Exercises on the Directed Landscape

Based on Lectures and the Paper by Dauvergne, Ortmann, and Virág

Abstract

These exercises are based on lectures discussing the paper "The Directed Landscape" by Dauvergne, Ortmann, and Virág. They explore properties of the directed landscape using Tracy-Widom tail bounds, Brownian absolute continuity, and modulus of continuity bounds.

Definition (Tracy-Widom Tail Bounds)**.** Let TW denote a Tracy-Widom GUE random variable. There exist constants $d > 0$ such that for all $a > 0$,

$$\mathbb{P}(TW \leq -a) \leq \exp(1 - da^{3/2}), \quad \mathbb{P}(TW \geq a) \leq \exp(1 - da^3).$$

These bounds apply to the scaled fluctuations of the directed landscape $\mathcal{L}(p; q)$ at fixed times or positions.

Definition (Brownian Absolute Continuity)**.** For the directed landscape $\mathcal{L}(p; q)$, the process $(\mathcal{L}(0, 0; x/2, t) - \mathcal{L}(0, 0; 0, t), x \in [0, x_0])$ is absolutely continuous with respect to a standard Brownian motion with variance one, with a Radon-Nikodym derivative bounded by $e^{cx_0^3}$ for some constant $c > 0$.

Definition (Modulus of Continuity Bound)**.** For the directed landscape $\mathcal{L}(p; q)$, define the stationary version $\mathcal{R}(x, s; y, t) = \mathcal{L}(x, s; y, t) + \frac{(x-y)^2}{t-s}$. For two points $u_i = (x_i, s_i; y_i, t_i)$, $i = 1, 2$, let

$$\xi = \xi(u_1, u_2) = \|(x_1, y_1) - (x_2, y_2)\|, \quad \tau = \tau(u_1, u_2) = \|(s_1, t_1) - (s_2, t_2)\|.$$

Then for any $b \geq 2$, $\epsilon \leq 1$, and $u_1, u_2 \in K_\epsilon^b$ with $\tau \leq \epsilon^{3/b^3}$, we have

$$|\mathcal{R}(u_1) - \mathcal{R}(u_2)| \leq C \left(\tau^{1/3} \log^{2/3}(\tau^{-1}) + \xi^{1/2} \log^{1/2}(4b\xi^{-1}) \right),$$

with a random constant C satisfying $\mathbb{P}(C > m) \leq cb^{10}\epsilon^{-6}e^{-dm^{3/2}}$, where c, d are universal constants.

Exercise 1. Use the Tracy-Widom tail bounds to show that

$$\mathbb{P}\left(|\mathcal{L}(u + (0, 0; 0, t)) - \mathcal{L}(u)| < -at^{1/3}\right) \leq \exp(1 - da^3)$$

for some constant $d > 0$ and all u and $a, t > 0$.

Exercise 2. Use Brownian absolute continuity to show that for $u = (0, 0; 0, 1)$,

$$\mathbb{P}\left(|\mathcal{L}(u + (0, 0; x, 0)) - \mathcal{L}(u)| > ax^{1/2}\right) \leq \exp(1 - da^2)$$

for some constant $d > 0$ and all $a, x > 0$.

Exercise 3. Let $X = \arg\max_x \mathcal{L}(0,0;x,1)$. Show that there exists a constant $d > 0$ such that

$$\mathbb{P}(|X| > a) \leq \exp(1 - da^3)$$

for all $a > 0$. Use the Tracy-Widom tail bounds to handle integer x, and Brownian absolute continuity to handle x in between.

Exercise 4. Show that for the unique $(0,0;0,1)$ geodesic γ in the directed landscape,

$$\mathbb{P}(\gamma(1/2) > a) \leq \exp(1 - da^3)$$

for some constant $d > 0$. (Hint: Essentially the same proof as in Exercise 3 can be applied.)

Exercise 5. Show that for $u = (0,0;0,1)$,

$$\mathbb{P}\left(|\mathcal{L}(u + (0,0;0,t)) - \mathcal{L}(u)| > at^{1/3}\right) \leq \exp(1 - da^{3/2})$$

for some constant $d > 0$ and all $a > 0$ and $t \in [0,1]$. (Note: This direction is much harder than Exercise 1.)

Exercise 6. Use Brownian absolute continuity and scaling to show that

$$(\mathcal{L}(0,0;x/2,t) - \mathcal{L}(0,0;0,t),\ x \geq 0)$$

converges in law to a standard Brownian motion $(B(x),\ x \geq 0)$ as $t \to \infty$.

Exercise 7. Use the modulus of continuity bounds for L to show that the $(0,0;0,1)$-geodesic γ is Hölder-α continuous for all $\alpha < 2/3$.