Modelling flocks and prices: jumping particles with an attractive interaction

Joint work in progress with Miklós Zoltán Rácz

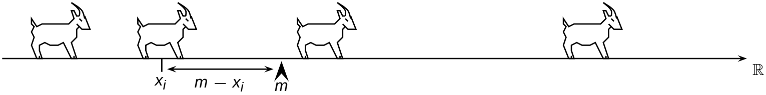
Márton Balázs

Budapest University of Technology and Economics

Large Scale Stochastic Dynamics
Oberwolfach, November 11, 2010.

The model (Bálint Tóth)

Goats jump on $\mathbb{R}$.



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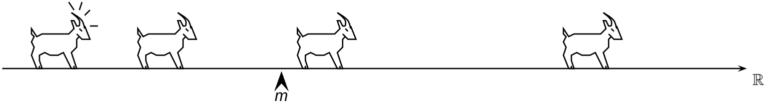

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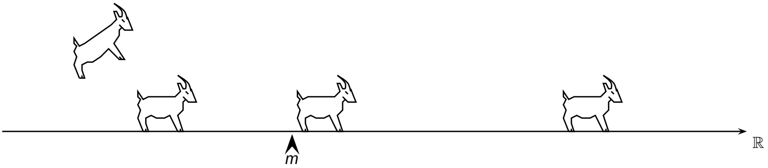

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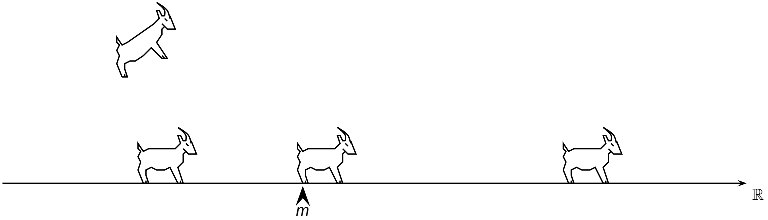

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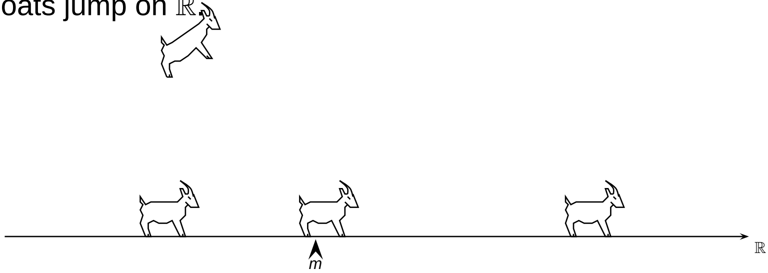

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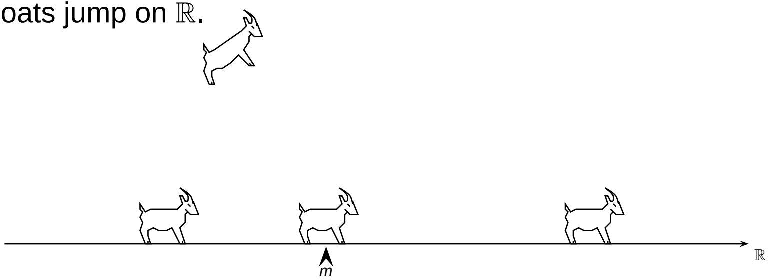

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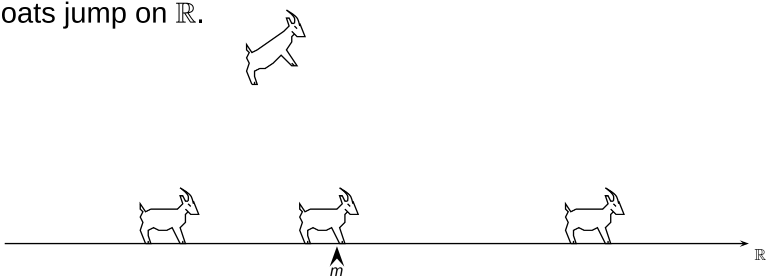

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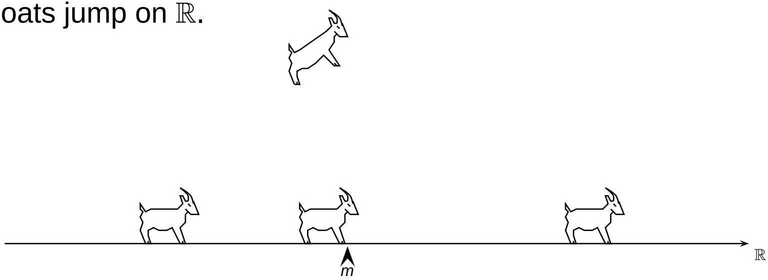

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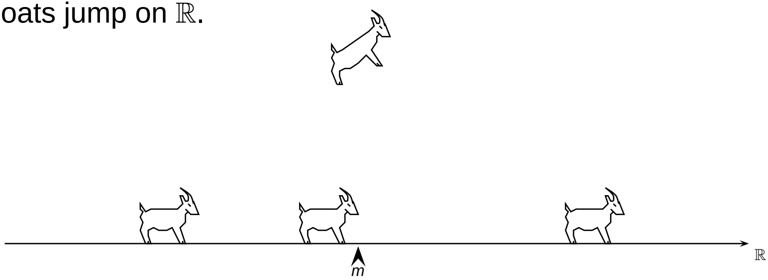

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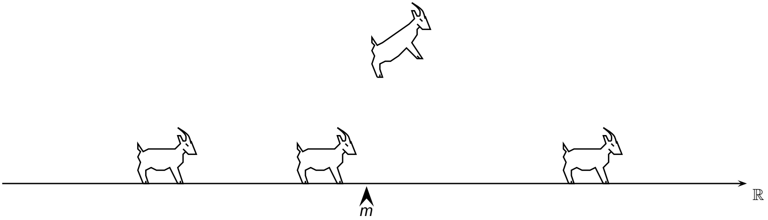

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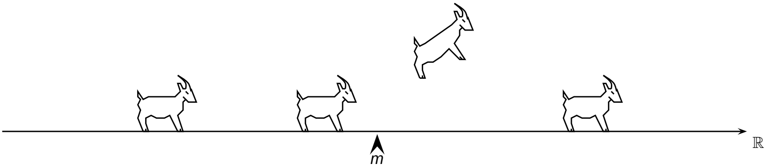

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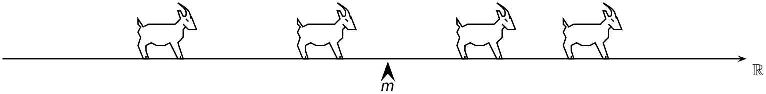

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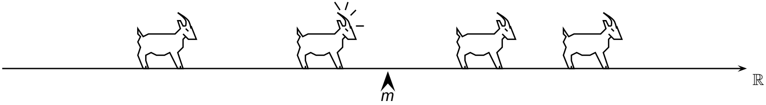

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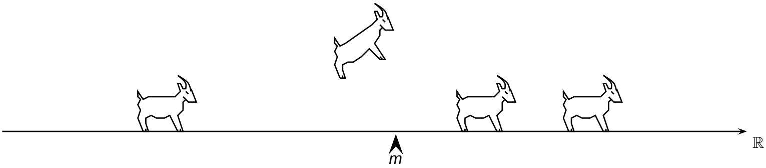

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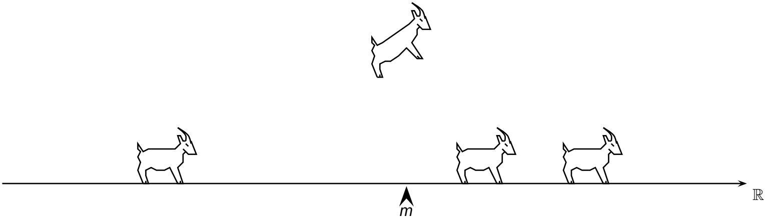

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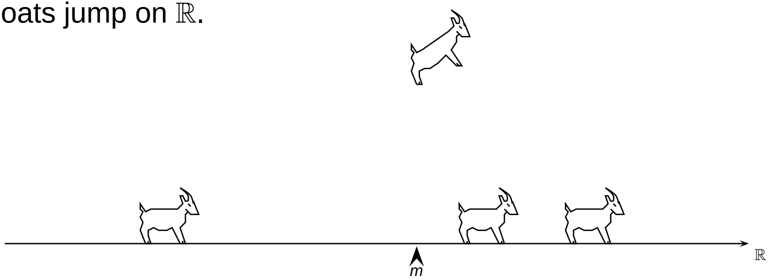

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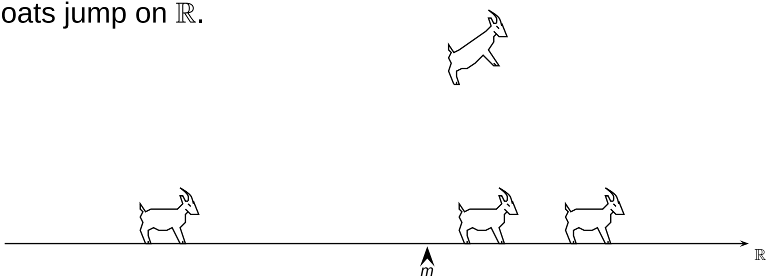

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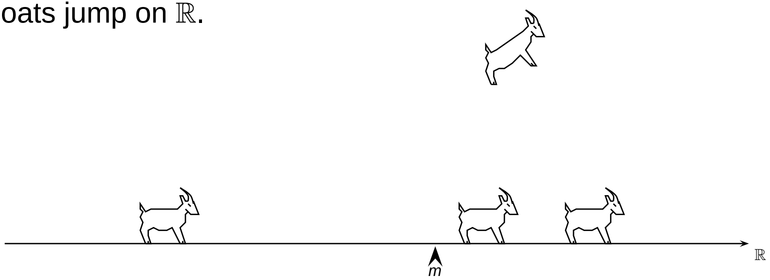

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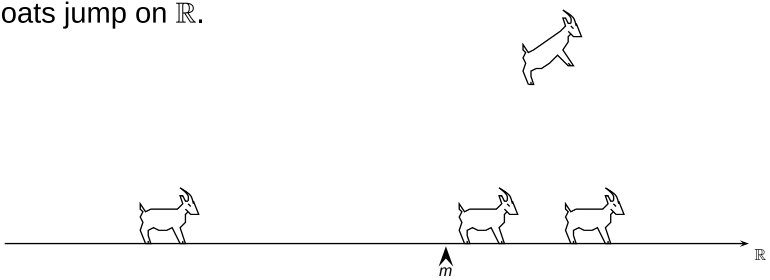

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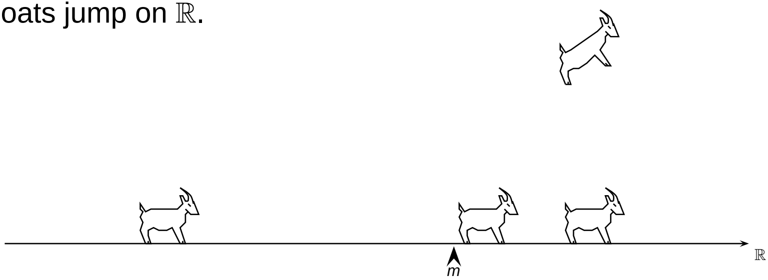

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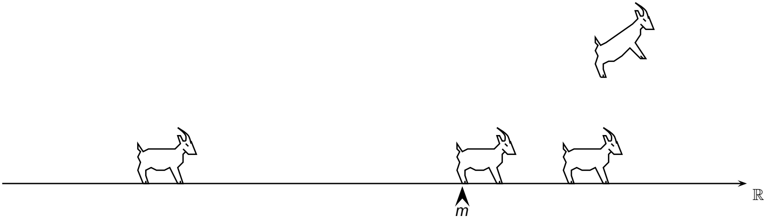

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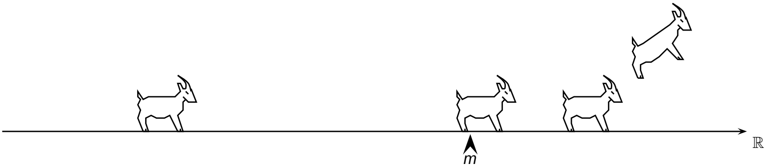

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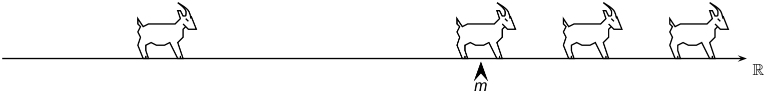

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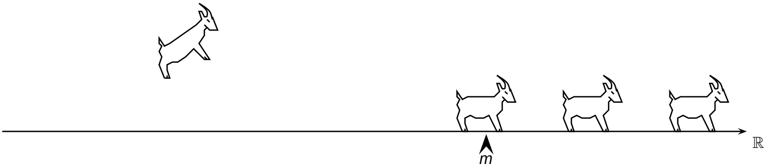

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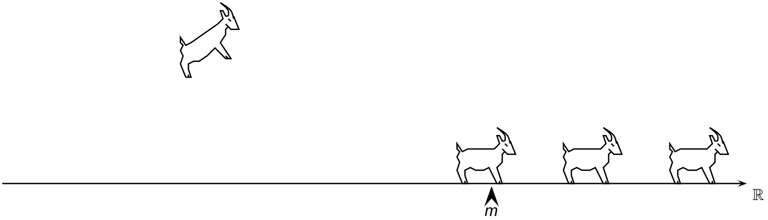

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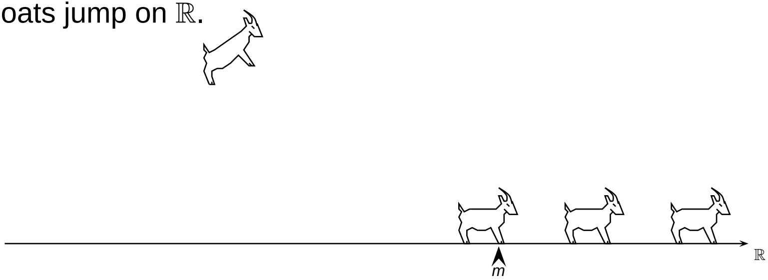

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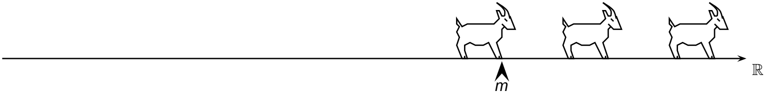

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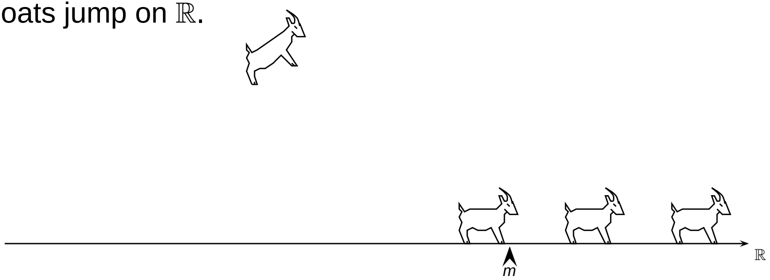

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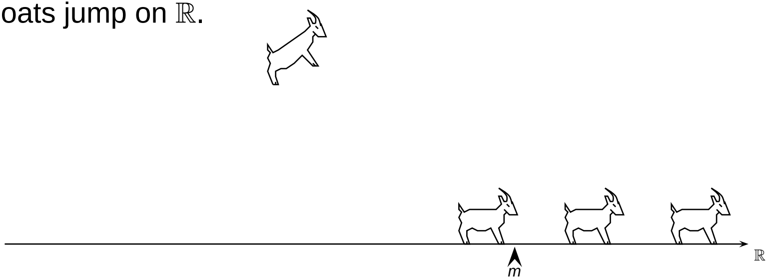

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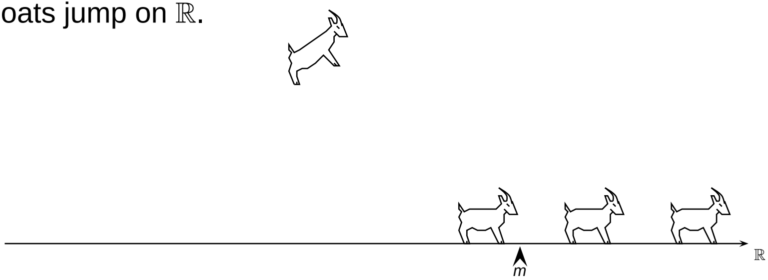

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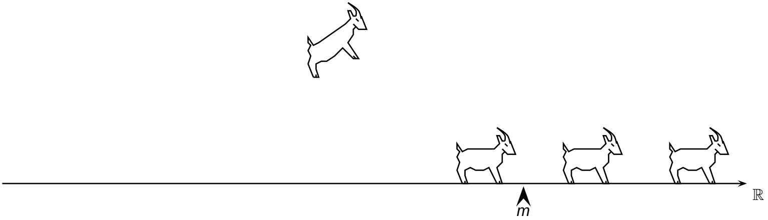

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- Given a configuration $x_1, x_2, \ldots, x_n$ of goats the center of mass is $m = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- Goat i jumps with rate $w(x_i - m)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jumps are positive, random, independent of everything, and are of density φ, mean one.

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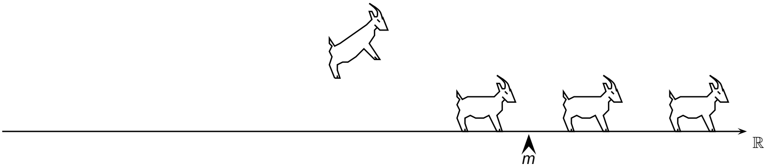

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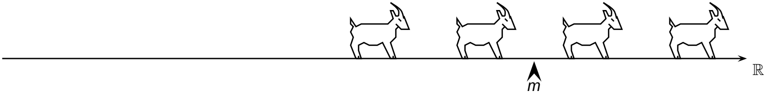

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Found results of the types:

- interacting diffusions with linear drift (A. Greven et. al.),
- rank dependent drift of Brownian motions (S. Chatterjee, S. Pal 2007, S. Pal, J. Pitman 2007),
- relocation of random walking particles (A. Manita, V. Shcherbakov 2005),
- reordering and steps by a joint Gaussian (A. Ruzmaikina, M. Aizenman 2005, L.P. Arguin 2008, L.P. Arguin, M. Aizenman 2009),
- multiplicative steps as well (I. Grigorescu, M. Kang 2009).

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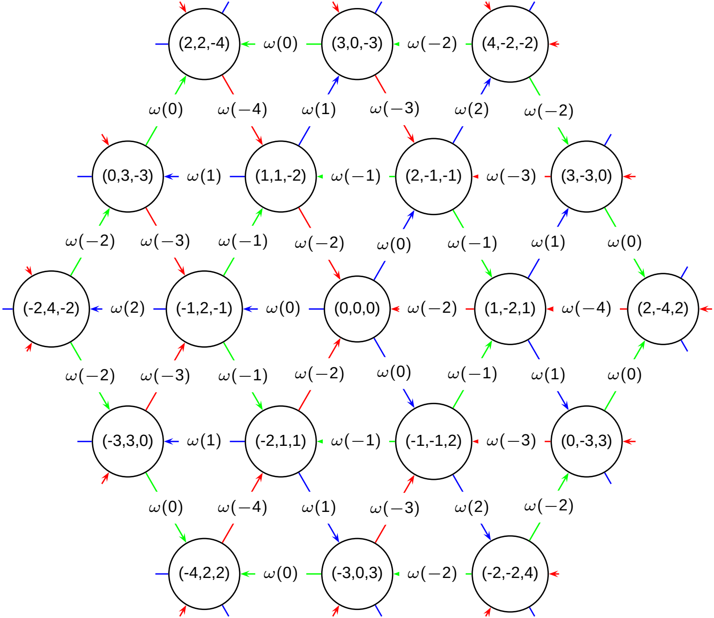
$n = 3$ particles: already seems hopeless. The process is “very irreversible”.

$n = 3$ particles, jump lengths are deterministically 1



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$$+\int_{-\infty}^{x} w(y-m(t)) \cdot \varrho(y,t) \cdot \varphi(x-y) \, \mathrm{d}y,$$

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These equations conserve $1 = \int \varrho(x,t)\,\mathrm{d}x$ and give $\dot{m}(t) = \int w(x-m(t)) \cdot \varrho(x,t)\,\mathrm{d}x$.

Fluid limit: a mean field equation

We look for stationary solution of this equation as seen from the center of mass.

Idea: as $n \to \infty$, in a stationary distribution $m(t)$ would stabilize. So assume

$$m(t) = ct \qquad \text{and}$$
$$\varrho(x, t) = \varrho(x - ct).$$

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Plug this in to get

$$-c\varrho'(x) = -w(x)\varrho(x) + \int_{-\infty}^{x} w(y)\varrho(y)\varphi(x-y)\,\mathrm{d}y.$$

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Extreme value statistics (Attila Rákos)

When the jumps are Exp(1): $\varphi(x) = \mathrm{e}^{-x}$,
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Between t and $t + \mathrm{d}t$, $\mathrm{d}N(t) = \mathrm{e}^{ct}\,\mathrm{d}t$ many new Exp(1) particles try to break the record. So the probability that $Y(t)$ jumps is

$$1 - \left(1 - \mathrm{e}^{-Y(t)}\right)^{\mathrm{e}^{ct}\,\mathrm{d}t} \simeq \mathrm{e}^{ct-Y(t)}\,\mathrm{d}t \qquad \text{(for large } Y(t)\text{)}.$$

And when it jumps, it jumps Exp(1). But we know that $Y(t) - ct + \log c$ converges to standard Gumbel.

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- When $w(x) = \mathrm{e}^{-\beta x}$ is exponential: take Fourier transform to get

  $$ci\tau\widehat{\varrho}(\tau) = \big(\widehat{\varphi}(\tau) - 1\big)\cdot\widehat{\varrho}(\tau + i\beta).$$

  Hope to solve the recurrence relation on the $\mathfrak{Im}$ line, then analytic continuation gives a hint on the form of $\widehat{\varrho}$, to be verified.

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  - Method tested when $\varphi(x) = \mathrm{e}^{-x}$ (also seen before), hope to work with other φ's too.

Taking the fluid limit

Recall the original fluid equation:

$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x - m(t)) \cdot \varrho(x,t) + \int_{-\infty}^{x} w(y - m(t)) \cdot \varrho(y,t) \cdot \varphi(x-y)\, \mathrm{d}y,$$

or, for all f testfunctions:

$$\langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle = \int_0^t \Big\langle \big\{ \mathbf{E}[f(x+Z)] - f(x) \big\} w(x - m(s)),\, \mu(s) \Big\rangle \, \mathrm{d}s,$$

$$m(s) = \langle x,\, \mu(s)\rangle.$$

Here **E** refers to expectation of Z w.r.t. density φ.

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Problem: bounded functions and “just measures” are not enough!

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Dealing with the space of probability measures having first moments, and the Wasserstein 1 metric seems to work.

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