Modelling flocks and prices: jumping particles with an attractive interaction

Joint work with Miklós Zoltán Rácz and Bálint Tóth

Márton Balázs

Budapest University of Technology and Economics

Madison-Wisconsin, February 24, 2011.

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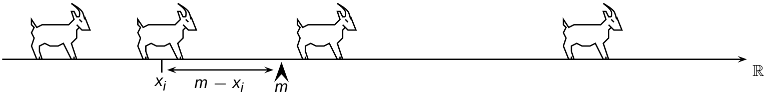
Goats jump on $\mathbb{R}$.



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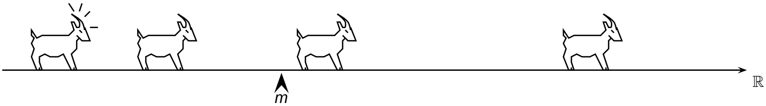

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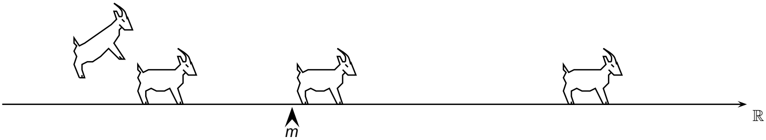

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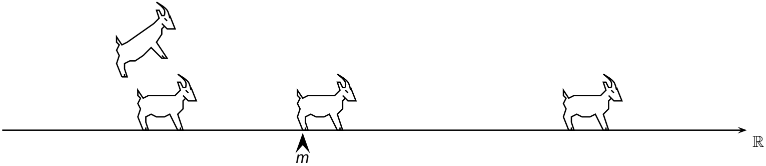

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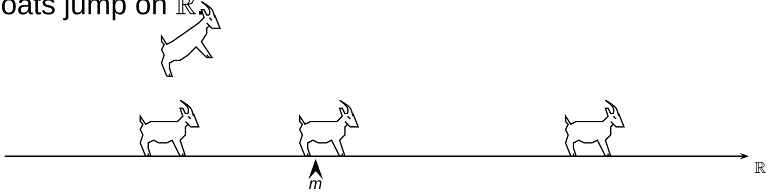

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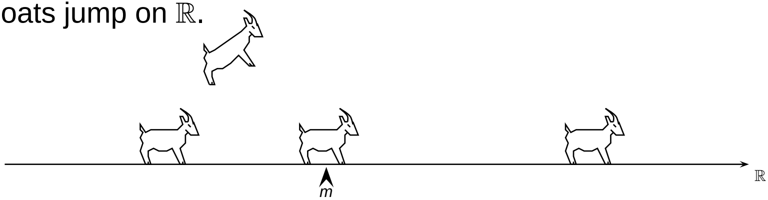

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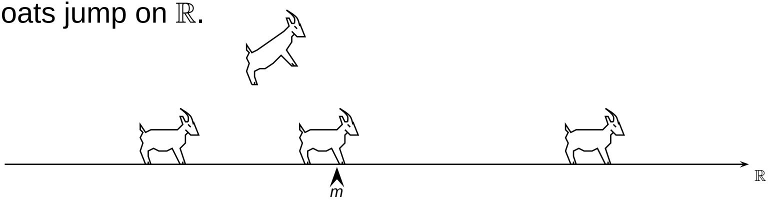

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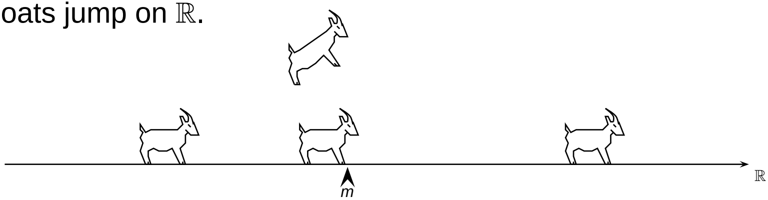

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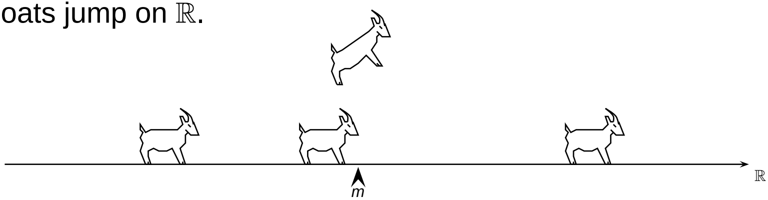

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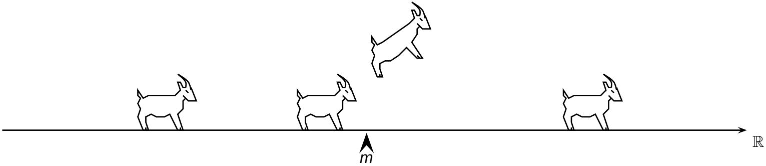

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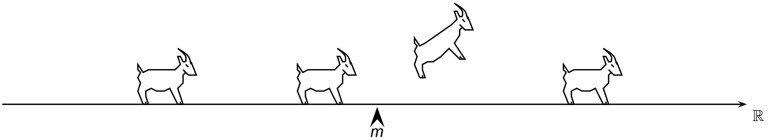

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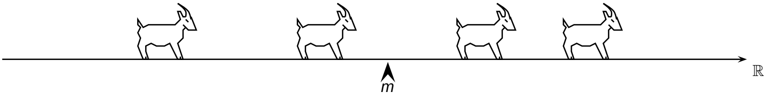

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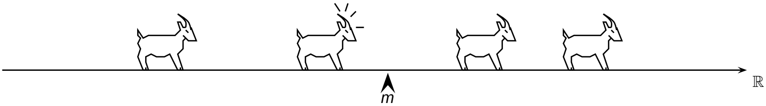

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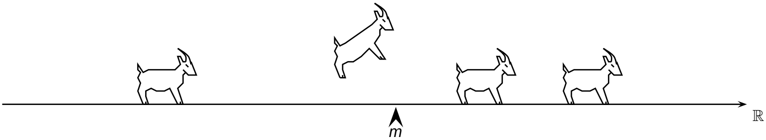

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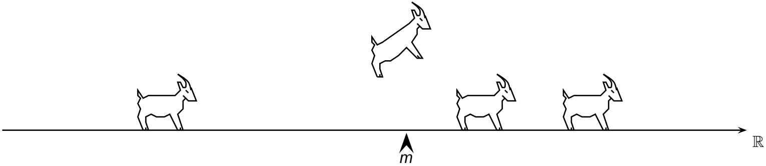

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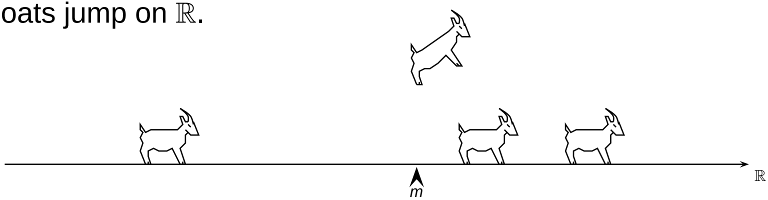

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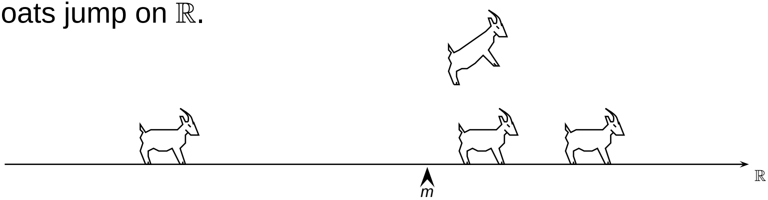

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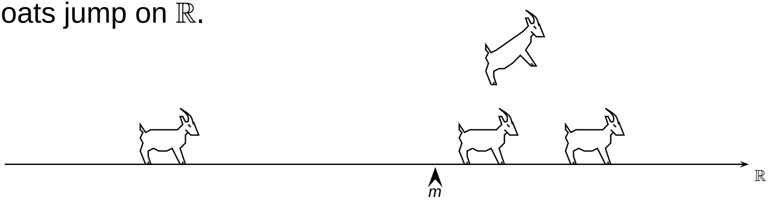

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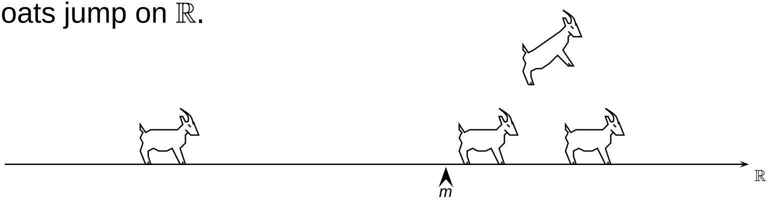

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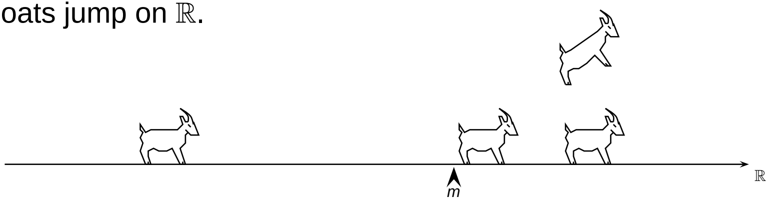

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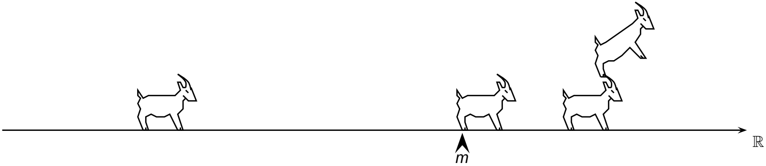

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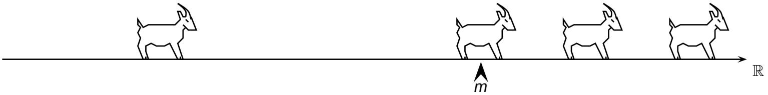

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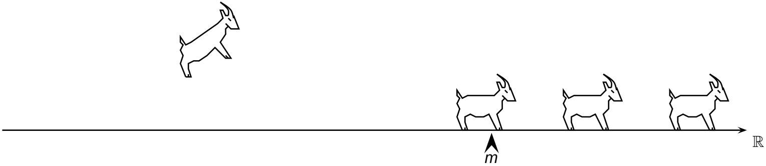

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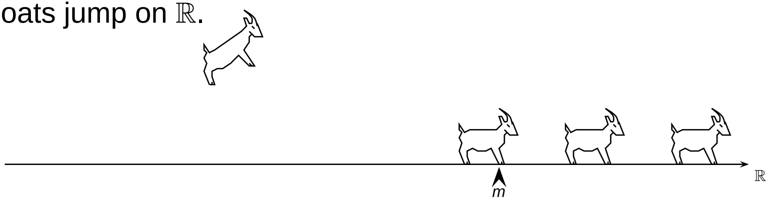

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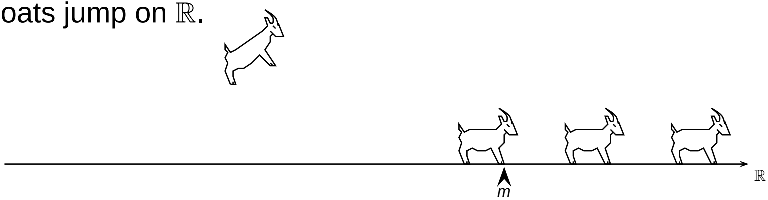

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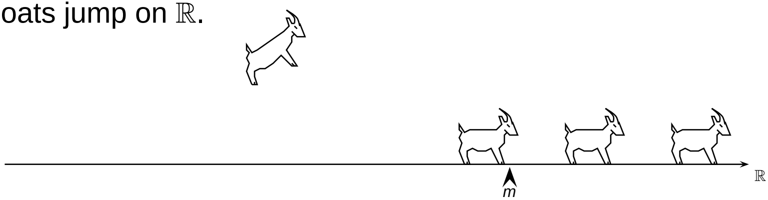

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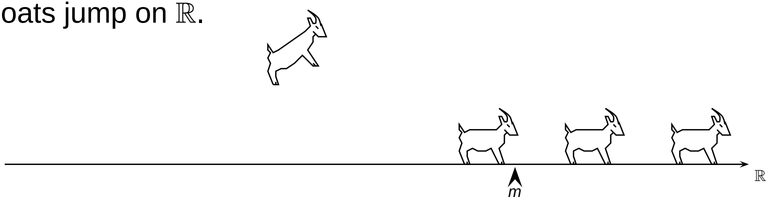

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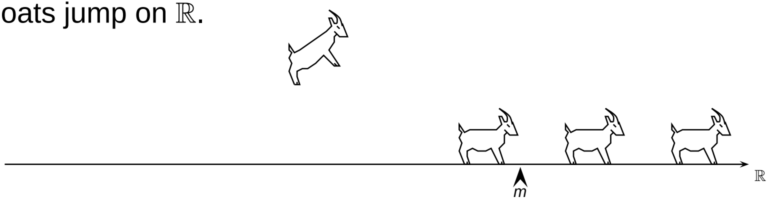

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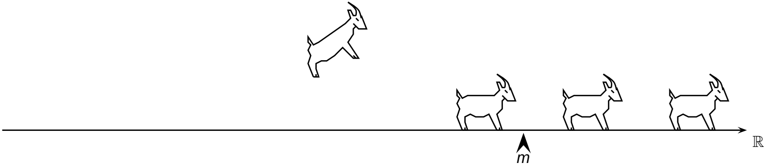

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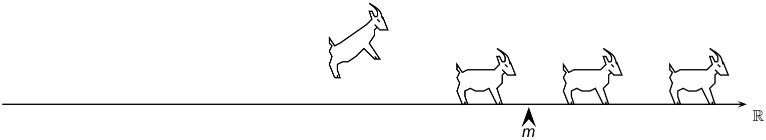

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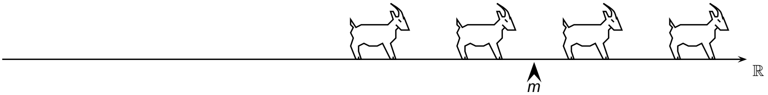
- Goat i jumps with rate $w(x_i - m)$, where w is the jump rate function: $\mathbb{R} \to \mathbb{R}^+$, decreasing.
- Jumps are positive, random, independent of everything, and are of density φ, mean one.

The model

Goats jump on $\mathbb{R}$.



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Found results of the types:

- rat race model (D. ben-Avraham, S.N. Majumdar, S. Redner 2007)
- interacting diffusions with linear drift (A. Greven et. al.),
- rank dependent drift of Brownian motions (S. Pal, J. Pitman 2008, S. Chatterjee, S. Pal 2009),
- relocation of random walking particles (A. Manita, V. Shcherbakov 2005),
- interacting jump processes (A. Greenberg, V.A. Malyshev, S.Yu. Popov 1995)
- multiplicative steps as well (I. Grigorescu, M. Kang 2010).

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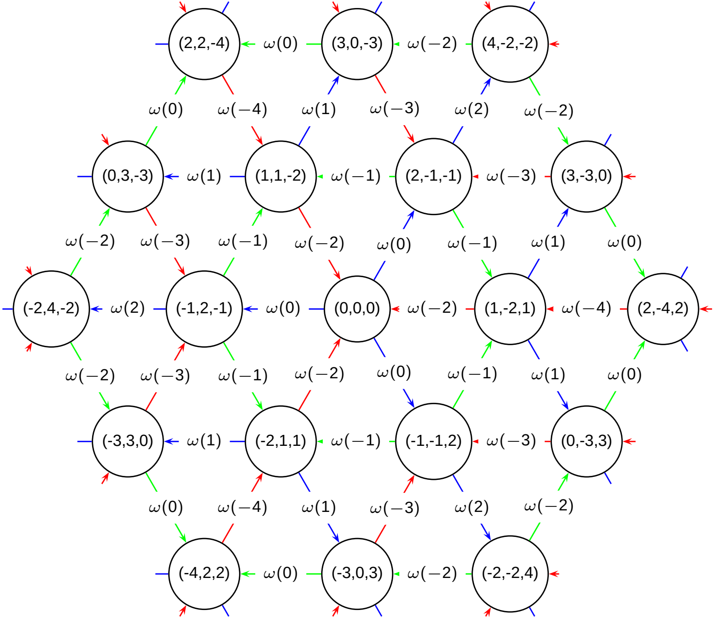
$n = 3$ particles: already seems hopeless. The process is "very irreversible".

$n = 3$ particles, jump lengths are deterministically 1



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These equations conserve $1 = \int \varrho(x,t)\,\mathrm{d}x$ and give $\dot m(t) = \int w(x-m(t)) \cdot \varrho(x,t)\,\mathrm{d}x$.

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We look for stationary solution of this equation as seen from the center of mass.

Idea: as $n \to \infty$, in a stationary distribution $m(t)$ would stabilize. So assume

$$m(t) = ct \qquad \text{and}$$
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Plug this in to get

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Between t and $t + \mathrm{d}t$, $\mathrm{d}N(t) = \mathrm{e}^{ct}\,\mathrm{d}t$ many new Exp(1) particles try to break the record. So the probability that $Y(t)$ jumps is

$$1 - \left(1 - \mathrm{e}^{-Y(t)}\right)^{\mathrm{e}^{ct}\,\mathrm{d}t} \simeq \mathrm{e}^{ct - Y(t)}\,\mathrm{d}t \qquad \text{(for large } Y(t)\text{)}.$$

And when it jumps, it jumps Exp(1). But we know that $Y(t) - ct + \log c$ converges to standard Gumbel.

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  Hope to solve the recurrence relation on the $\mathfrak{Im}$ line, then analytic continuation gives a hint on the form of $\widehat{\varrho}$, to be verified.

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  - Method tested when $\varphi(x) = \mathrm{e}^{-x}$ (also seen before), hope to work with other φ's too.

Taking the fluid limit

Recall the original mean field equation:

$$\frac{\partial \varrho(x,t)}{\partial t} = -w(x-m(t))\cdot\varrho(x,t) + \int_{-\infty}^{x} w(y-m(t))\cdot\varrho(y,t)\cdot\varphi(x-y)\,\mathrm{d}y,$$

or, for all f test functions:

$$\langle f,\mu(t)\rangle - \langle f,\mu(0)\rangle = \int_0^t \big\langle \{\mathbf{E}[f(x+Z)] - f(x)\}\, w(x-m(s)),\ \mu(s)\big\rangle\,\mathrm{d}s,$$

$$m(s) = \langle x,\ \mu(s)\rangle.$$

Here **E** refers to expectation of Z w.r.t. the jump length distribution.

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Problem: bounded functions and “just measures” are not enough!

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Goal: convergence of the n-particle empirical measures $\mu_n(t)$ in the Skohorod space $D([0, \infty), \mathcal{P}_1)$.

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Method for these bounds: introduce *ghost goats*: they jump with rate $\sup_x w(x)$, they have the same jump length distribution as their planetary counterparts. Couple such that ghost goat$_i$ can jump without goat$_i$, but not vice-versa. $\rightsquigarrow$ increments of ghosts dominate increments of the planetary goats.

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Perkins' theorem originally was about checking C-relative compactness in $D([0,\infty],\mathcal{M})$ by checking that of appropriate integrals $\langle f,\mu_n(t)\rangle$ in $D([0,\infty],\mathbb{R})$. Our job here was to slightly generalize from finite measures $\mathcal{M}$ to measures with finite first moment $\mathcal{P}_1$.

2. The limit solves the mean field eq.

Let

$$\begin{aligned} A_{t,f}(\mu) :&= \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle \\ &\quad - \int_0^t \Big\langle \big\{\mathbf{E}[f(x+Z)] - f(x)\big\} w(x - m(s)),\ \mu(s)\Big\rangle \, \mathrm{d}s \\ &= \langle f, \mu(t)\rangle - \langle f, \mu(0)\rangle - \int_0^t L\langle f, \mu(s)\rangle \, \mathrm{d}s, \\ m(s) &= \langle x,\ \mu(s)\rangle. \end{aligned}$$

Recall that the mean field equation was

$$A_{t,f}(\mu) = 0.$$

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For the second, convergence in $D([0,\infty], \mathcal{P}_1)$ with the Wasserstein metric d_1 is just right for our test functions (including the center of mass!).

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$\rightsquigarrow d_H(\mu(t), \nu(t)) \leq d_H(\mu(0), \nu(0)) + c \int_0^t d_H(\mu(s), \nu(s)) \,\mathrm{d}s$, apply Grönwall's inequality.

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$$\mathbf{Var}(m_n(t)) \sim \frac{t^\gamma}{n^\alpha}.$$

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Thank you.