessness of $\mathring{A}$) proves that $\langle \Im G(z_1)\mathring{A}\Im G(z_2)\mathring{A}\rangle \lesssim 1$ with very high probability, uniformly in z_1, z_2 with $\Im z_1, \Im z_2 \sim N^{-1+\xi}$, then (2) follows from (3).

Applications 2: Thermalisation

H is the Hamiltonian of a quantum system and $A, B, \ldots$ are deterministic observables. Let

$$A(t) = e^{-itH} A e^{itH}$$

be the Heisenberg (quantum) time evolution of A. How much $A(t)$ and B become orthogonal (independent) at large time?

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$$\langle A(t)B \rangle = \langle A \rangle \langle B \rangle + \theta(t)^2 \frac{\langle \mathring{A}\mathring{B} \rangle}{t^3} + O\Big(\frac{t^2}{N}\Big) \qquad \text{w.v.h.p.} \tag{4}$$

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Similar results can be derived for more than two observables, for example for three observables and two different times t, s with $t \geq s \gg 1$, $t - s \gg 1$ we have

$$\begin{aligned} \langle A(t)B(s)C \rangle = \langle A \rangle \langle B \rangle \langle C \rangle &+ \theta(s)^2 \frac{\langle A \rangle \langle \mathring{B}\mathring{C} \rangle}{s^3} + \theta(t)^2 \frac{\langle B \rangle \langle \mathring{A}\mathring{C} \rangle}{t^3} + \theta(t-s)^2 \frac{\langle C \rangle \langle \mathring{A}\mathring{B} \rangle}{(t-s)^3} \\ &+ \theta(s)\theta(t)\theta(t-s) \frac{\langle \mathring{A}\mathring{B}\mathring{C} \rangle}{s^{3/2} t^{3/2} (t-s)^{3/2}} + O\Big(\frac{t^3}{N}\Big) \qquad \text{w.v.h.p.} \end{aligned} \tag{5}$$

A related object is the *out-of-time-ordered correlator (OTOC)*

$$\mathcal{C}_{A,B}(t) := \frac{1}{2}\langle \left|[A(t), B]\right|^2 \rangle$$

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In all these problems, we use contour integral

$$e^{itH} = \frac{1}{2\pi i}\oint_\gamma \frac{e^{itz}}{H-z}dz = \frac{1}{2\pi i}\oint_\gamma e^{itz}G(z)dz$$

where γ encircles the spectrum of H. For example

$$\langle A(t)B\rangle = -\frac{1}{4\pi^2}\oint_\gamma\oint_\gamma e^{itz_1}e^{-itz_2}\langle G(z_1)AG(z_2)B\rangle dz_1 dz_2.$$

If we find a deterministic approximation $M = M(A, B, z_1, z_2)$ to $G(z_1)AG(z_2)B$, then we can compute the leading term by explicit contour integration.

Applications 3: Eigenvector overlaps (Hermitian case)

Consider a Wigner matrix W with two different deformations

$$H_1 = W + D_1, \qquad H_2 = W + D_2$$

where D_1, D_2 are deterministic (hermitian) matrices, $\langle D_i \rangle = 0$. Let

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$$\left|\langle \boldsymbol{u}_i^{(1)}, \boldsymbol{u}_j^{(2)} \rangle\right|^2 \lesssim \frac{N^\xi}{N} \frac{1}{\langle (D_1 - D_2)^2 \rangle + |\lambda_i^{(1)} - \lambda_j^{(2)}|^2 + \dots} \qquad \text{w.v.h.p.} \qquad (6)$$

This shows that eigenvectors can become decorrelated in two ways: either their energies are at distance or the deformations are far away in Hilbert-Schmidt norm sense.

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$$\left|\langle \boldsymbol{u}_i^{(1)}, \boldsymbol{u}_j^{(2)} \rangle\right|^2 \leq \eta \langle \Im G^{(1)}(\gamma_i^{(1)} + i\eta) \Im G^{(2)}(\gamma_j^{(2)} + i\eta) \rangle, \qquad \eta \sim N^{-1+\xi},$$

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The result (6) is essentially used in our papers on the *decorrelation transition* and the *Law of Fractional Logarithm* in the Wigner minor process.

Applications 4: Law of Fractional Logarithm

Let $(x_{ij})_{i,j\in\mathbf{N}}$, be a double infinite array of i.i.d. random variables with $x_{ij} = \bar{x}_{ji}$, $\mathbf{E}\, x_{ij} = 0$, $\mathbf{E}\, |x_{ij}|^2 = 1$ (and $\mathbf{E}\, x_{ij}^2 = 0$ in the complex case).

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Let $X^{(N)}$ be its $N \times N$ upper left minor and define

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Then, almost surely, we have

$$\liminf_{N\to\infty} \frac{\widetilde{\lambda}_1^{(N)}}{(\log N)^{1/3}} = -\Big(\frac{8}{\beta}\Big)^{1/3}, \quad \text{and} \quad \limsup_{N\to\infty} \frac{\widetilde{\lambda}_1^{(N)}}{(\log N)^{2/3}} = \Big(\frac{1}{2\beta}\Big)^{2/3},$$

(previous partial results for GUE by Paquette and Zeitouni, and Baslingker et.al.)

Applications 5: Gumbel distribution for the rightmost eigenvalue

Let X be an $N \times N$ complex i.i.d. random matrix, $\mathbf{E}\, x_{ij} = 0$, $\mathbf{E}\, |x_{ij}|^2 = \frac{1}{N}$

Let $\sigma_j, j = 1, 2, \ldots, N$ be the eigenvalues of X. We have the circular law:

$$\frac{1}{N} \sum_j f(\sigma_j) = \frac{1}{\pi} \int f(z) dz + O(N^\xi / N)$$

for a smooth N-independent test function f (there are also local versions). Also

$$\max |\sigma_j| \leq 1 + \frac{N^\xi}{\sqrt{N}}, \qquad \text{w.v.h.p.}$$

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Goal: identify more precisely the behavior of $\max_j \Re\sigma_j$.

One motivation for that is to study the standard ODE with random coefficients

$$\frac{d}{dt}\boldsymbol{v}(t) = -(I + gX)\boldsymbol{v}(t), \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0$$

Theorem [Cipolloni-E-Xu]

$$\sqrt{4\gamma_N N}\Big[\max_j \Re\sigma_j - 1 - \sqrt{\frac{\gamma_N}{4N}}\Big] \Longrightarrow G$$

where G is standard Gumbel random variable, i.e. $\mathbb{P}(G \le x) = \exp(-e^{-x})$ and

$$\gamma_N := \frac{1}{2}\Big[\log N - 5\log\log N - \log(2\pi^4)\Big].$$

There is a similar result (with slightly different γ_N) for $\max |\sigma_j|$, i.e. the spectral radius of X. We also have a statement that the few rightmost eigenvalues form a Poisson point process (with correct rescalings).

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How are these results related to local laws?

We look at linear statistics $\frac{1}{N}\sum_j f(\sigma_j)$ with a carefully chosen test function f

We use Girko's formula

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We need to compute, e.g. the second moment of this linear statistics, i.e.

$$\mathbf{E}\left|\frac{1}{N}\sum_j f(\sigma_j)\right|^2 = \frac{1}{16\pi^2}\iint\limits_{\mathbf{C}} \Delta f(z_1)\Delta f(z_2) \iint\limits_0^\infty \mathbf{E}\langle \Im G^{z_1}(i\eta_1)\rangle\langle \Im G^{z_2}(i\eta_2)\rangle d\eta_1 d\eta_2; dz_1 dz_2.$$

So we need to study the correlation of $\langle \Im G^{z_1}(i\eta_1)\rangle$ and $\langle \Im G^{z_2}(i\eta_2)\rangle$ for all regimes of η and this is given by a two resolvent local law $\langle \Im G^{z_1}(i\eta_1)\Im G^{z_2}(i\eta_2)\rangle$.

It is especially important to extract a decay in this correlation as $z_1 - z_2$ gets larger.