Construction of the zero range process and a deposition model with superlinear growth rates

Márton Balázs (MTA-BME Stochastics Research Group)

Joint work with
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Timo Seppäläinen (University of Wisconsin-Madison)
and
Sunder Sethuraman (Iowa State University)

March 1., 2007

1. The zero range process and the bricklayers' process

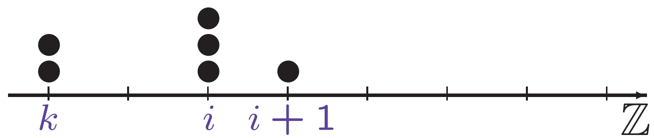
2. Construction materials and the construction

3. What have we constructed? Properties

4. What we didn't succeed in...

1. The zero range process (ZR):

$\omega_i \in \mathbb{Z}^+$



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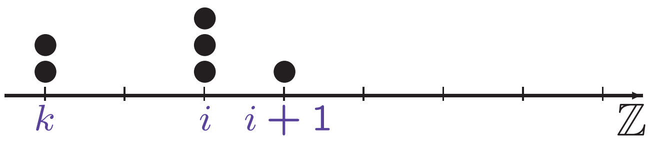

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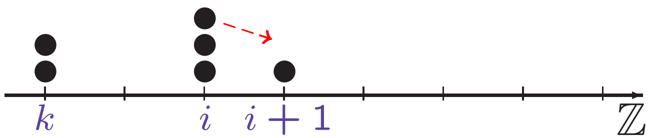

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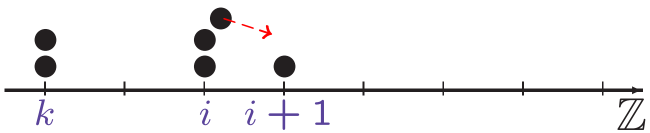

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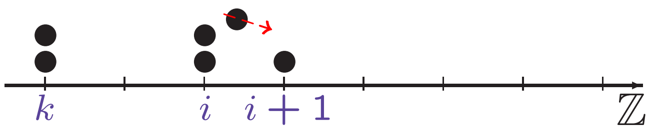

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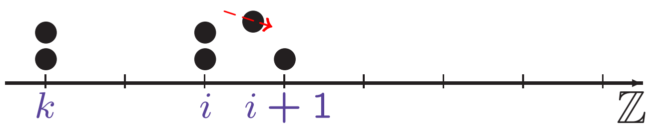

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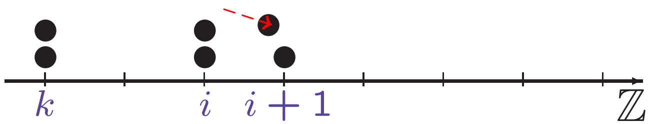

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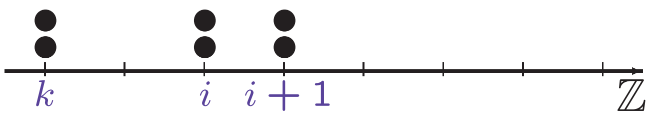

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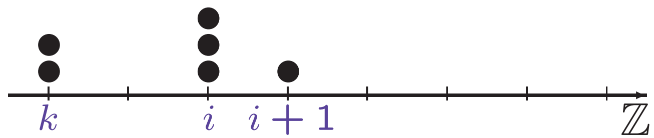

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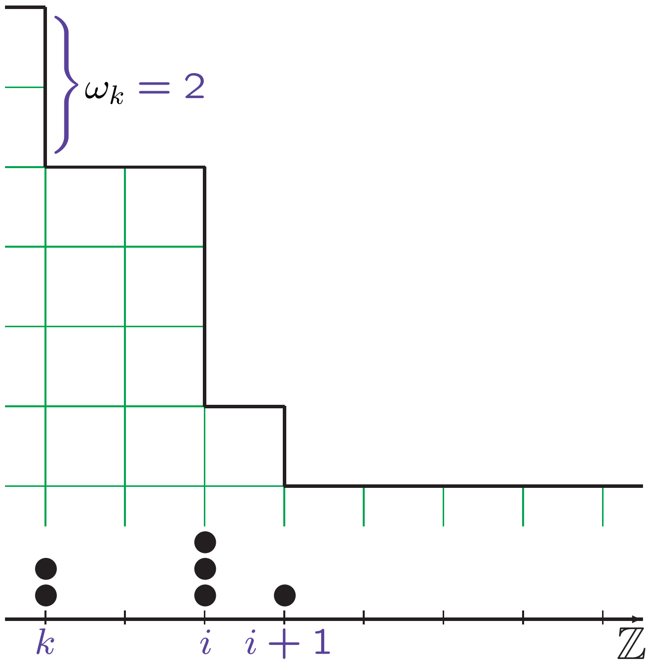

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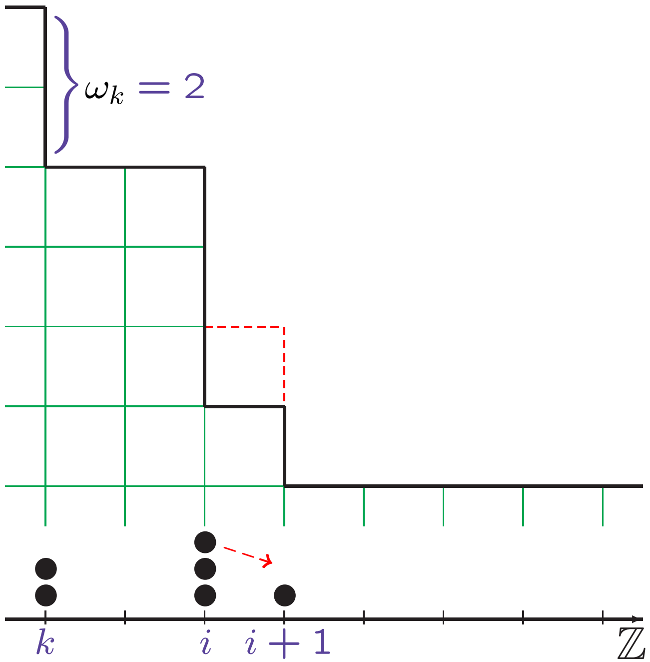

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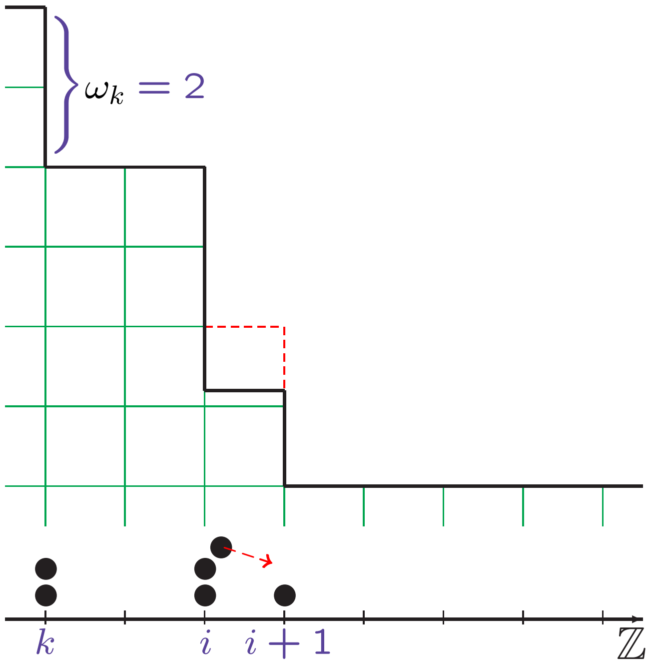

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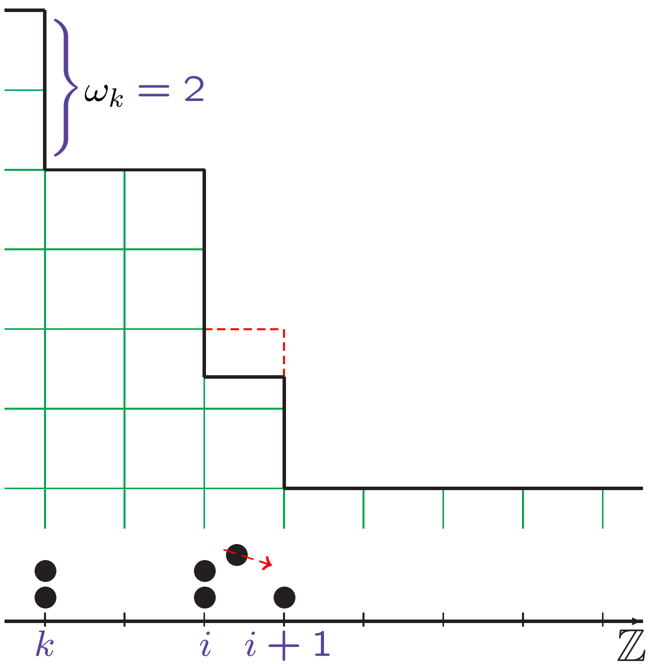

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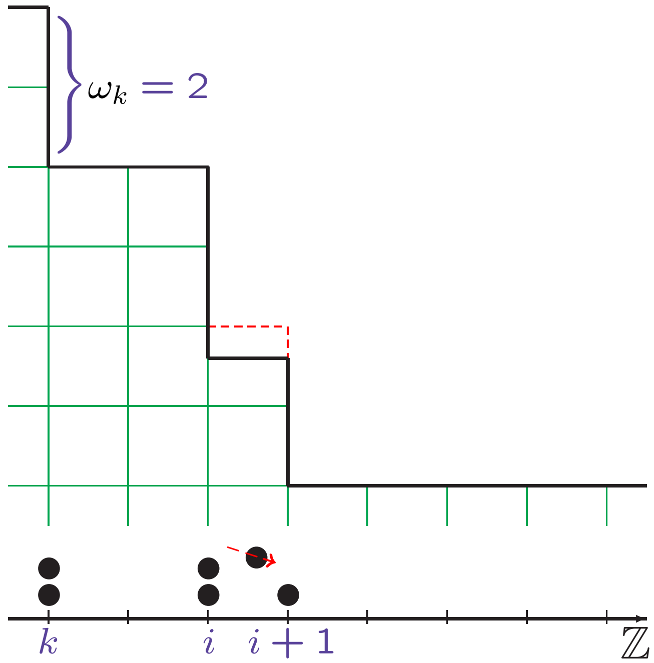

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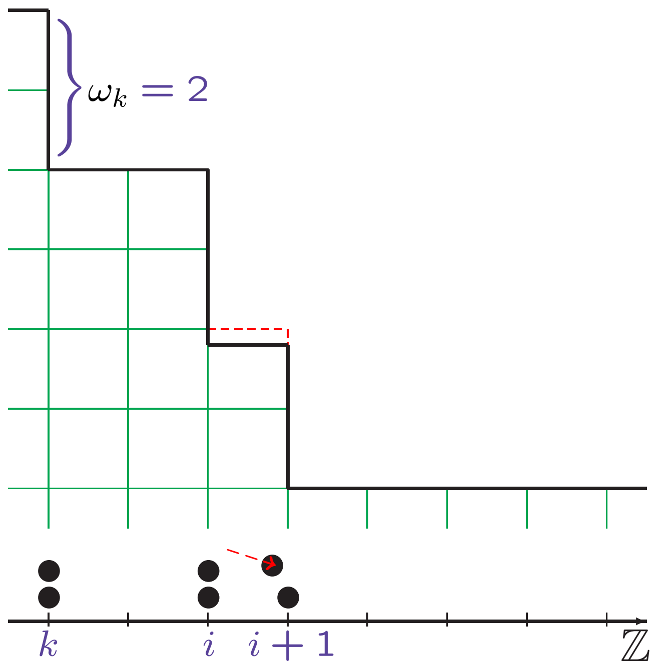

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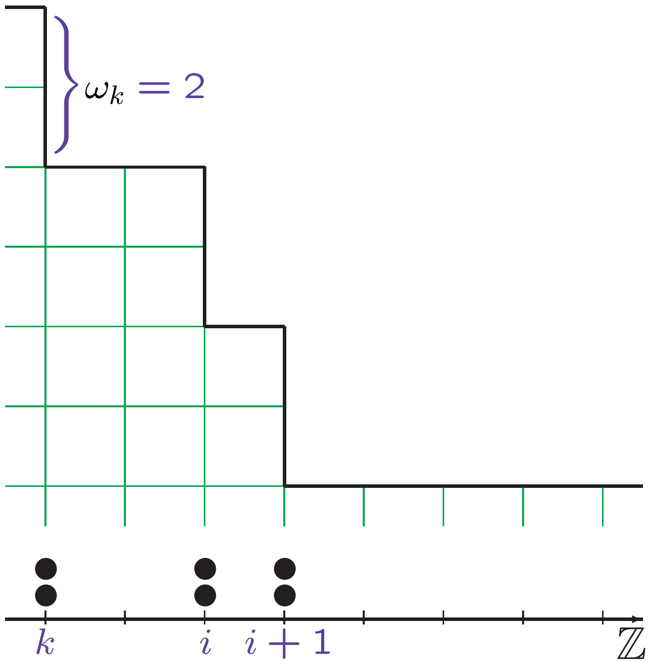

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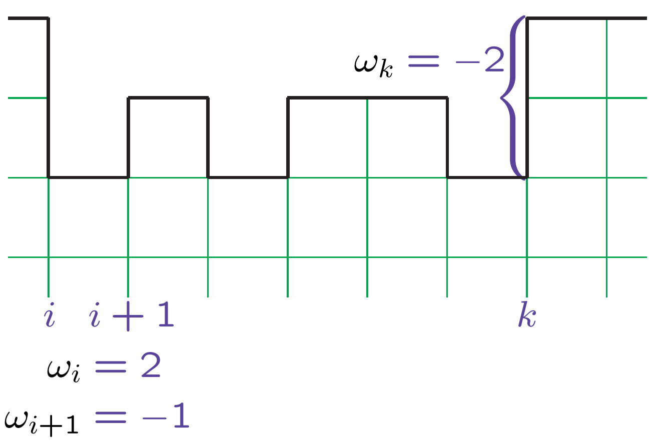

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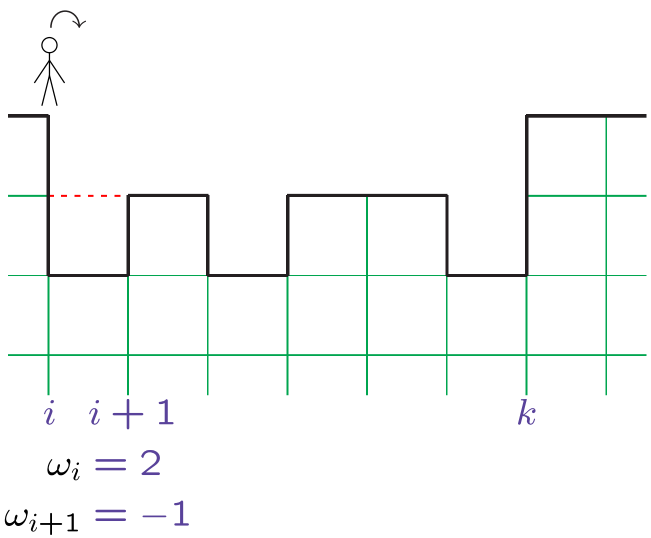

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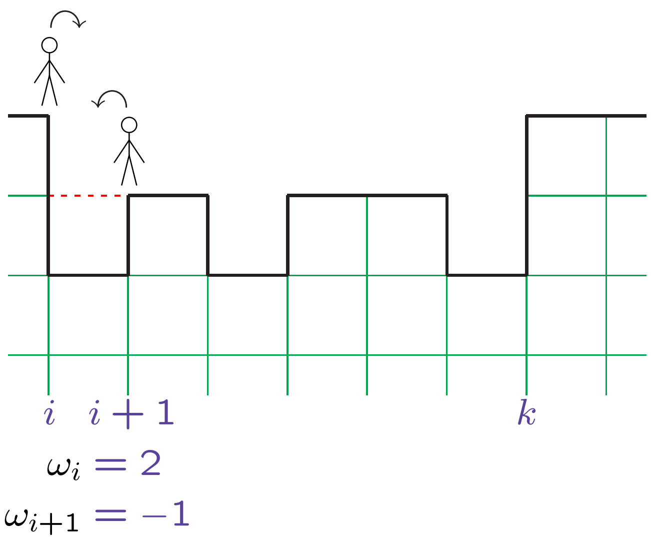

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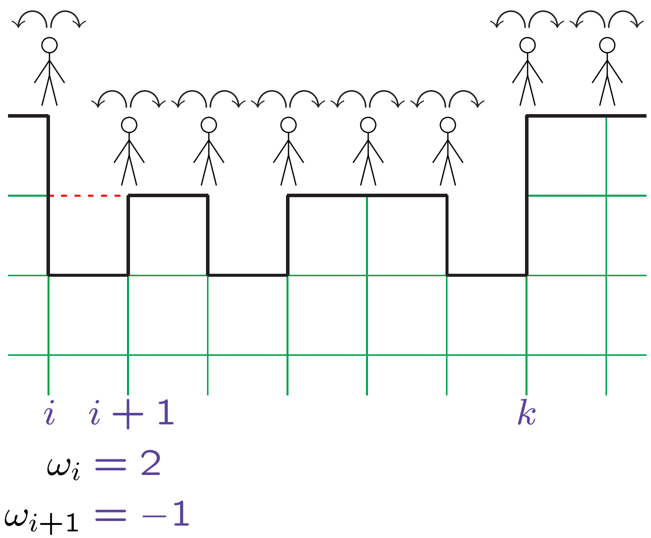

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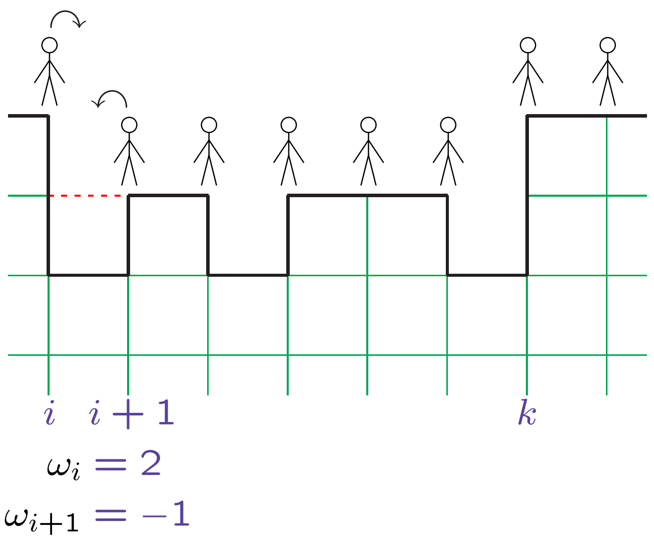

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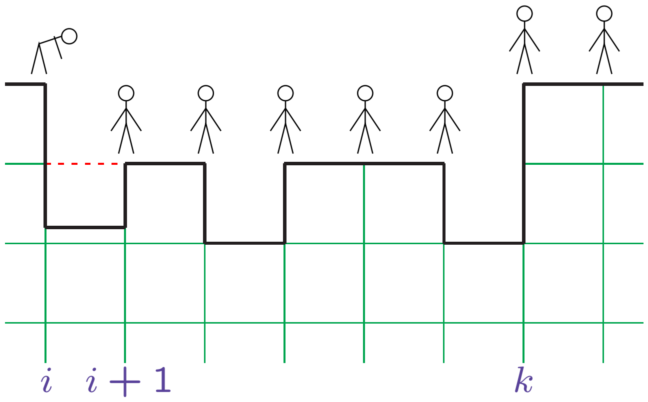

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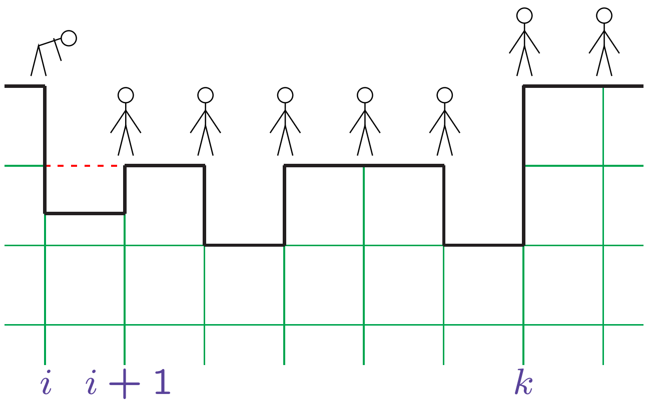

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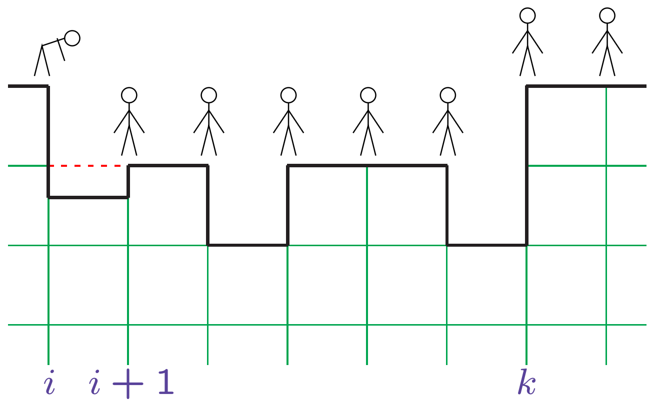

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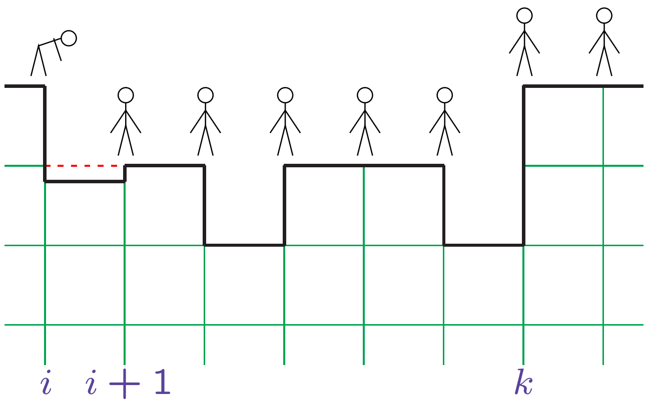

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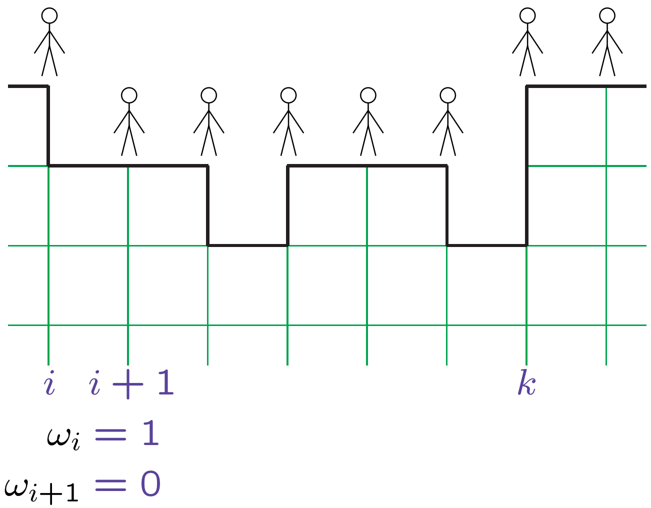

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$\rightsquigarrow$ Independent μ^θ-distributed ω_i's is (formally) the equilibrium of the process. The parameter θ sets the mean of ω_i's, that is, the slope of the wall.

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Unfortunately, the process is not constructed at that time.

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Estimates used by Andjel do not work.

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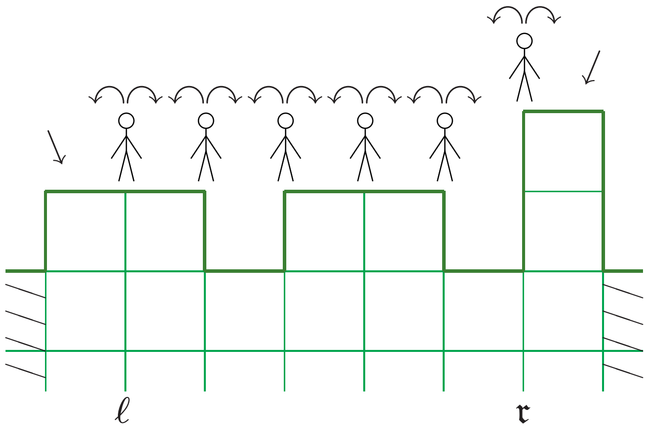

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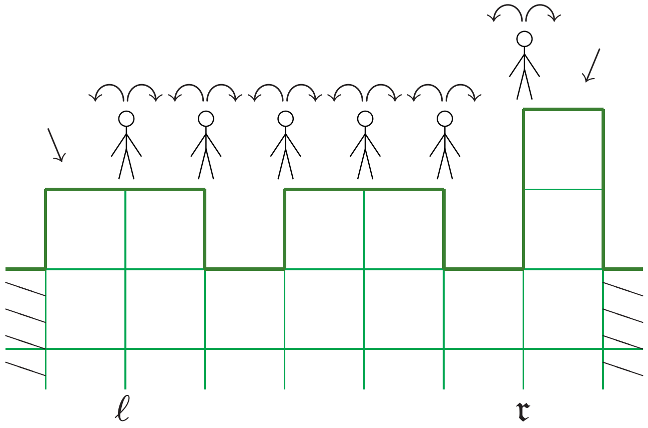

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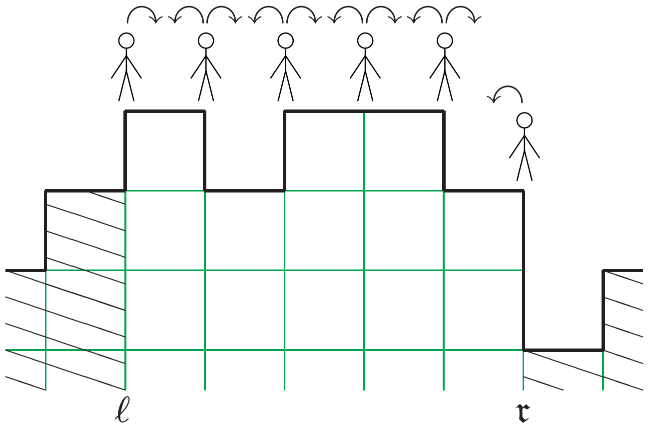

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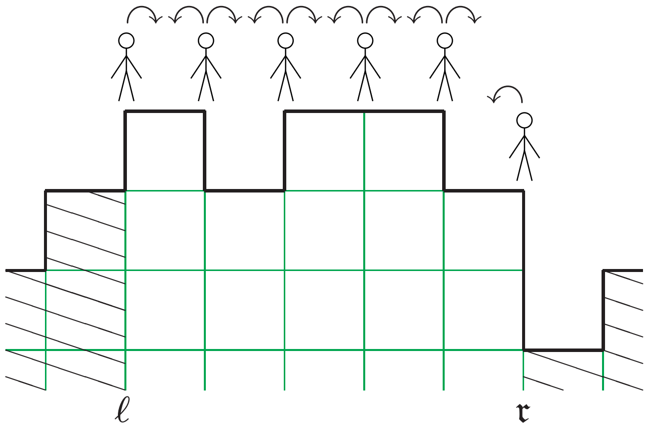

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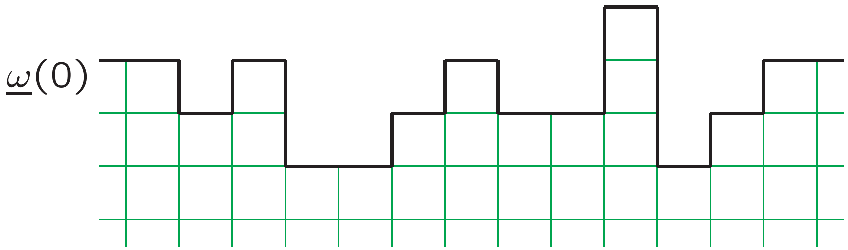
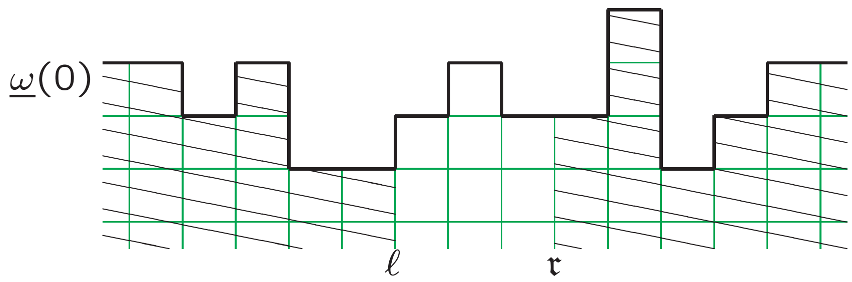
$\rightsquigarrow$ This process is far from equilibrium!!



- Fix a state $\underline{\omega}(0) \in \widetilde{\Omega}$.



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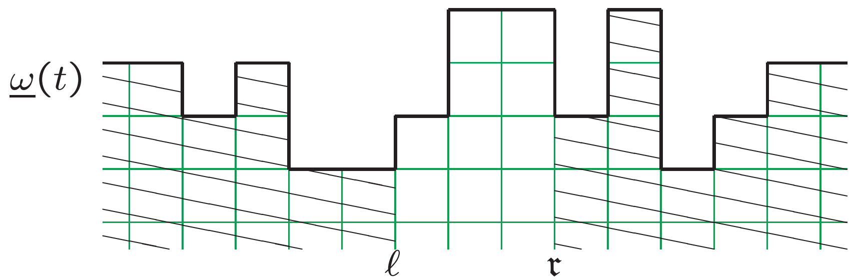

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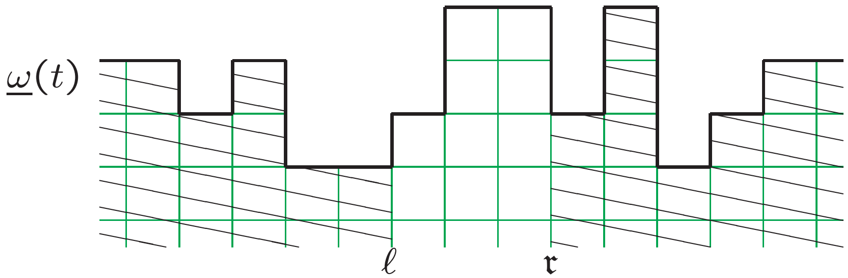

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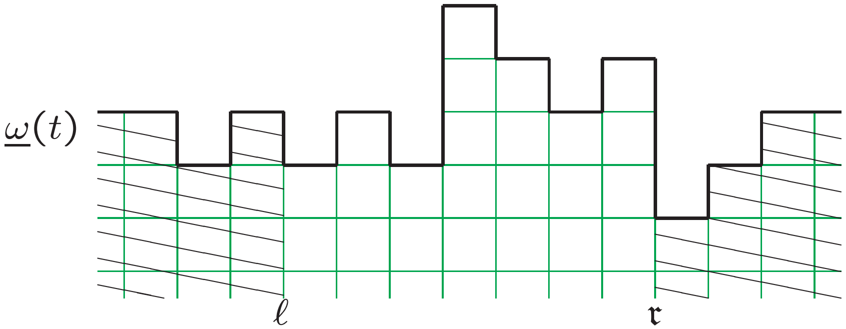

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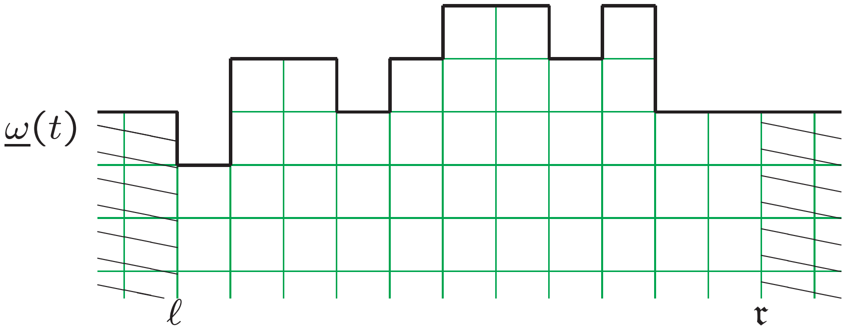

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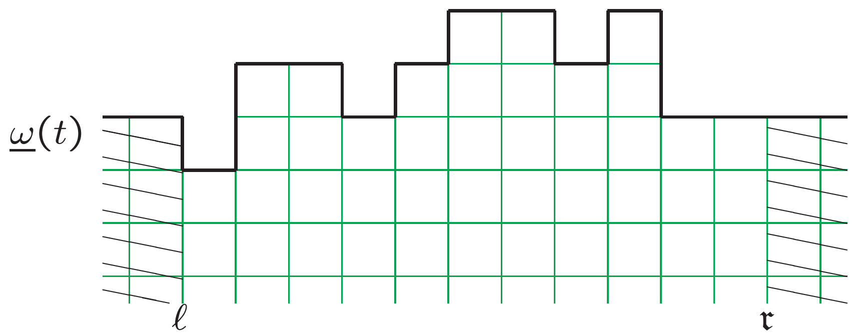

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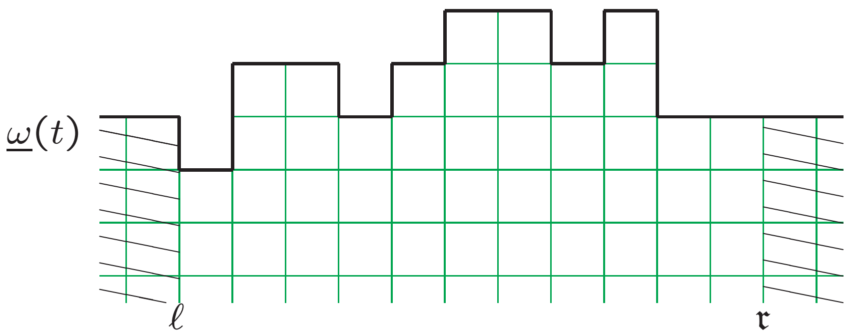

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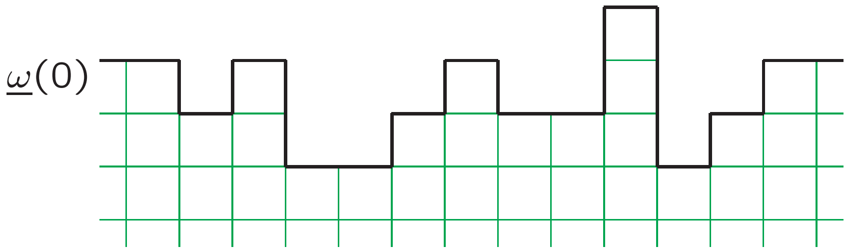

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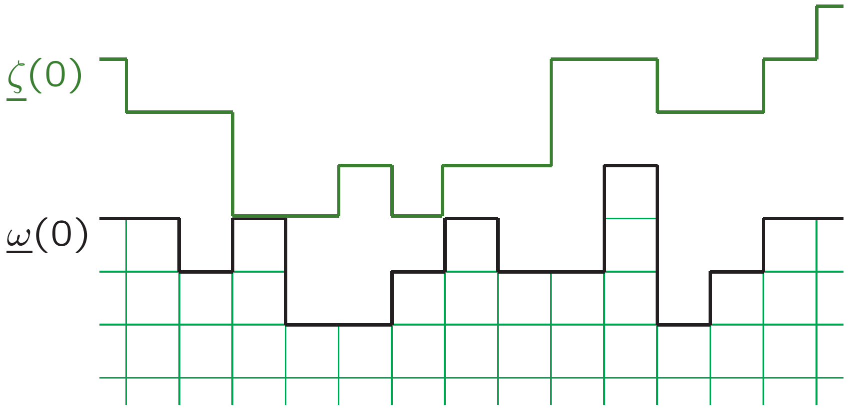

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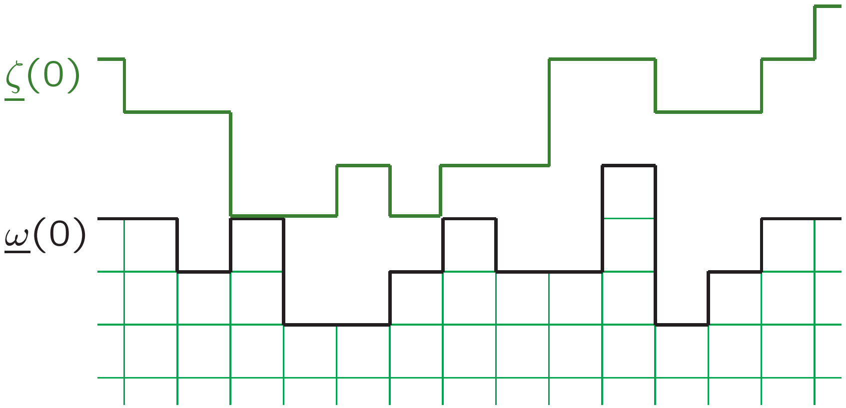

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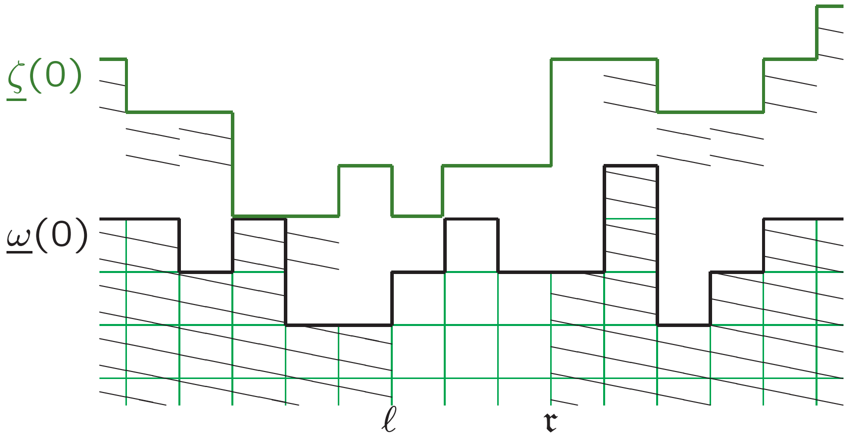
  $\Rightarrow$ With positive probability, each column of $\underline{\zeta}$ is higher than that column of $\underline{\omega}$.



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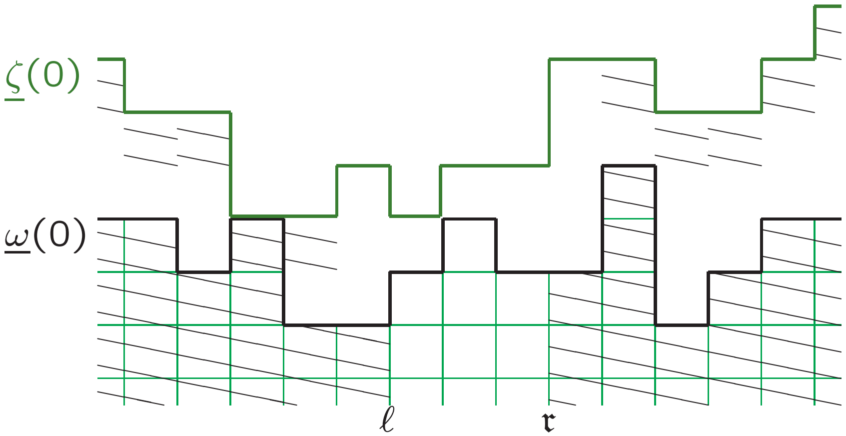

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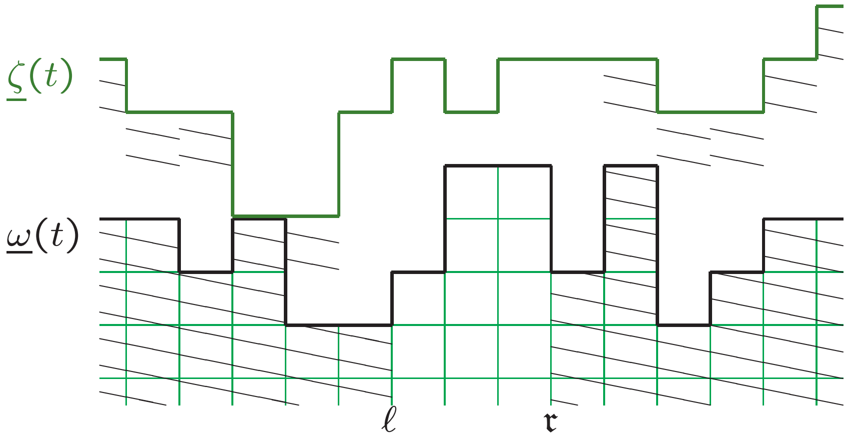

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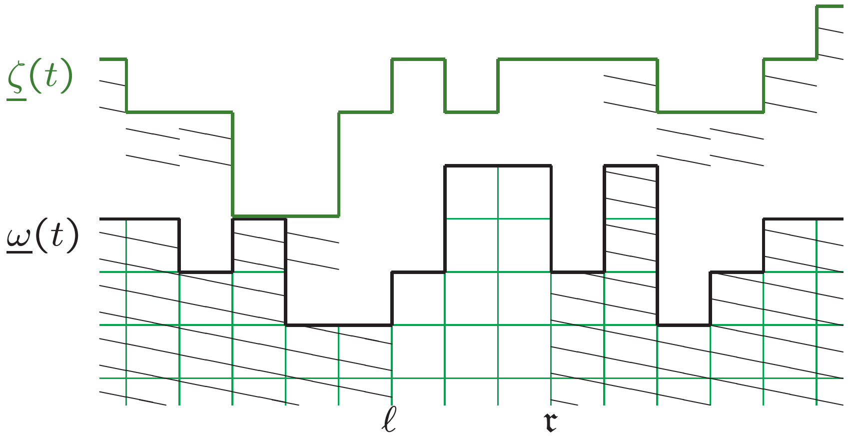

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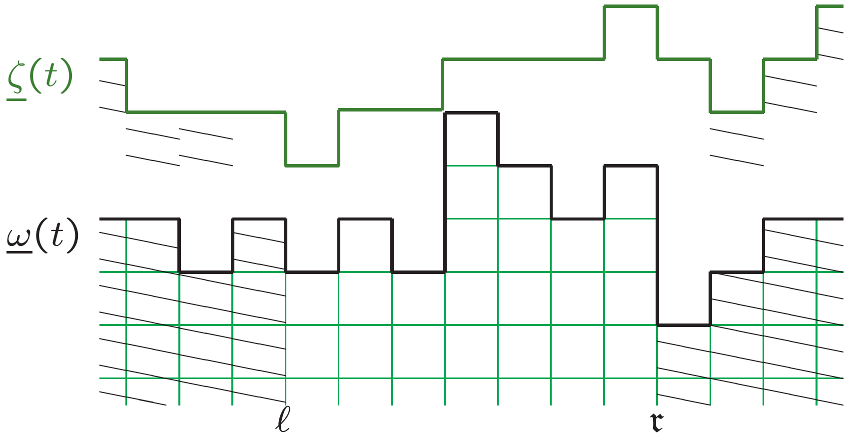

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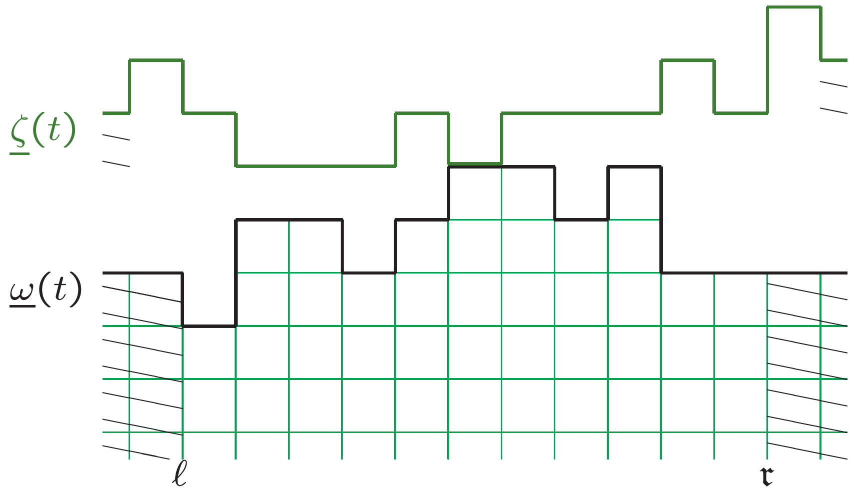

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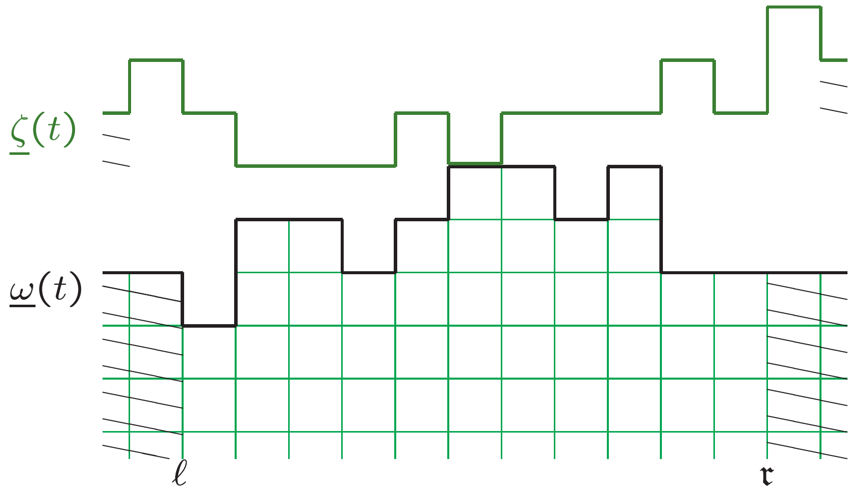

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⇝ Coupling 1: The *column heights* are monotone in ℓ and $\mathfrak{r}$.
$\Rightarrow$ We have a limit of the monotone processes. Is the limit finite? Yes, it is.

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For this *conditional coupling*, we need an appropriate state space:

$$\widetilde{\Omega} = \left\{ \underline{\omega} \;:\; \begin{cases} \limsup\limits_{i \to -\infty} \dfrac{1}{|i|} \displaystyle\sum_{j=i+1}^{0} |\omega_j| < \infty \\[2ex] \limsup\limits_{i \to \infty} \dfrac{1}{i} \displaystyle\sum_{j=1}^{i} |\omega_j| < \infty \end{cases} \right\}$$

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$\rightsquigarrow$ In this case we have a limiting process in infinite volume, for which it is almost impossible to blow up.

$\rightsquigarrow$ But what is this process?

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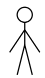
c.) True bricklayers are laying the bricks at each site 

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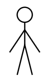
c.) True bricklayers are laying the bricks at each site 

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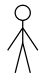
c.) True bricklayers are laying the bricks at each site  (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

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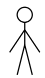
c.) True bricklayers are laying the bricks at each site  (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

d.) The product measure $\underline{\mu}^{\theta}$ is stationary, and the process in this distribution is ergodic.

3*a*.) Right-continuity and the Markov property

Idea: the limit for the space-time box $[a, b] \times [0, t]$ is already achieved in finite volume.

That is, a.s. there exist (random) ℓ and $\mathfrak{r}$, such that the heights $h_i^{[\ell, \mathfrak{r}]}(s)$ of columns of the $[\ell, \mathfrak{r}]$ monotone process agree to the column heights $h_i(s)$ of the limiting process for all $a \leq i \leq b$ and $s \in [0, t]$.

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Right-continuity is OK, and a bit of extra work yields the Markov property as well.

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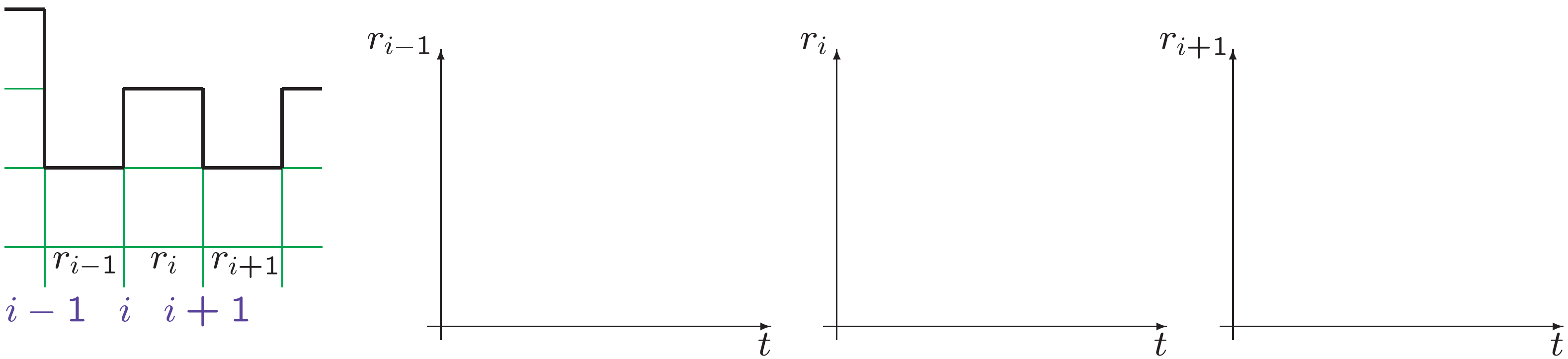

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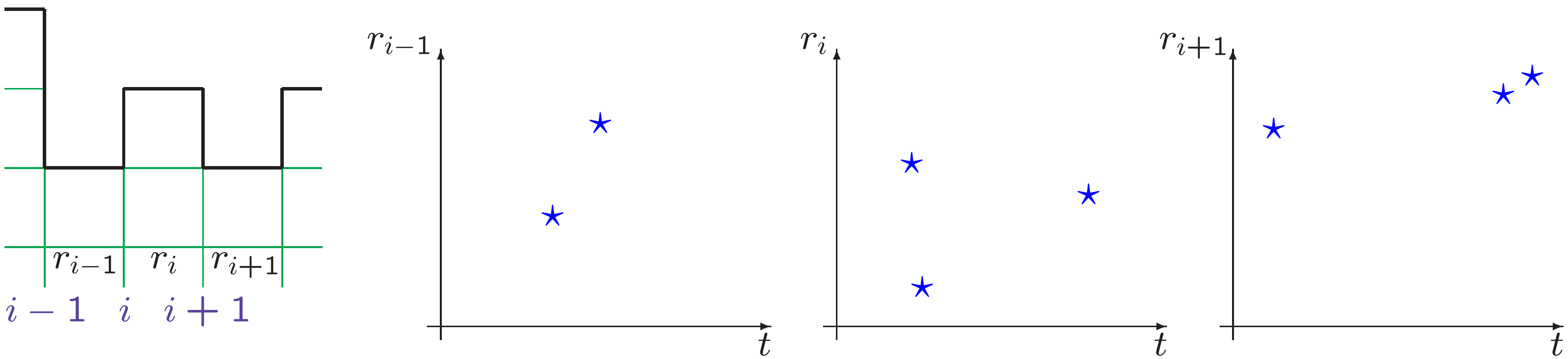

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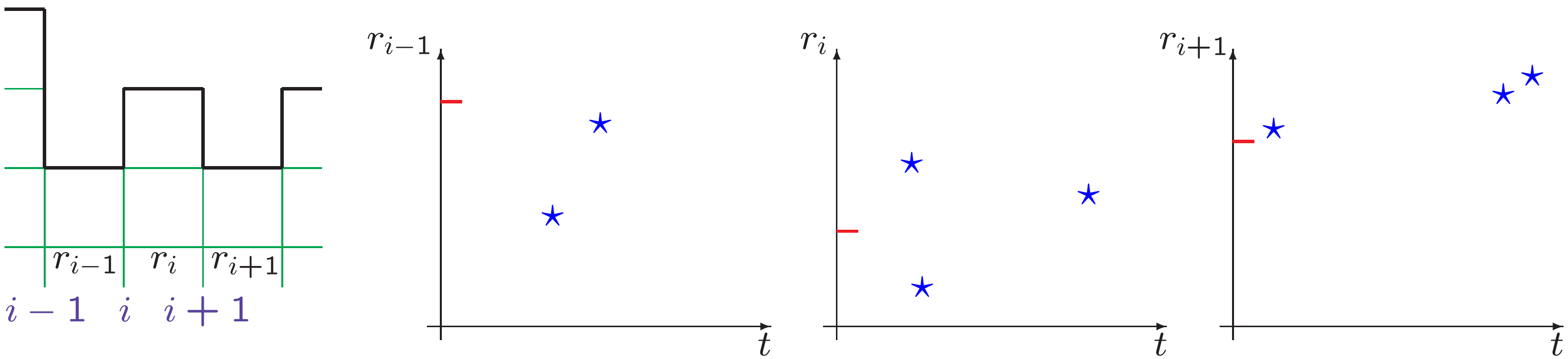

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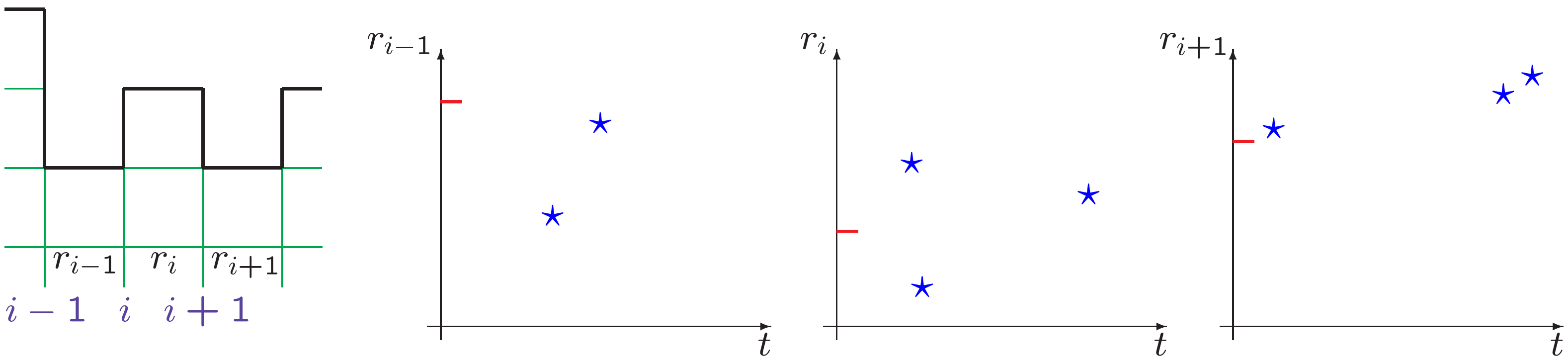

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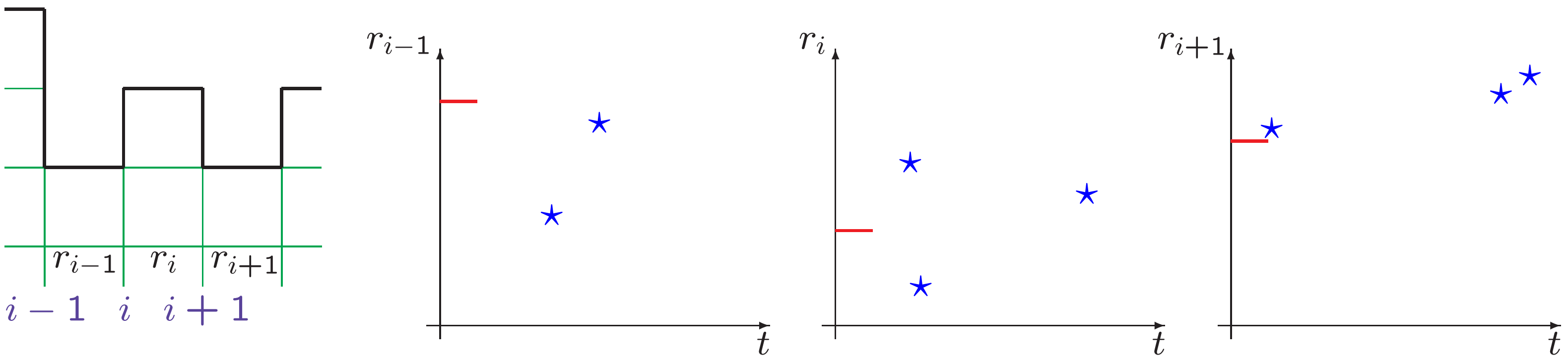

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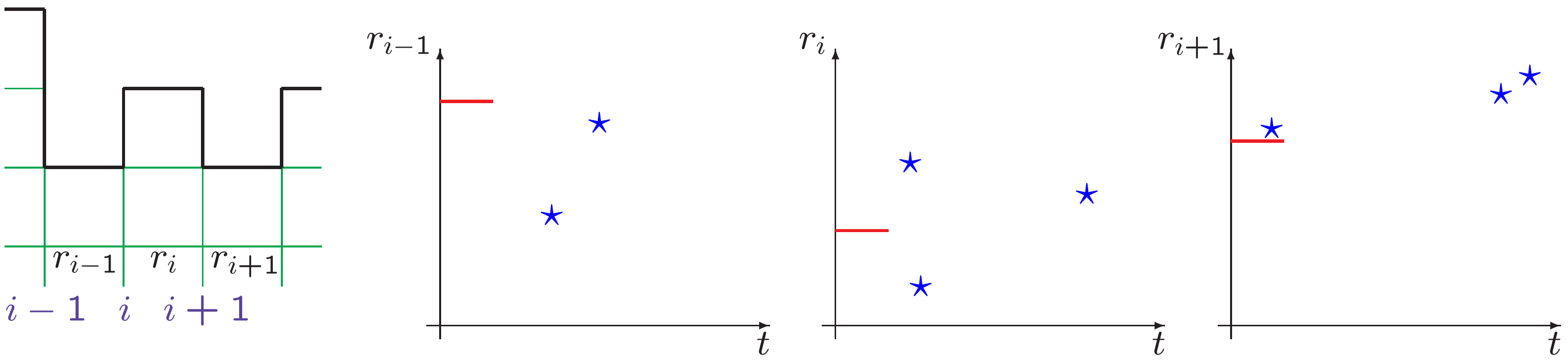

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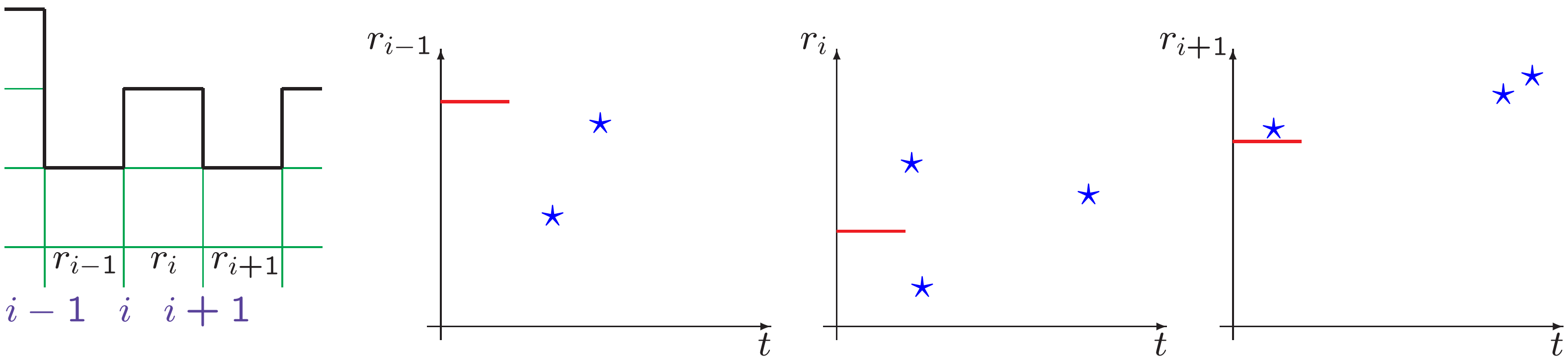

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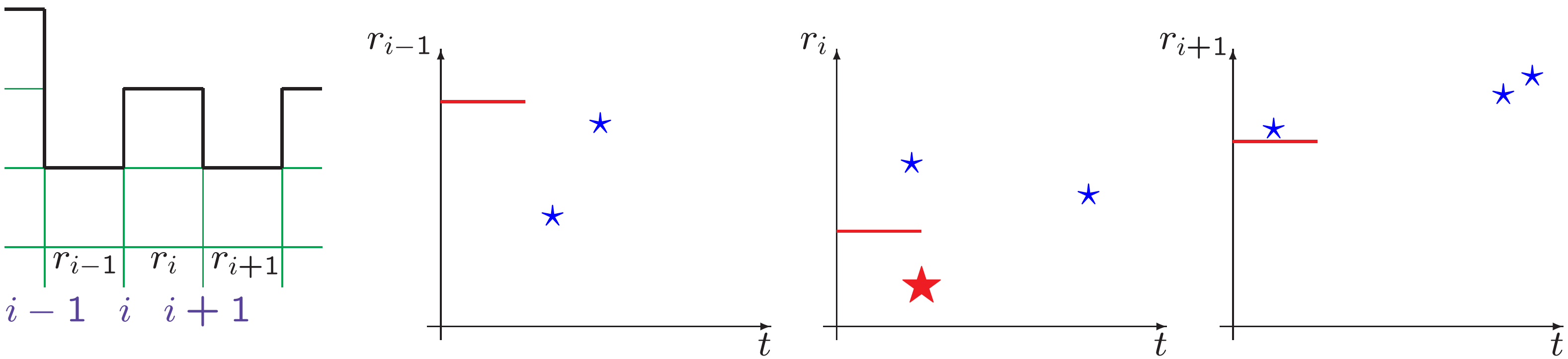

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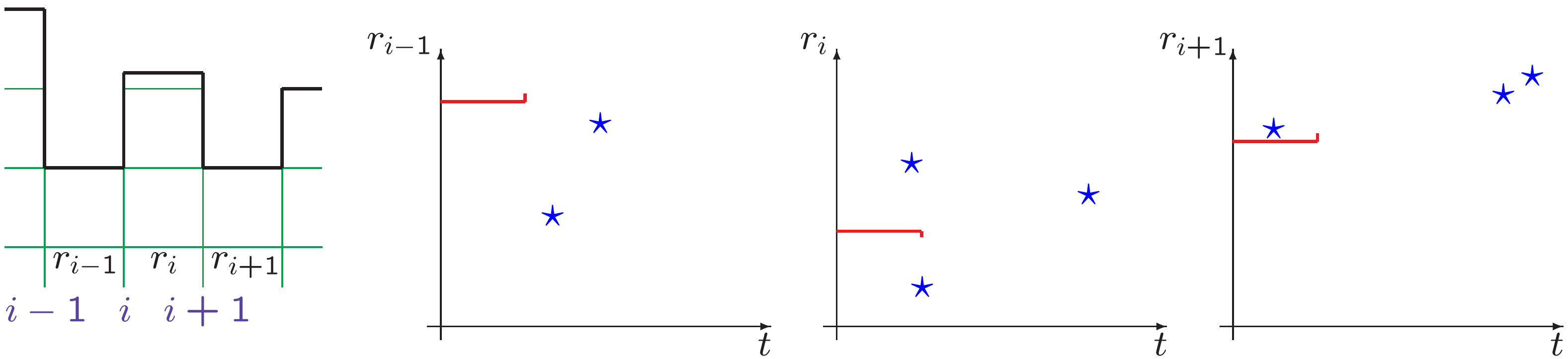

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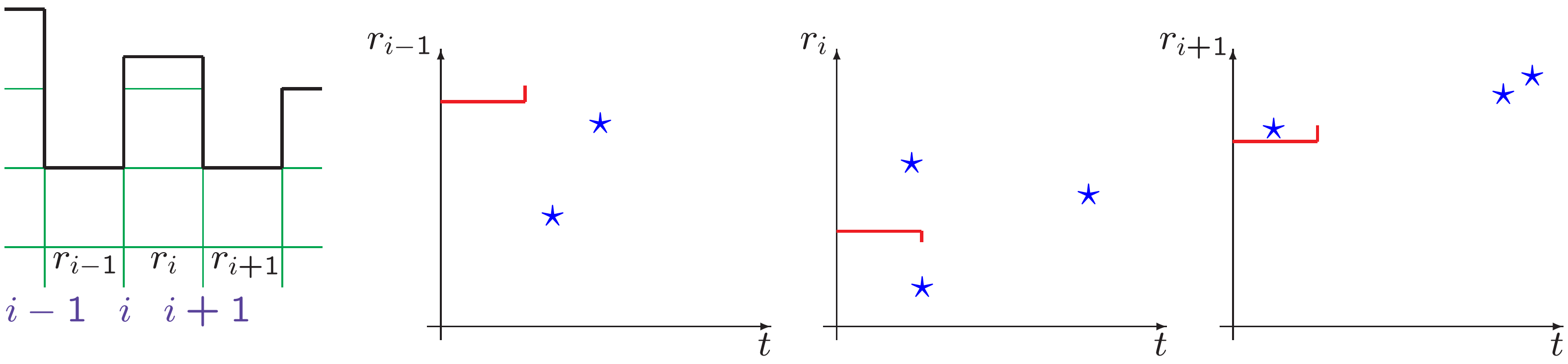

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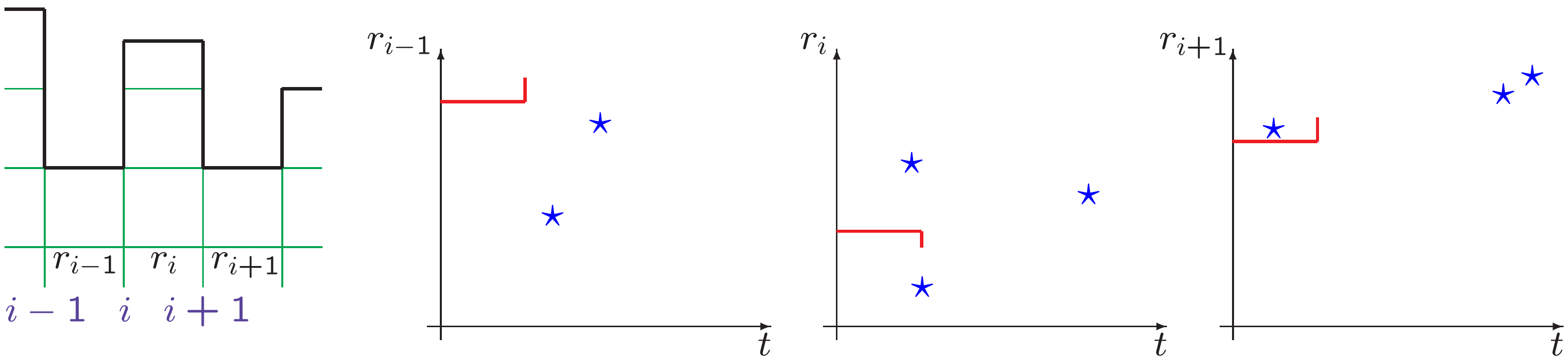

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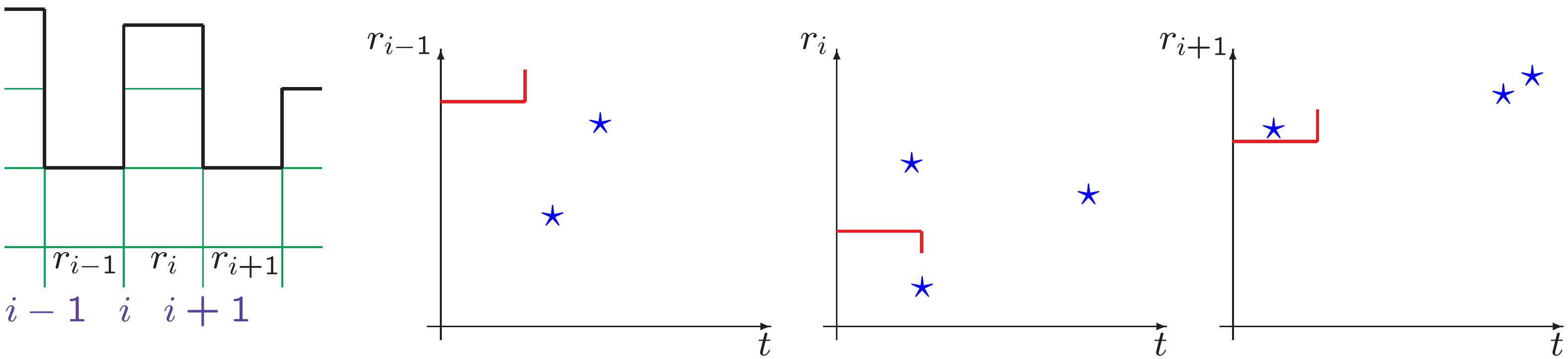

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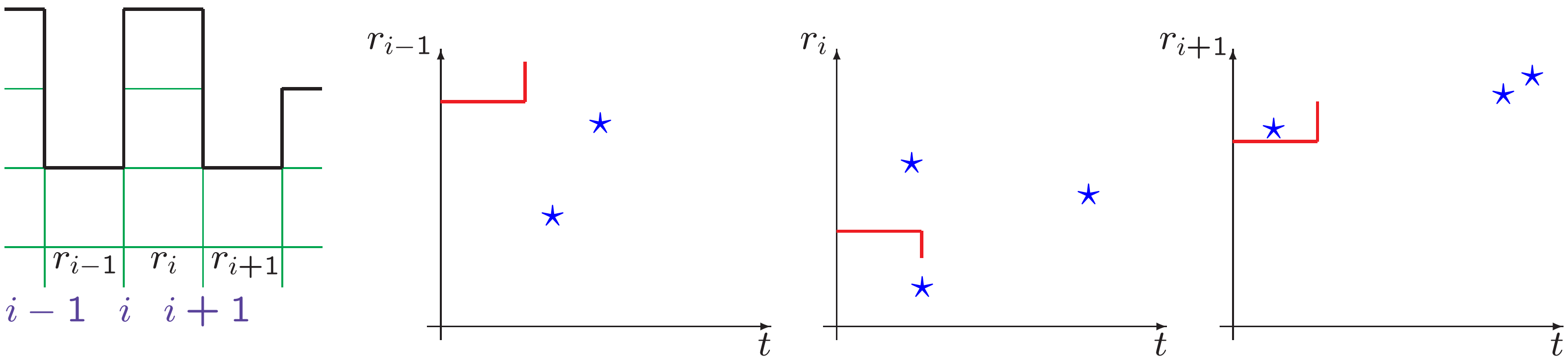

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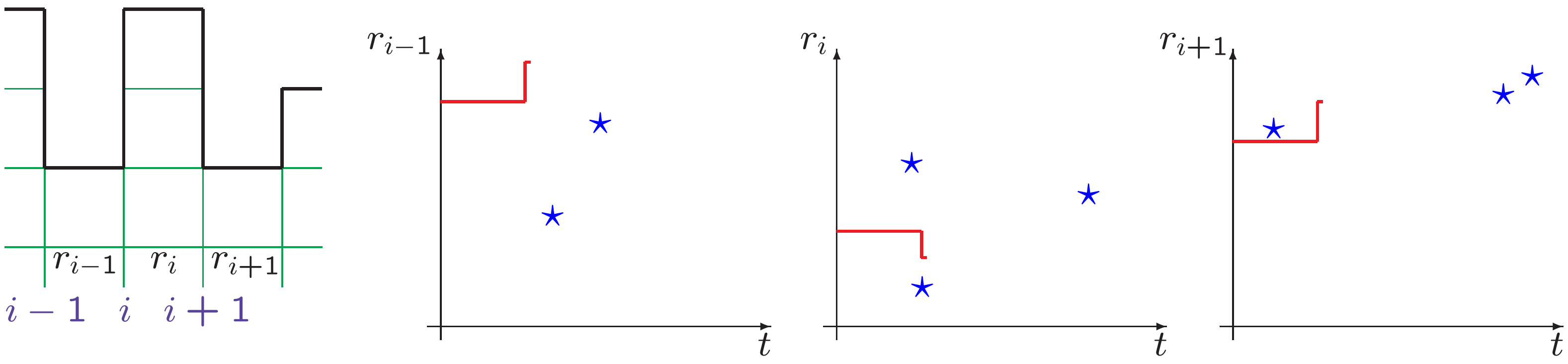

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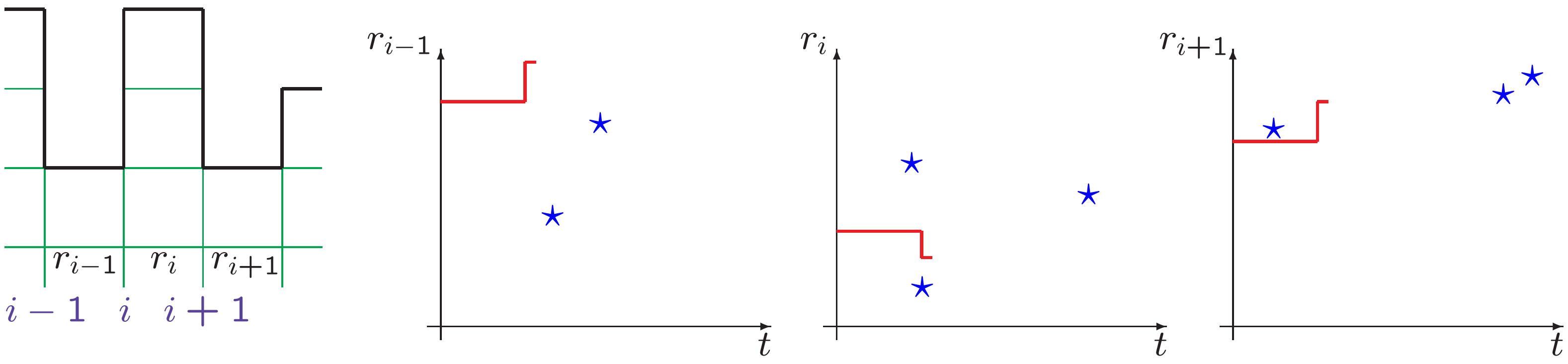

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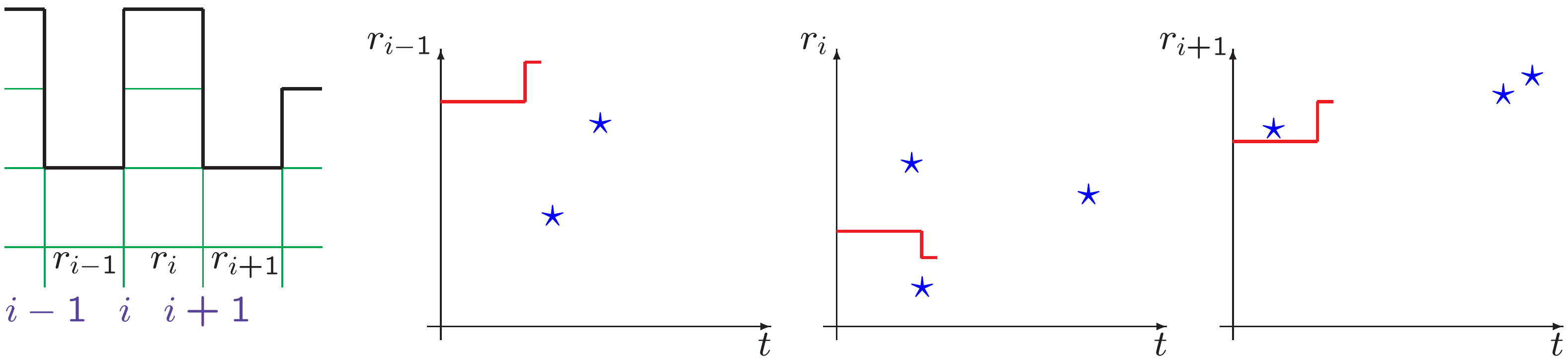

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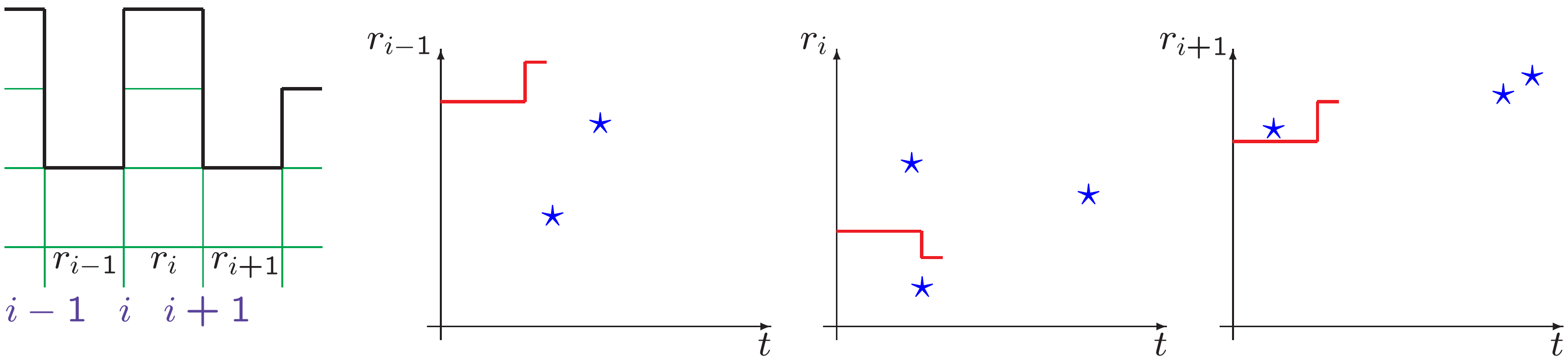

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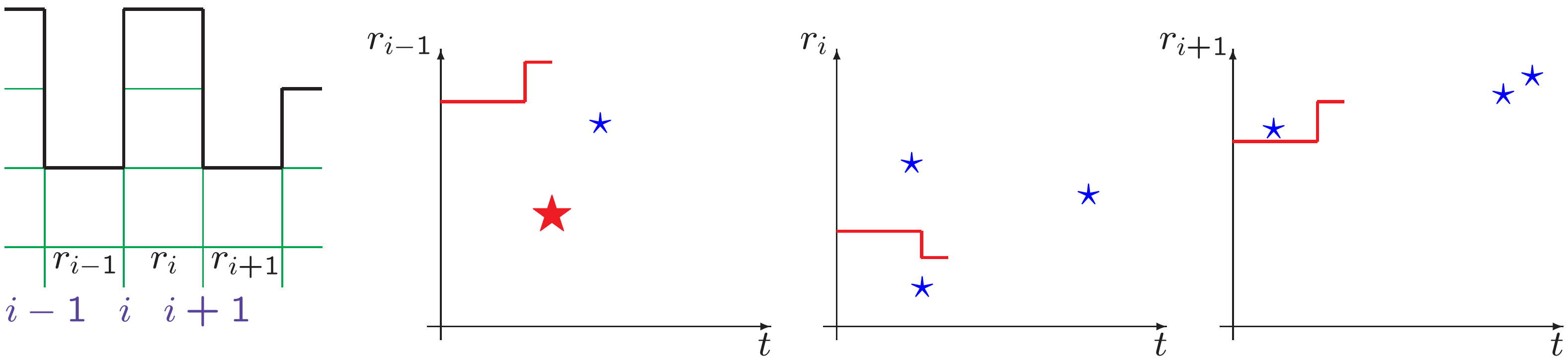

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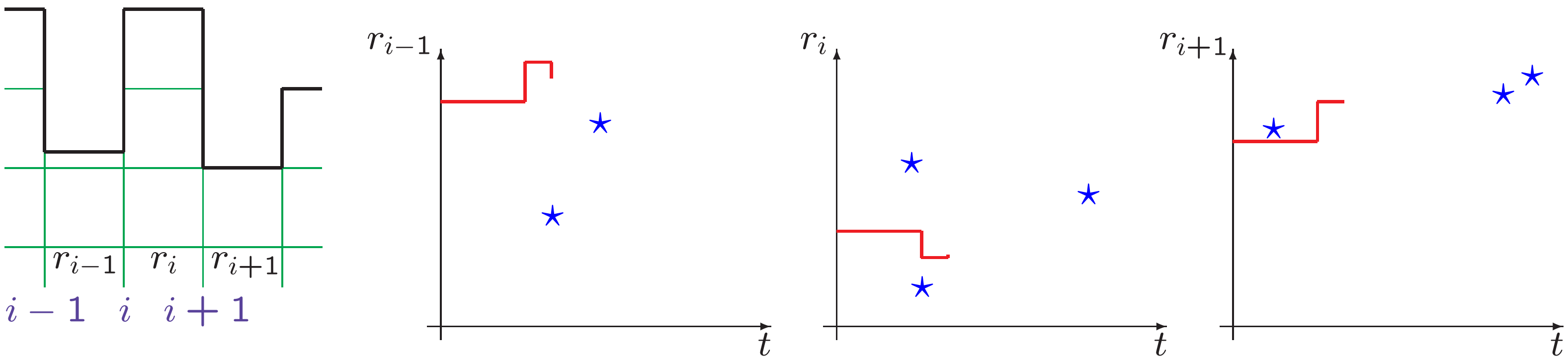

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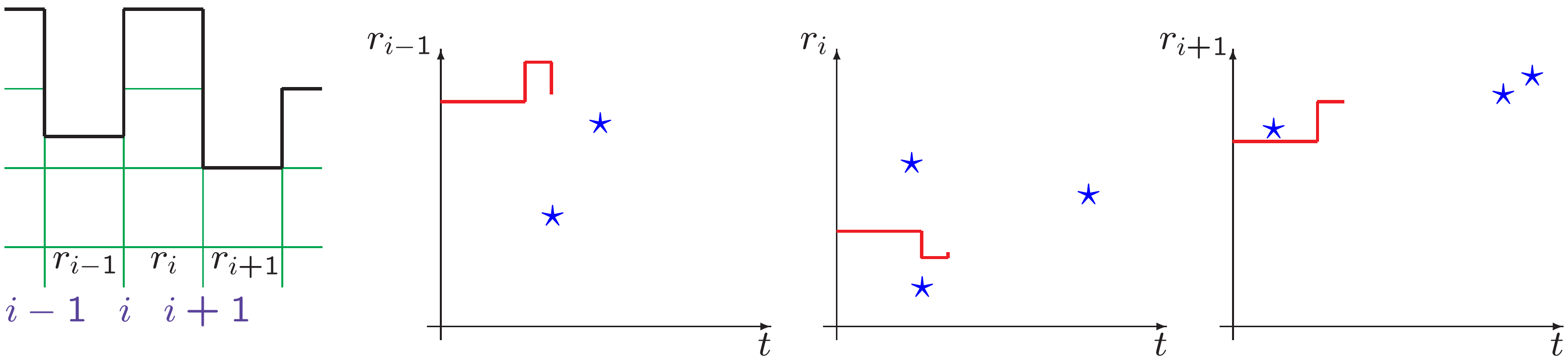

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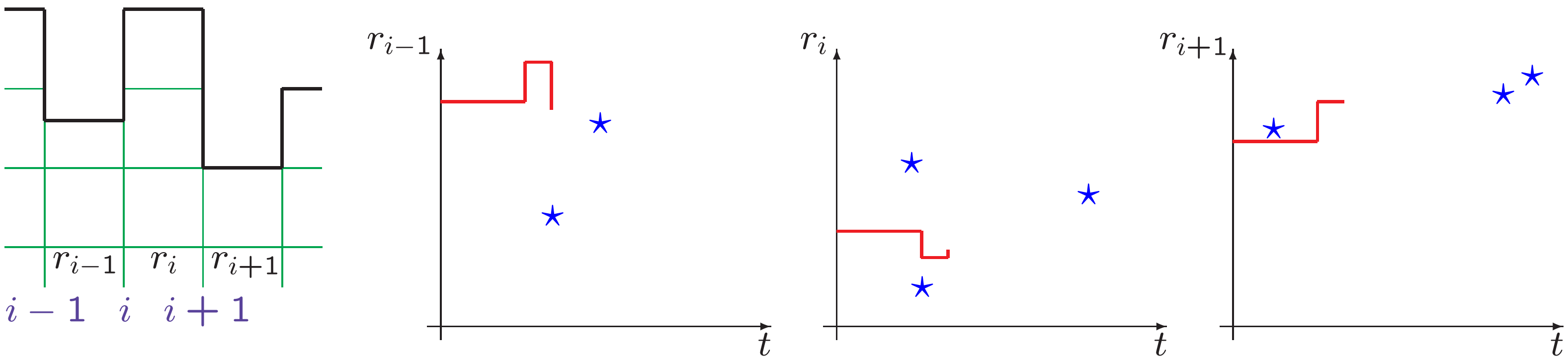

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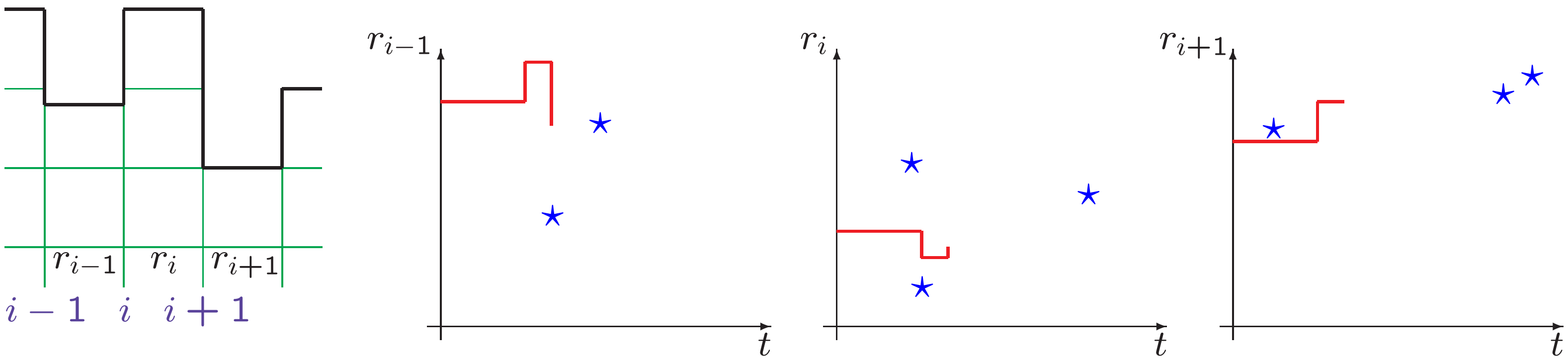

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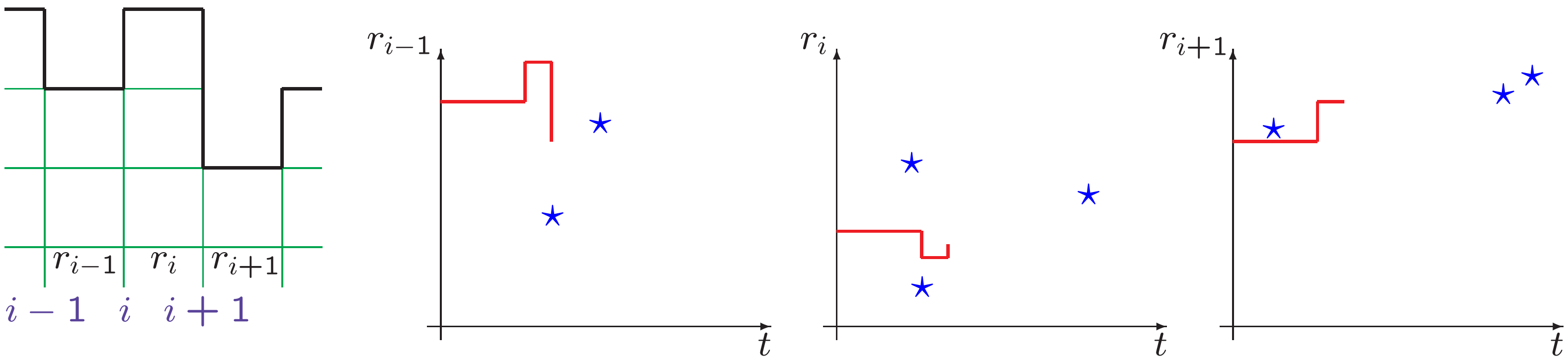

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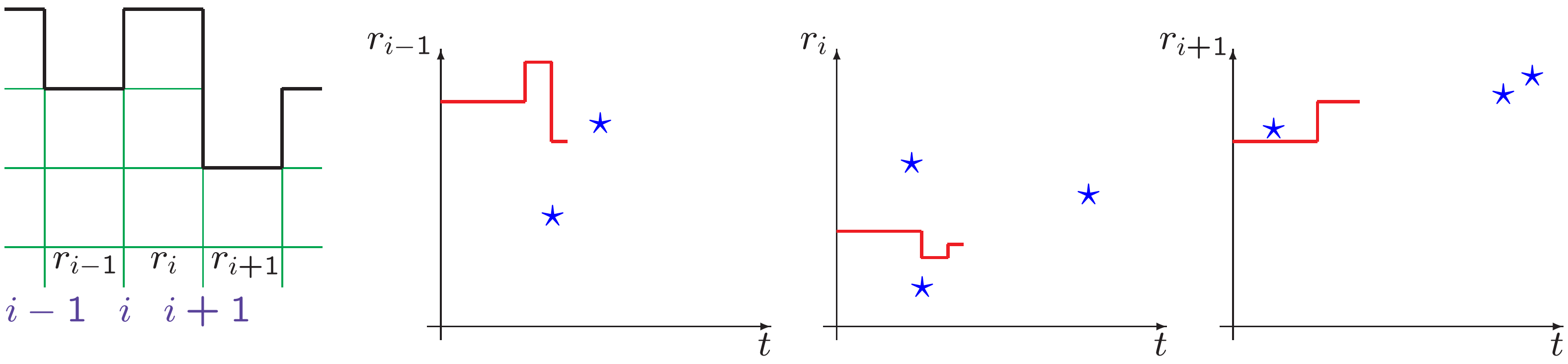

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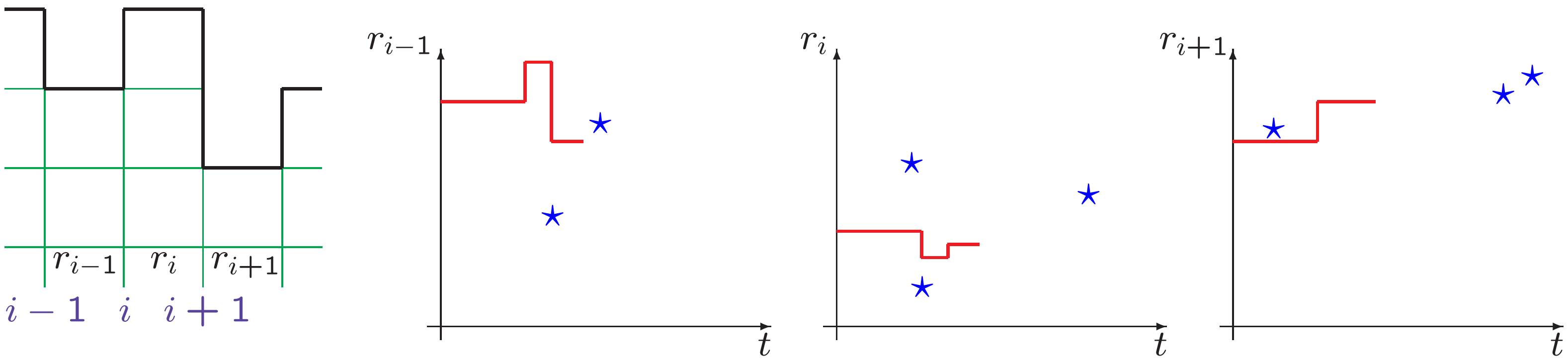

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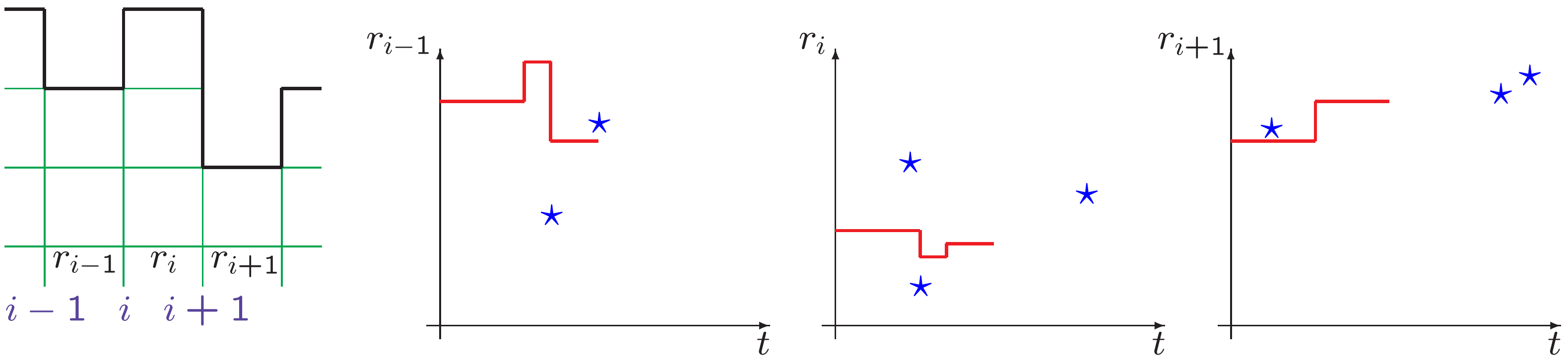

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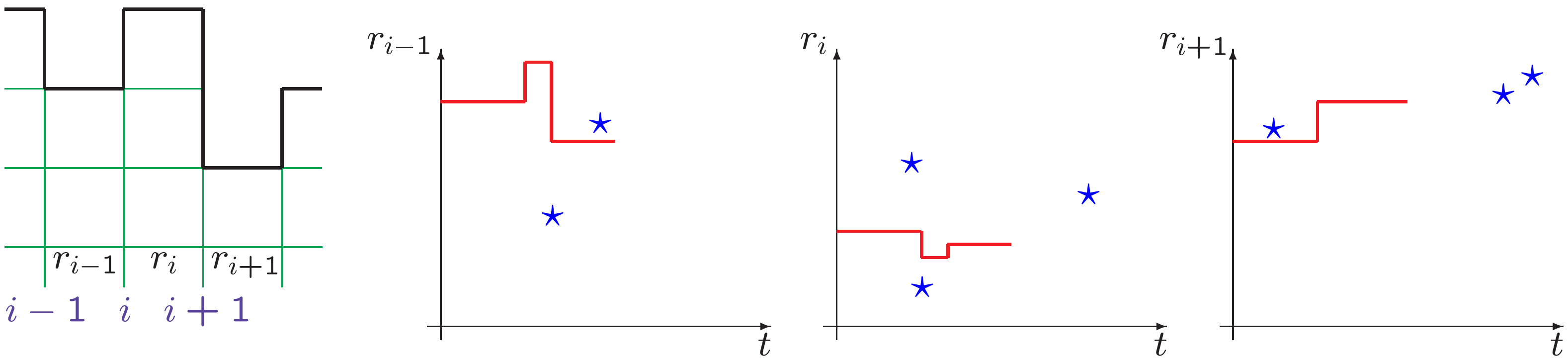

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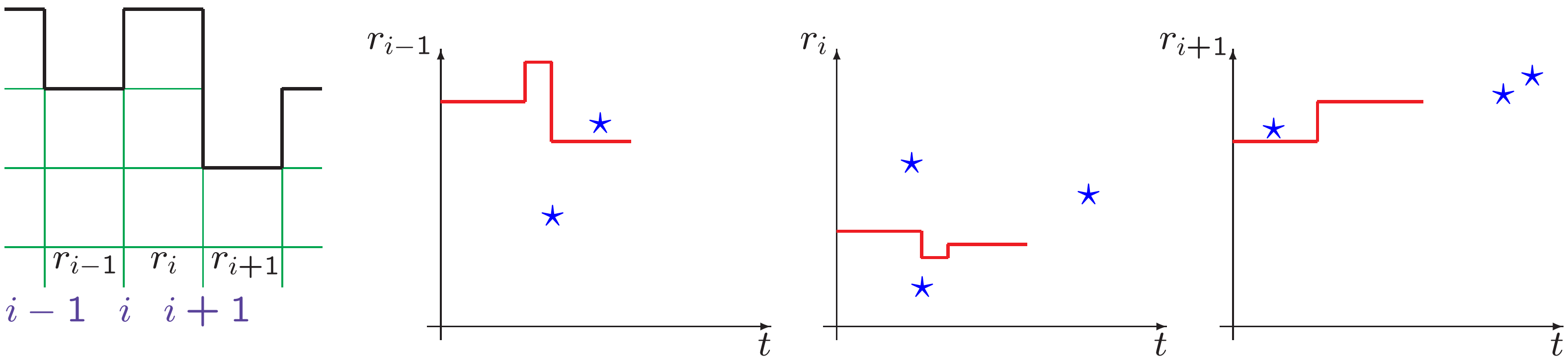

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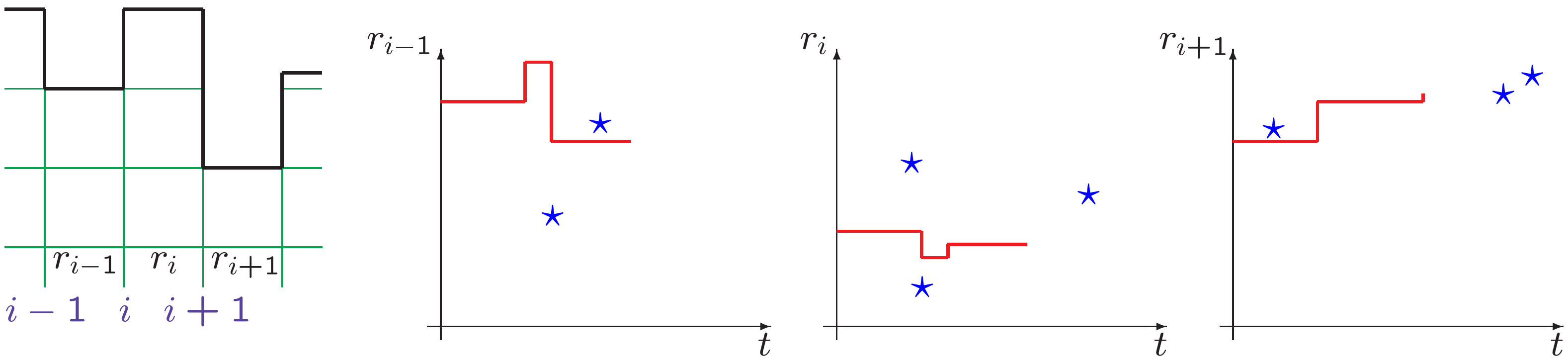

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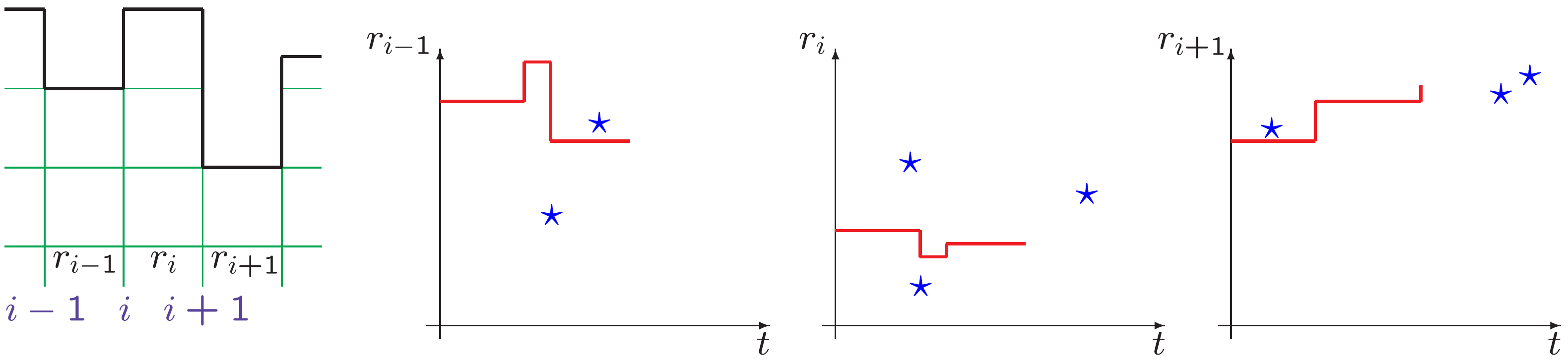

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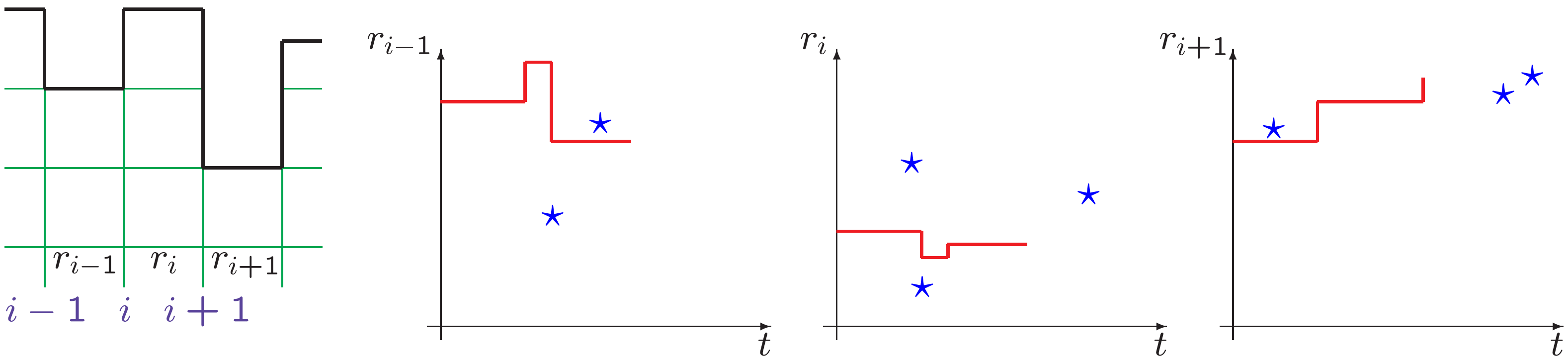

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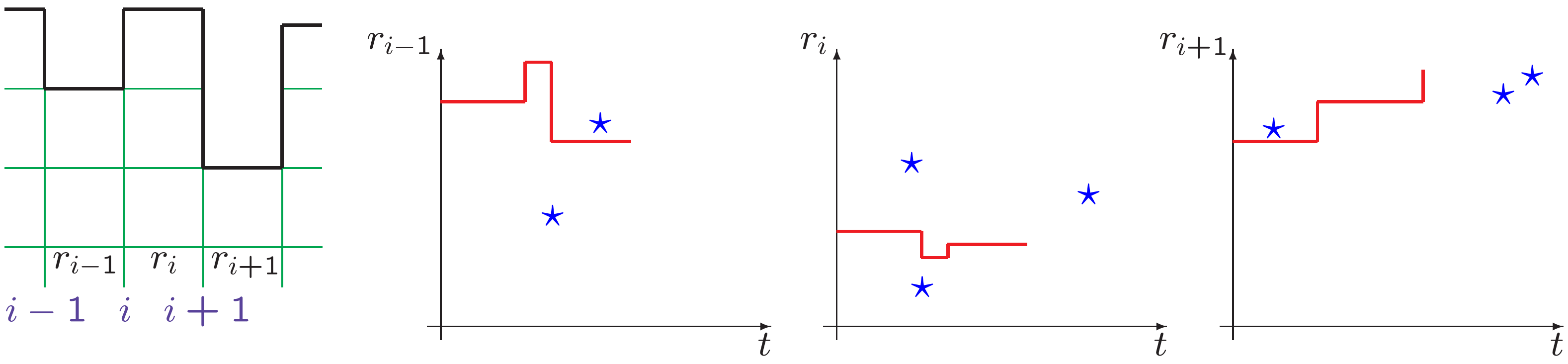

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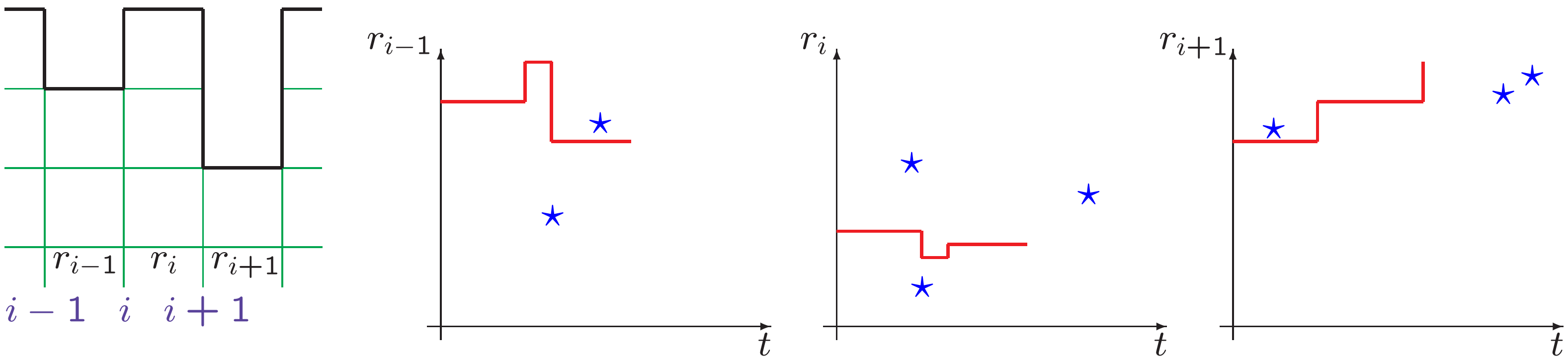

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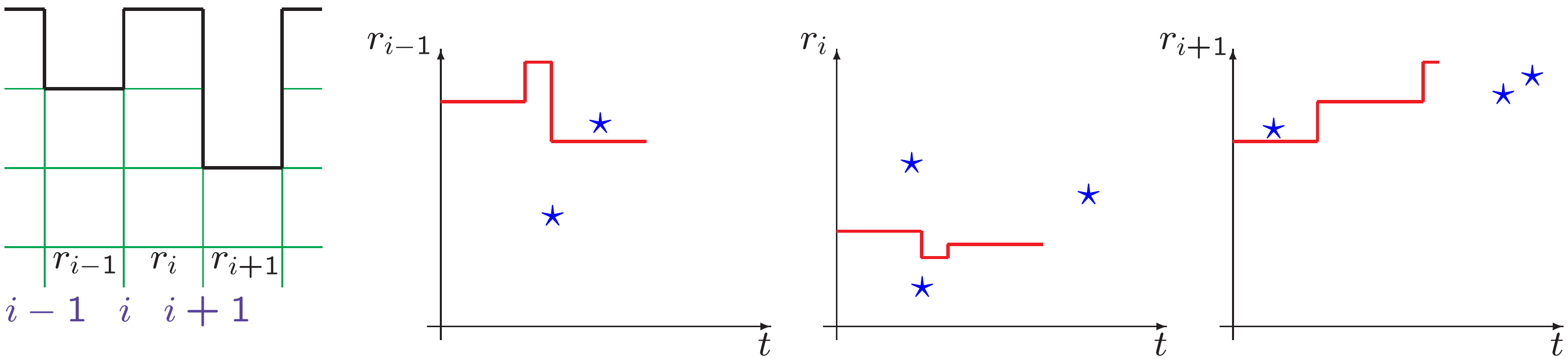

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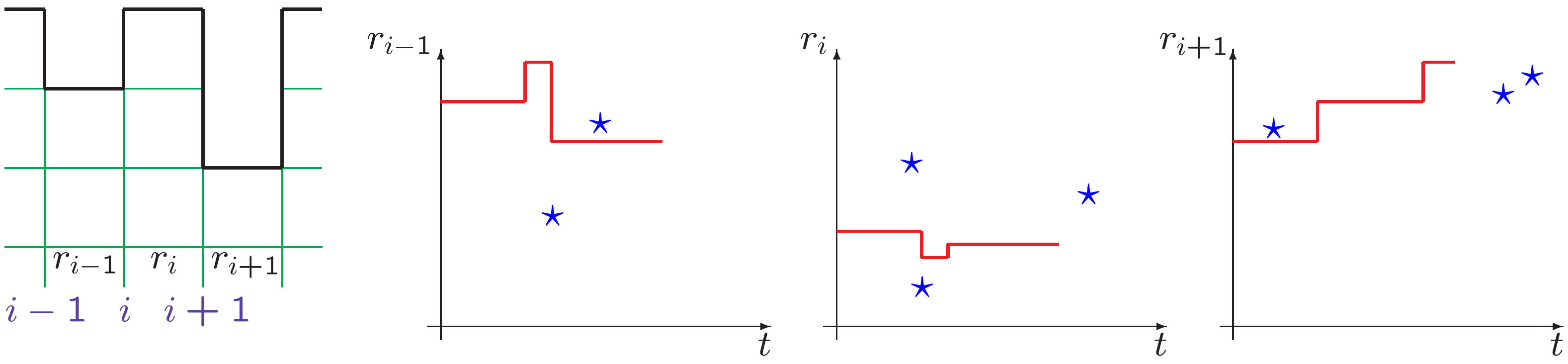

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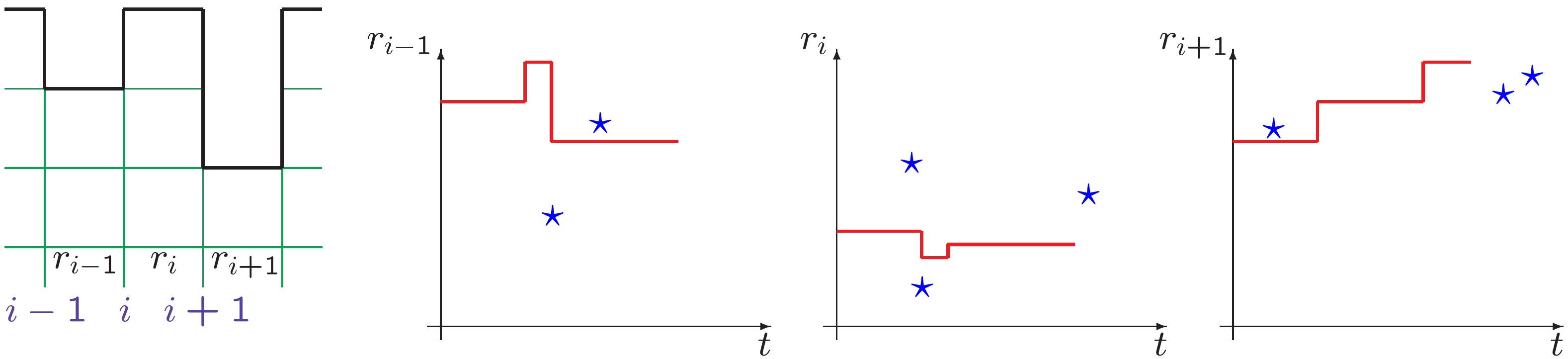

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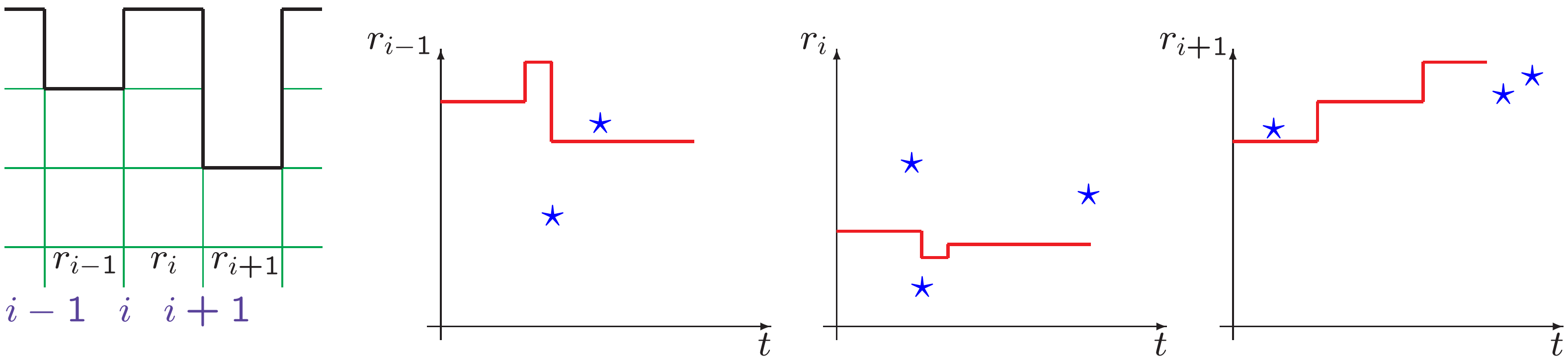

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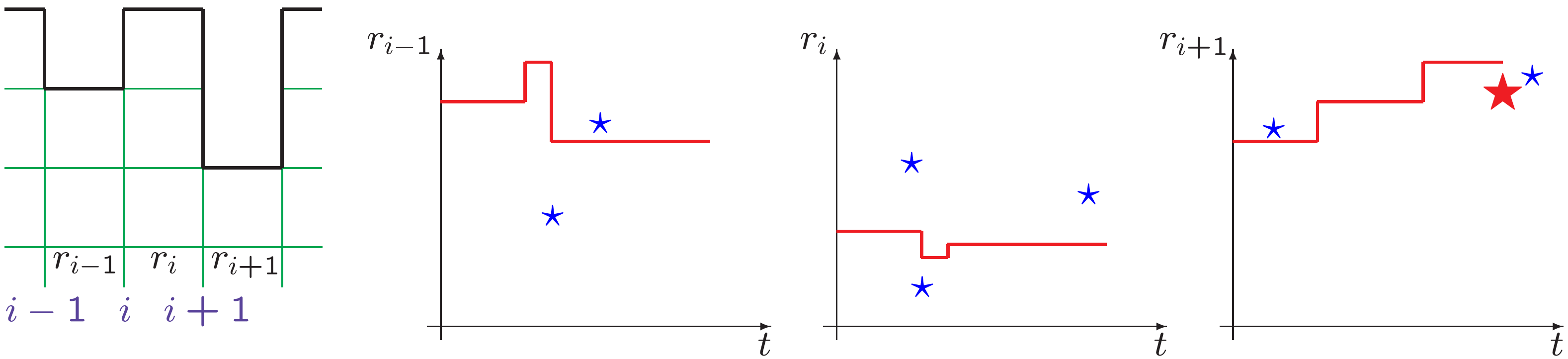

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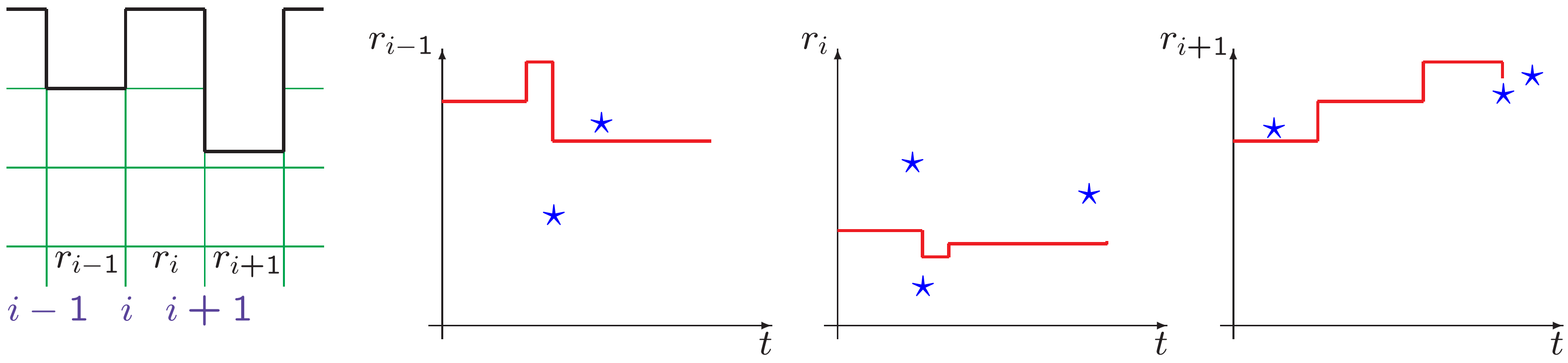

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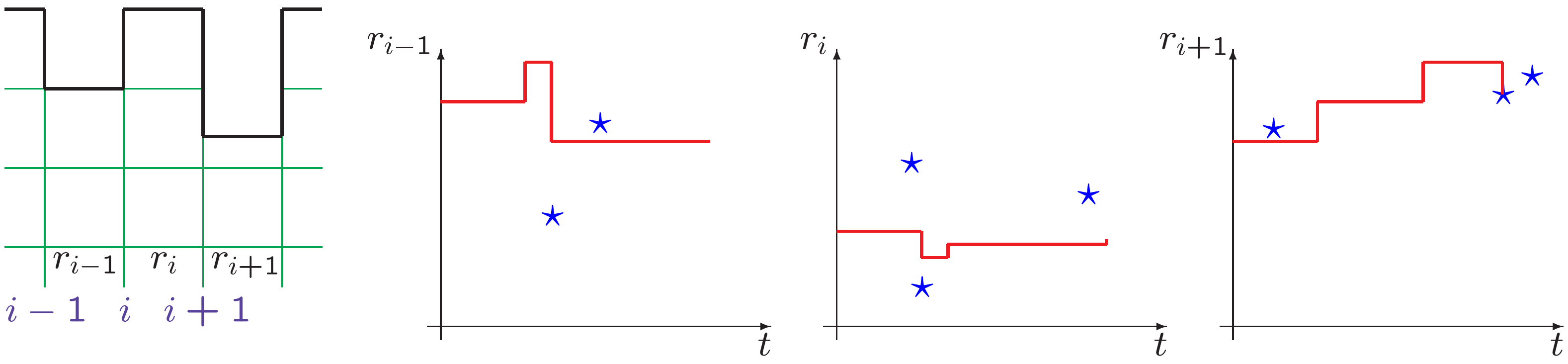

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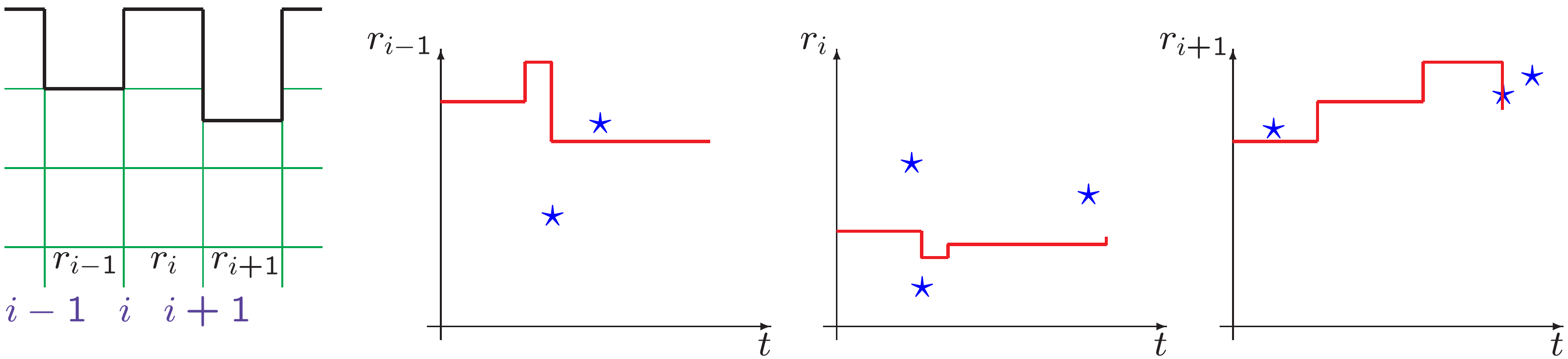

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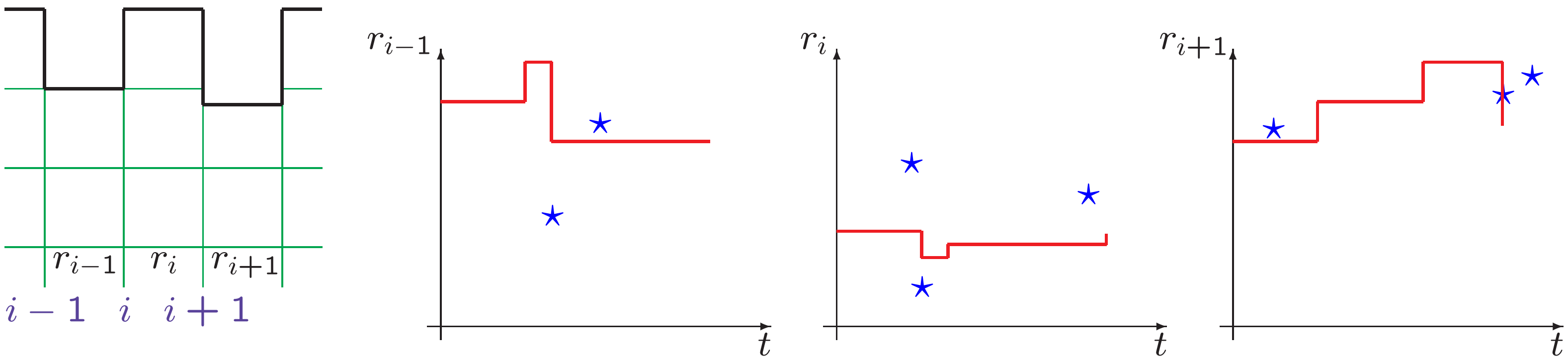

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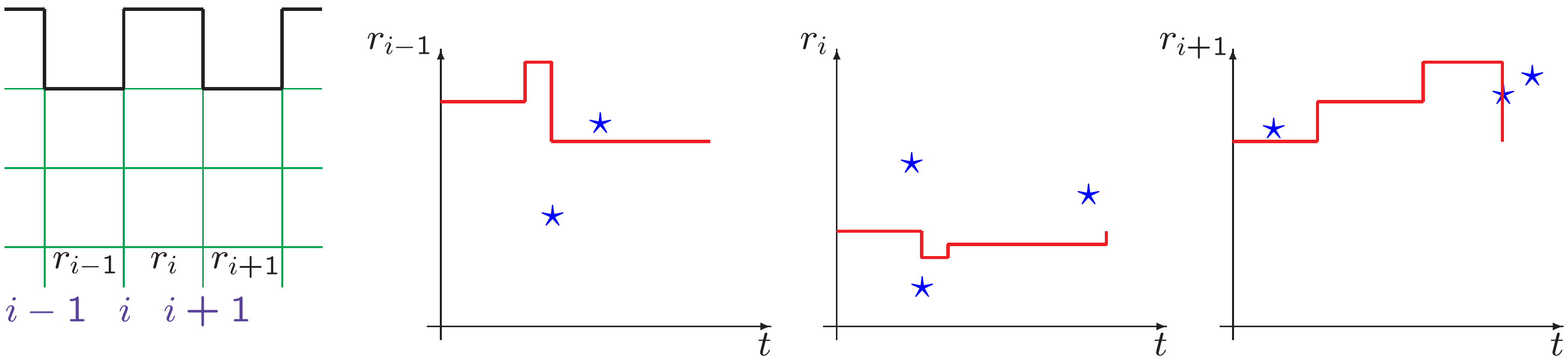

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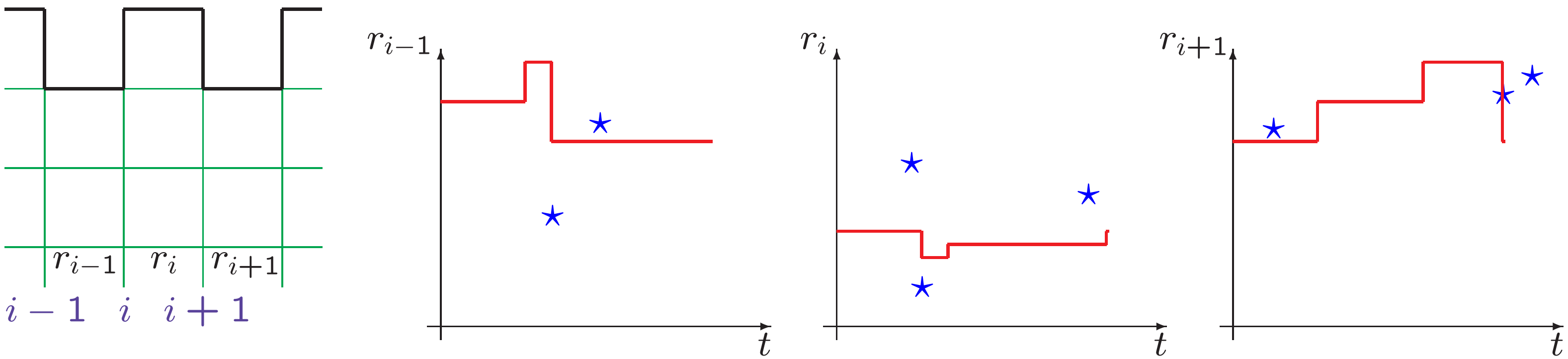

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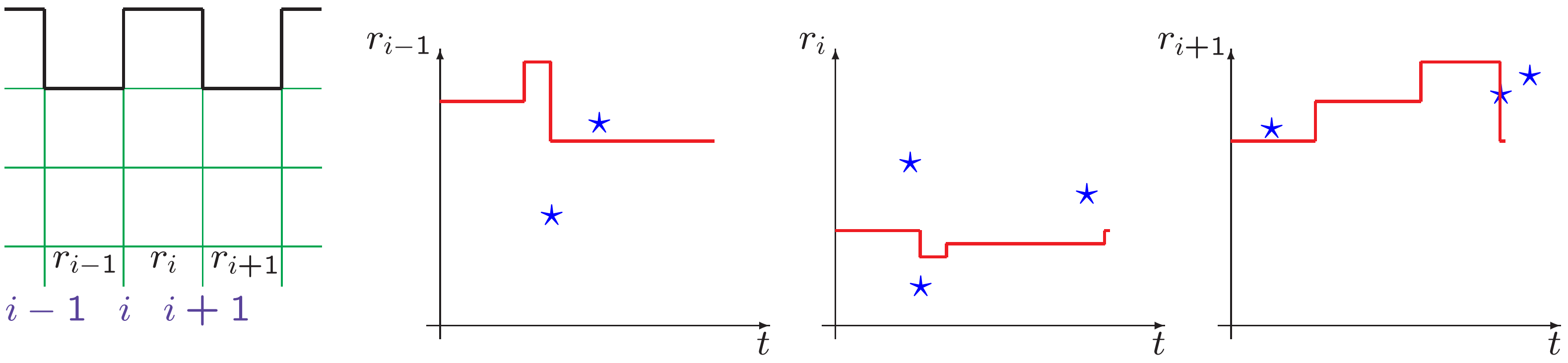

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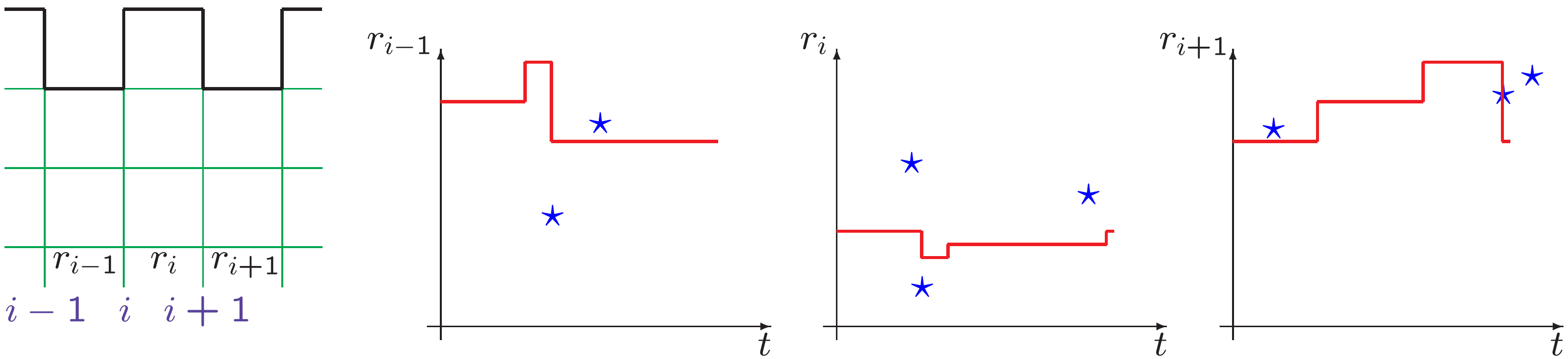

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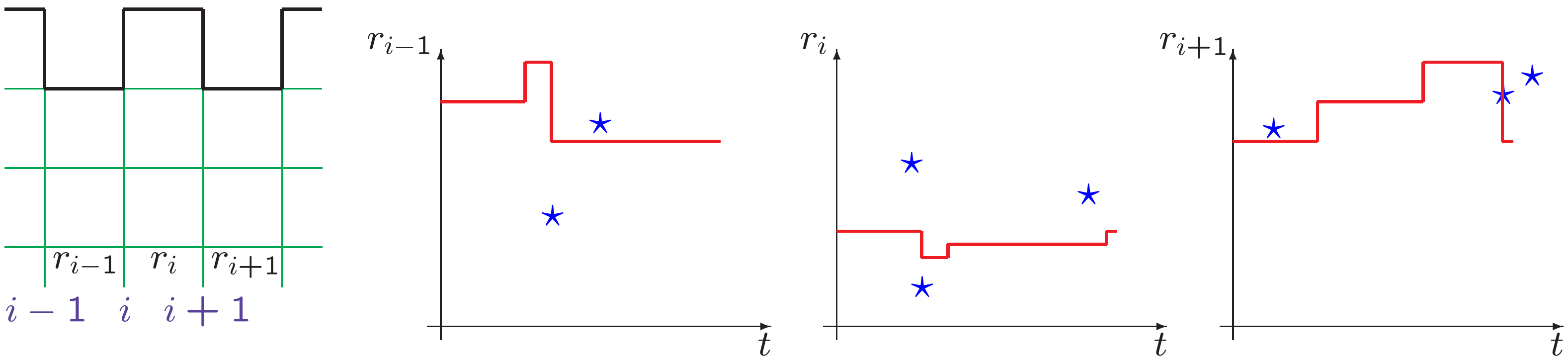

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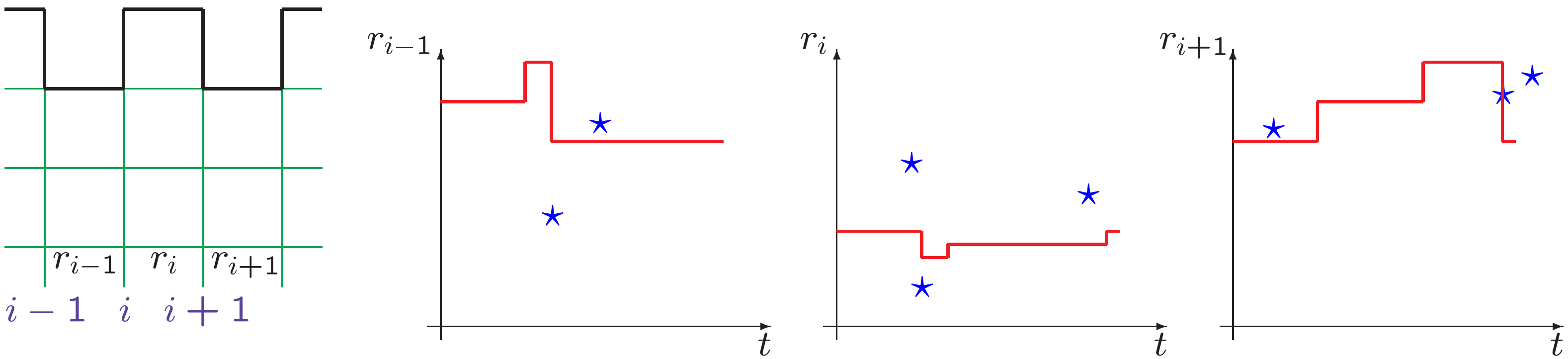

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$\rightsquigarrow$ finitely many $\star$'s govern *each* of the monotone processes in the space-time box $[a, b] \times [0, t]$.

$\rightsquigarrow$ The limit is already achieved in finite volume.

3. Properties: We should see that

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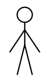
b.) The process (a.s.) stays in the state space $\widetilde{\Omega}$,

c.) True bricklayers are laying the bricks at each site  (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

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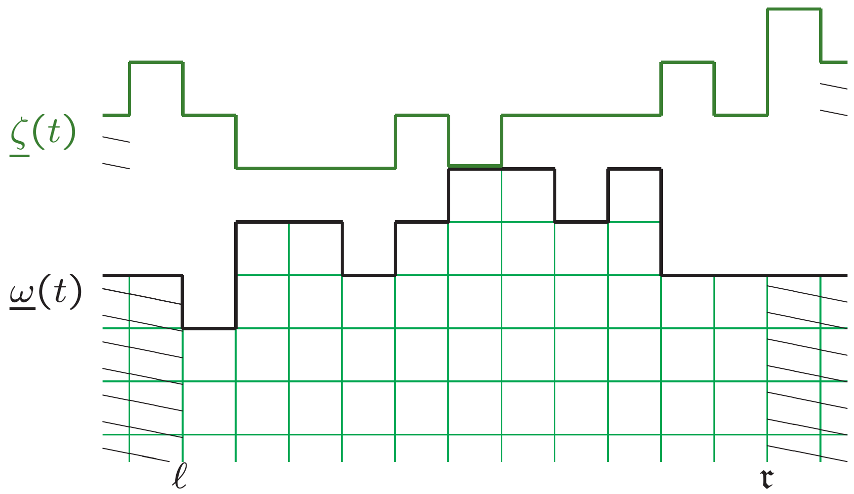

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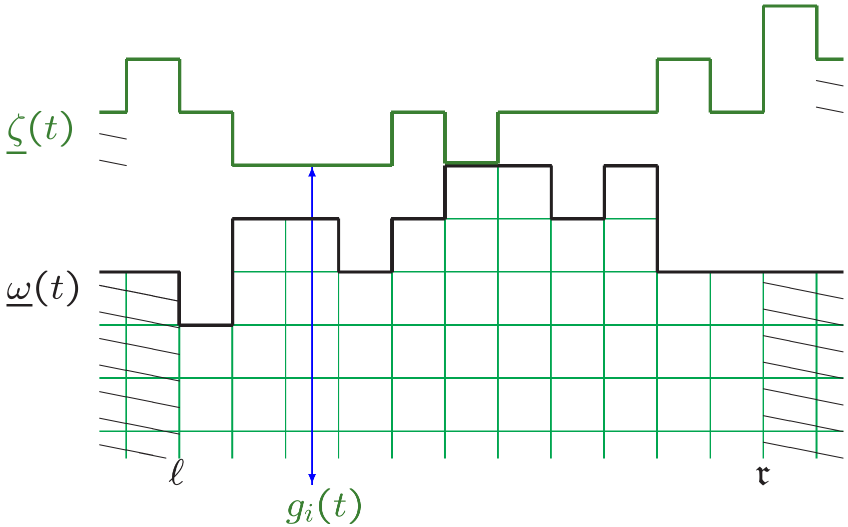

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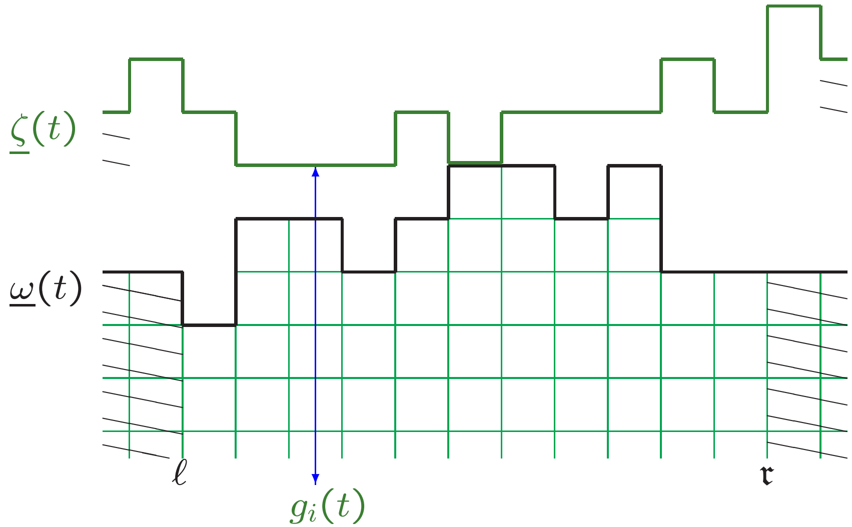

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Second moment:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\, g_i^2(t) = 2\mathbf{E}\, g_i(t) \cdot r_i(t) + \mathbf{E}\, r_i(t)$$

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Similar procedure works for higher moments as well.

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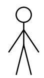
c.) True bricklayers are laying the bricks at each site  (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

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Semigroup (on measurable functions φ):

$$S(t)\varphi(\underline{\omega}) := \mathbf{E}[\varphi(\underline{\omega}(t)) \,|\, \underline{\omega}(0) = \underline{\omega}]$$

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Formal generator (on cylinder functions φ):

$$L\varphi(\underline{\omega}) = \sum_i r_i(\underline{\omega}) \cdot [\varphi(\underline{\omega}^{(i,\,i+1)}) - \varphi(\underline{\omega})],$$

where $\underline{\omega}^{(i,\,i+1)} = \underline{\omega} +$ one brick $= \ldots,\, \omega_{i-1},\, \omega_i - 1,\, \omega_{i+1} + 1,\, \omega_{i+2},\, \ldots$

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The moment computation above does not work.

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By contradiction, assume that *each* column has grown by time t in a large interval.

In the equilibrium process $\underline{\zeta}$, most of the $\zeta_i(t)$ values are not too large and not too small, hence the growth rates are not too large either.

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By contradiction, assume that *each* column has grown by time t in a large interval.

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$\rightsquigarrow$ So: For small enough t

$$\mathbf{P}\{\text{every column grew by time } t \text{ in } [0,\, i]\} \le \mathrm{e}^{-C \cdot i}.$$

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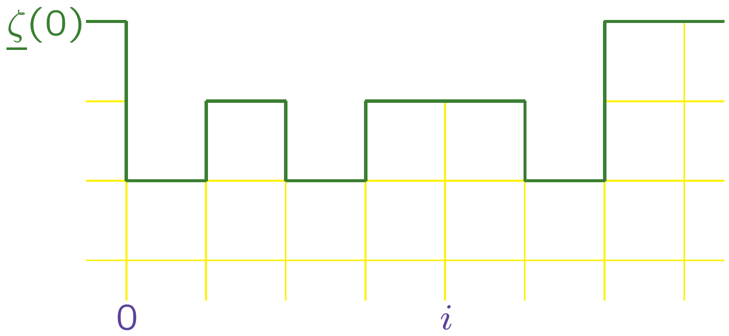

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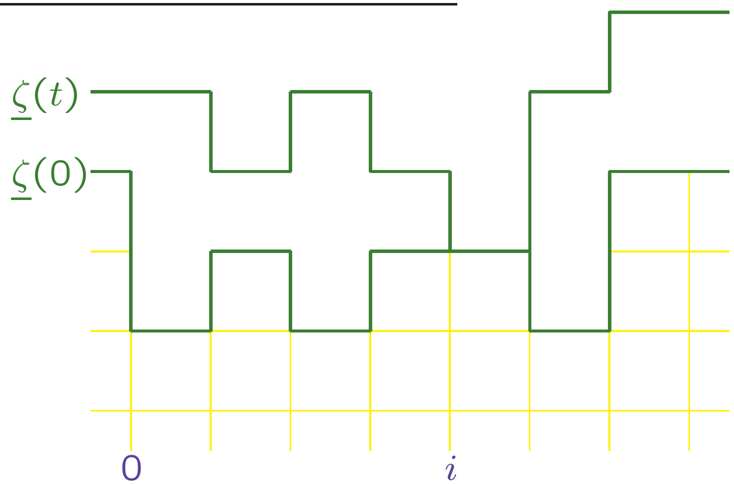

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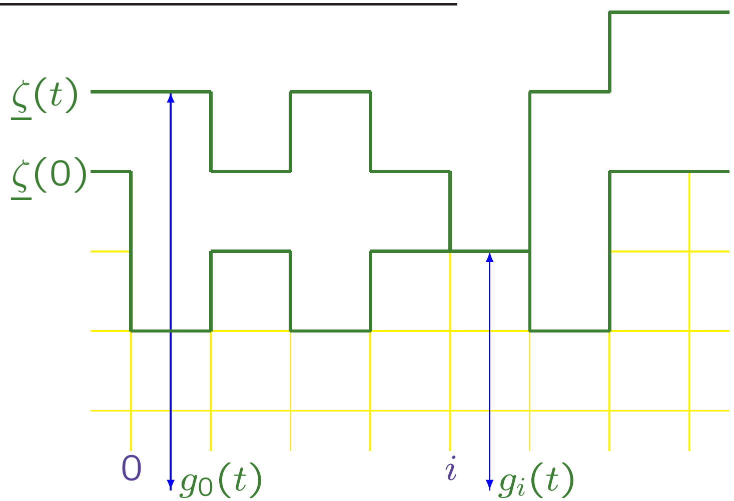

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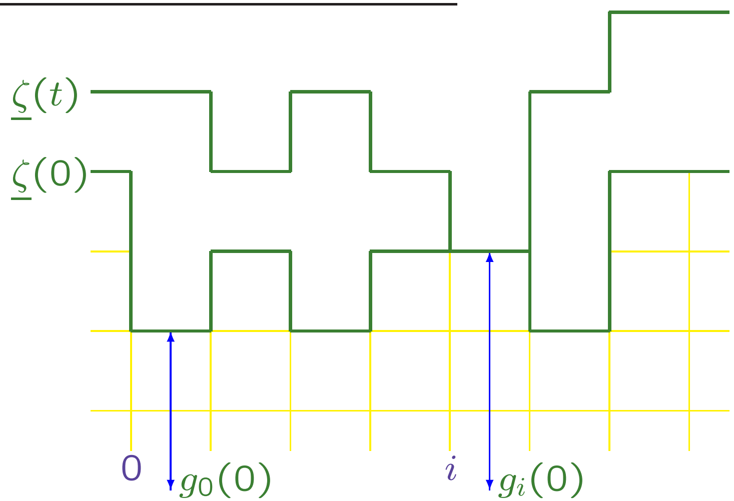

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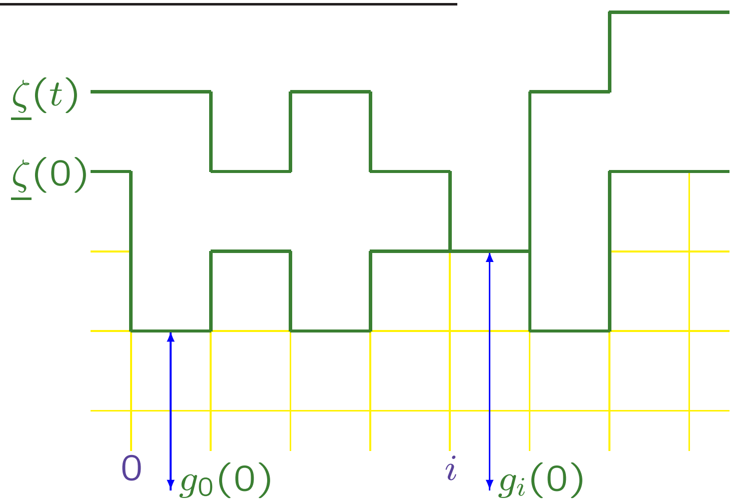

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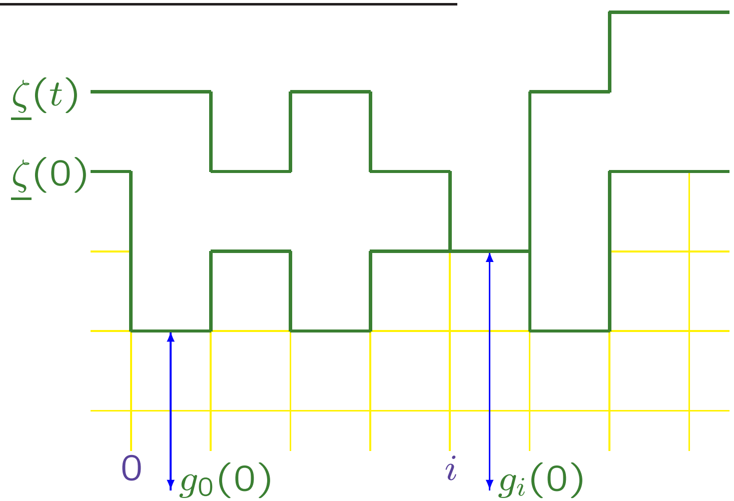

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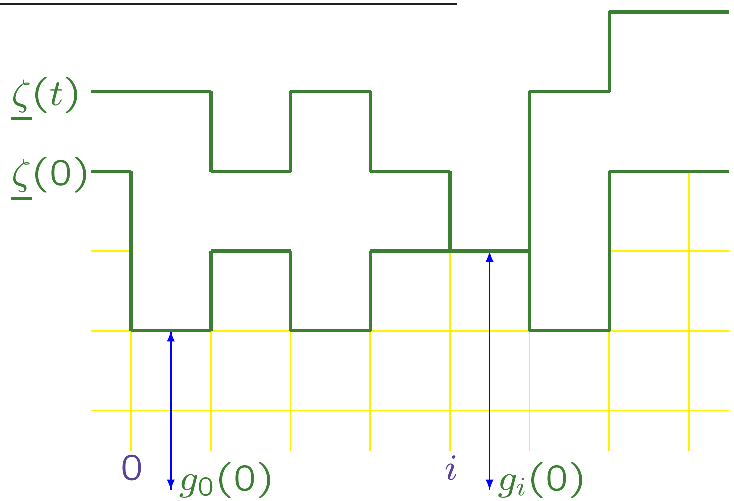

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Everyone on the right-hand side has exponential moments.

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$\rightsquigarrow$ Up to some time $T = T^{\underline{\omega}}$, we have the Kolmogorov forward and backward equations (also in differential form).

3. Properties: We should see that

a.) What we have is a right-continuous (in time) Markov process,

b.) The process (a.s.) stays in the state space $\widetilde{\Omega}$,

c.) True bricklayers are laying the bricks at each site (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

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Ergodicity is a bit more difficult. The *ergodicity* of such a non-countable state space process is characterized by:

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• Invariant functions are trivial: $S(t)\varphi = \varphi$ a.s. $\Rightarrow$ φ $\underline{\mu}^{\theta}$ is a.s. constant.

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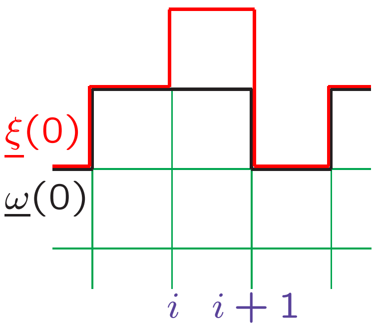
Start $\underline{\xi}$ with one extra brick compared to $\underline{\omega}$:



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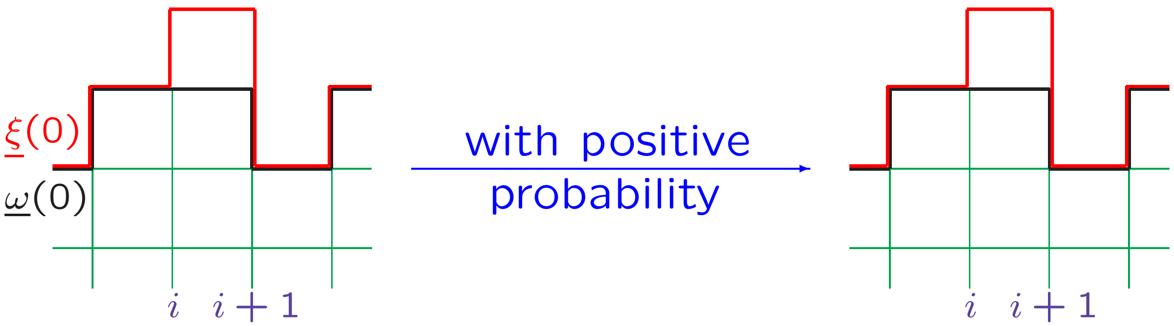

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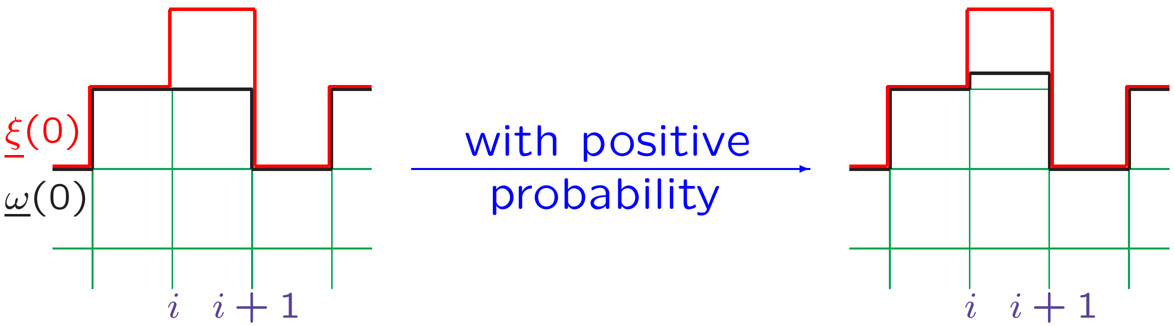

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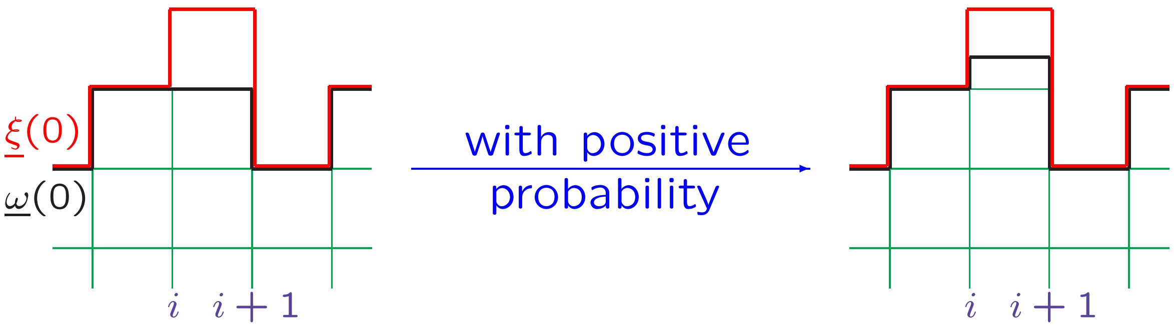

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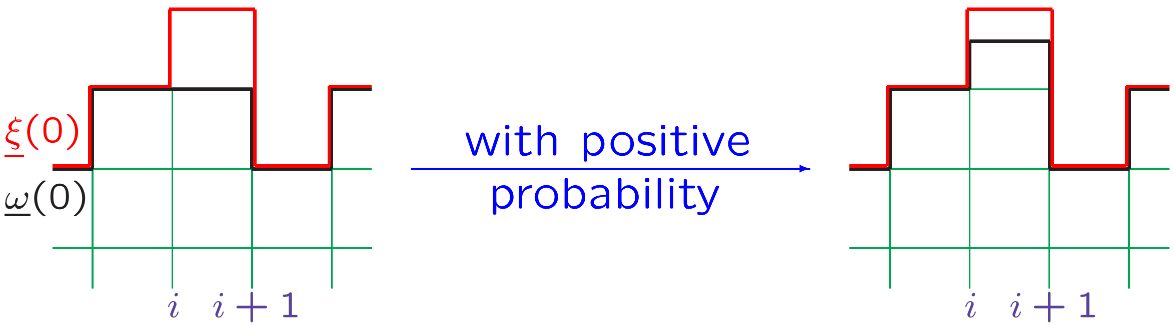

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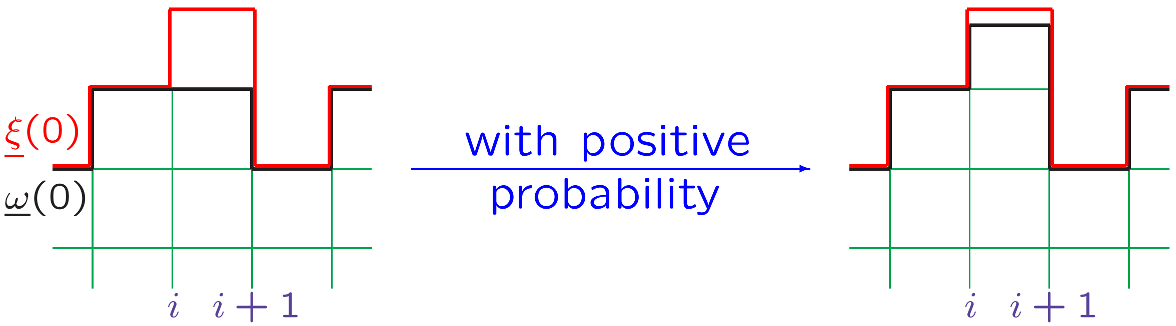

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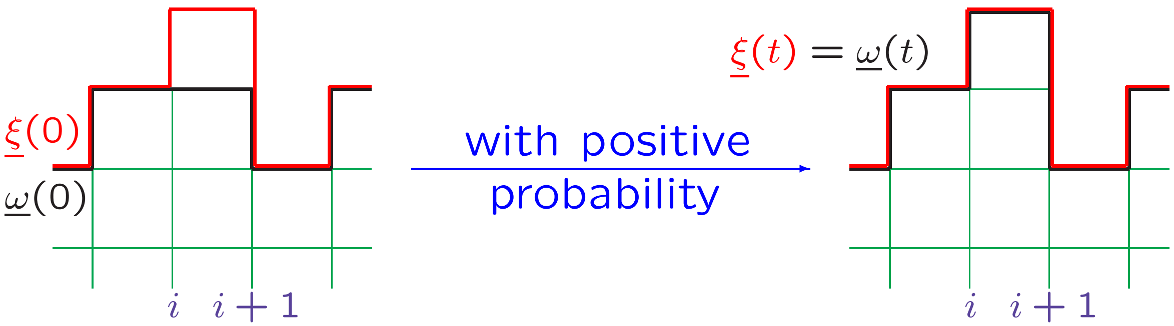

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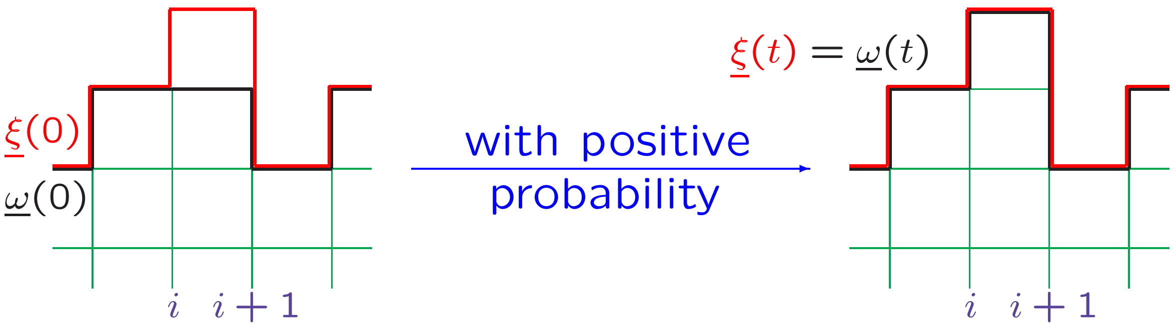

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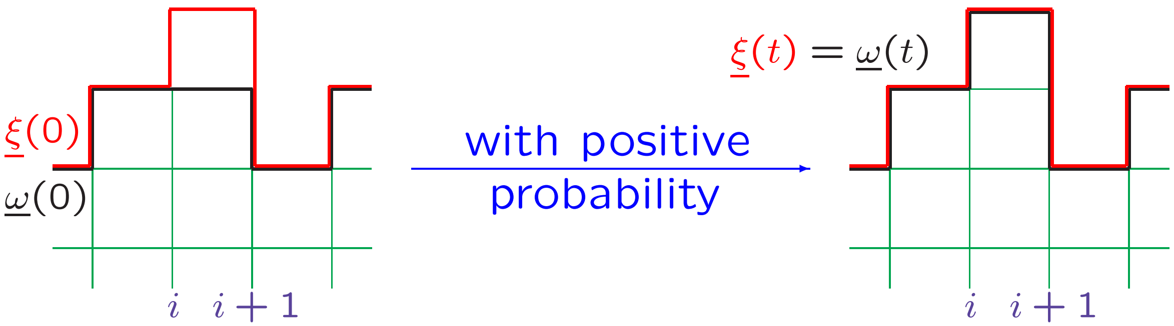

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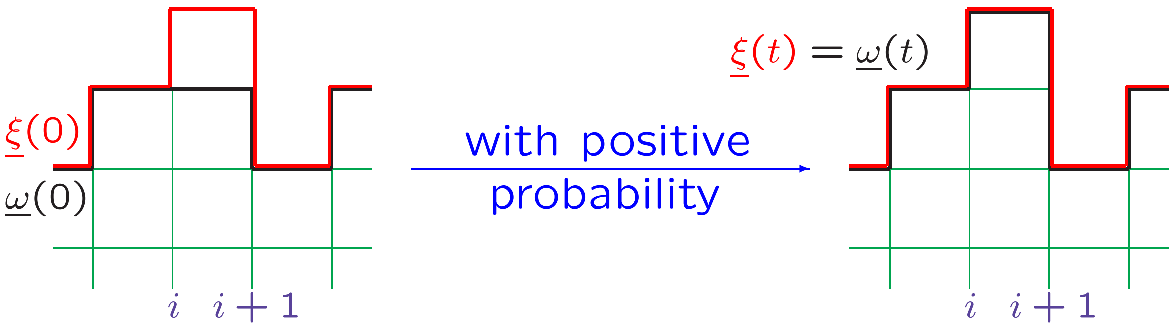

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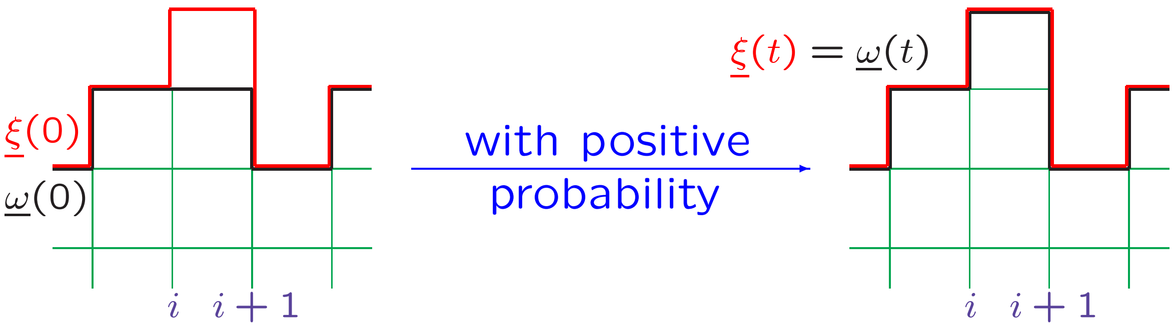

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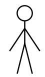
c.) True bricklayers are laying the bricks at each site  (that is, the Kolmogorov forward and backward equations hold with our favorite generator),

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- Constructing the version in which bricklayers are also allowed to remove bricks from columns (that is, particles are also allowed to jump to the left (ZR)). We haven't tried, but it didn't seem easy.

It exists.

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Thank you.