Current variance and the second class particle in particle systems

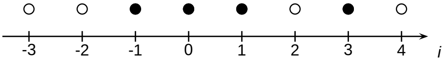
Supervised by Bálint Tóth
and then
Joint with Timo Seppäläinen

Márton Balázs

Budapest University of Technology and Economics

Oberwolfach, August 2007

Asymmetric simple exclusion



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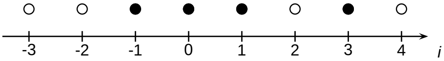

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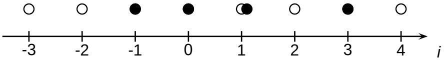

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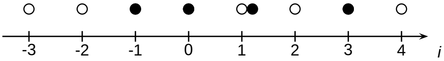

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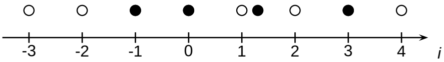

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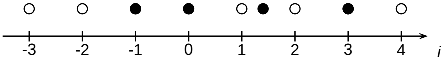

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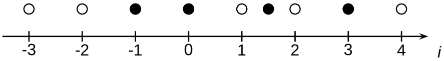

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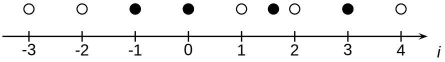

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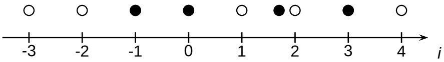

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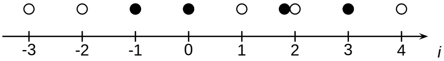

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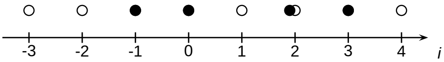

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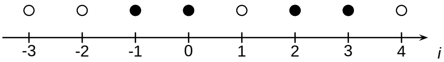

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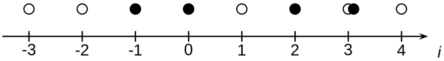

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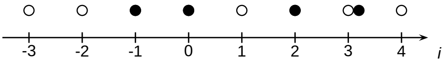

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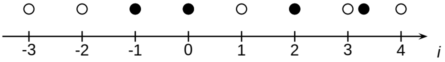

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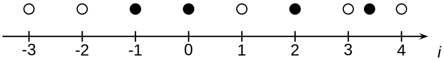

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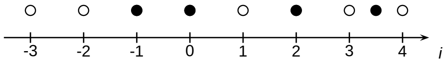

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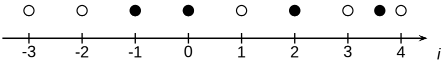

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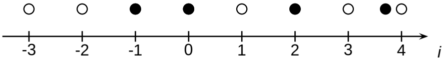

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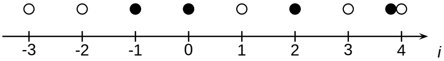

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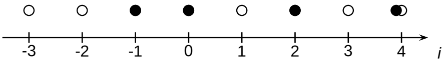

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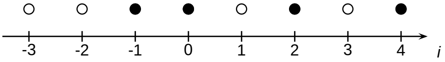

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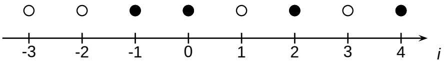

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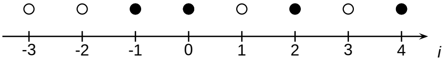

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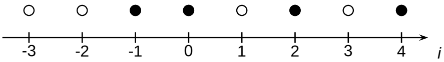

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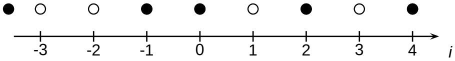

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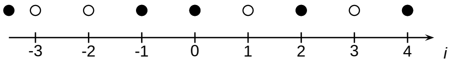

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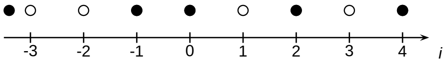

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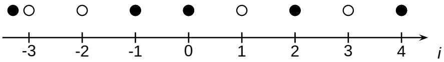

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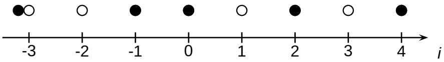

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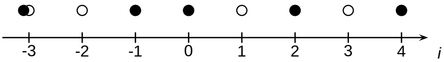

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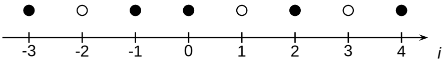

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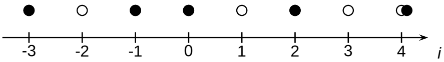

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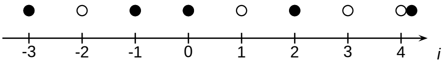

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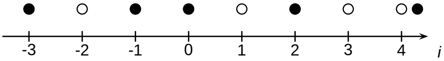

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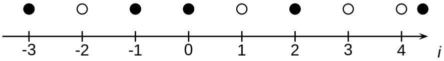

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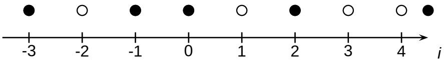

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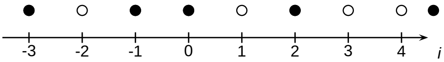

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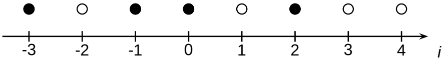

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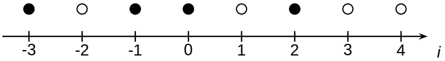

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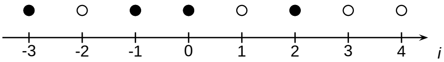

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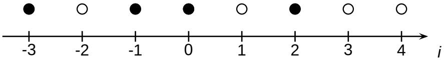

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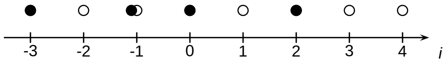

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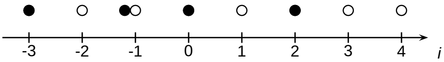

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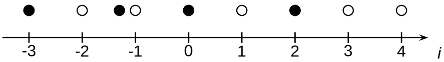

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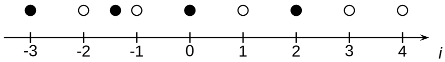

Bernoulli(ϱ) distribution; $\omega_i = 0$ or 1.

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

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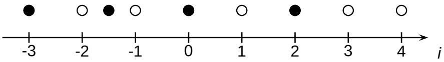

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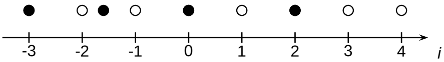

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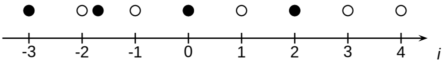

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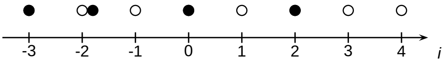

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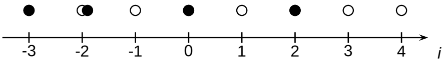

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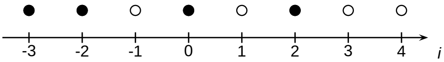

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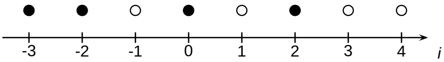

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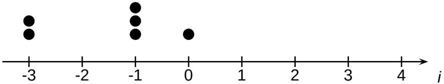
The Bernoulli(ϱ) distribution is time-stationary for any $(0 \le \varrho \le 1)$. Any translation-invariant stationary distribution is a mixture of Bernoullis.

The asymmetric zero range process



Poisson-type distribution; $\omega_i \in \mathbb{Z}^+$.

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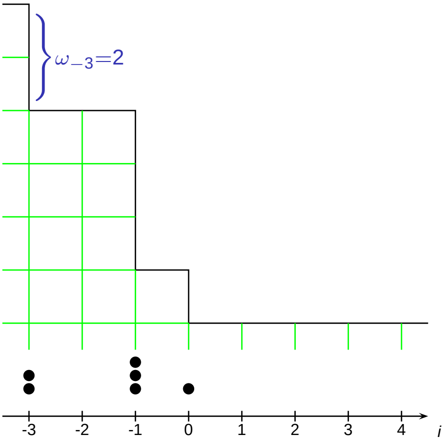

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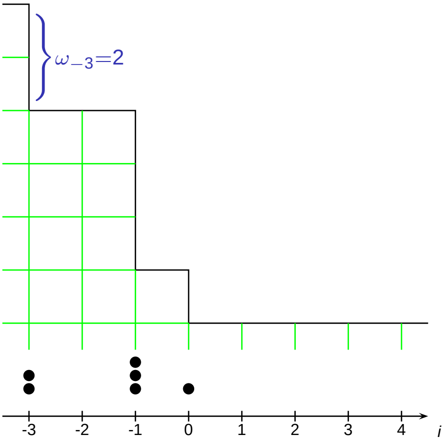

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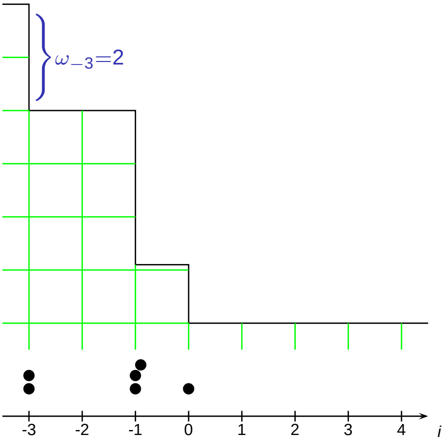

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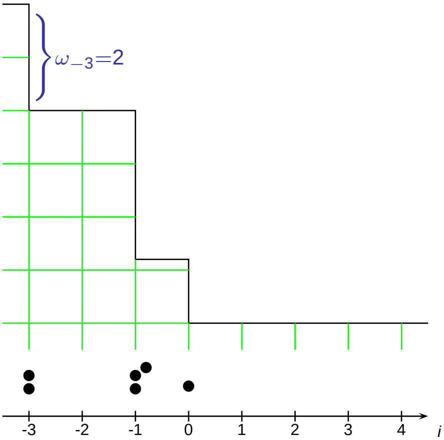

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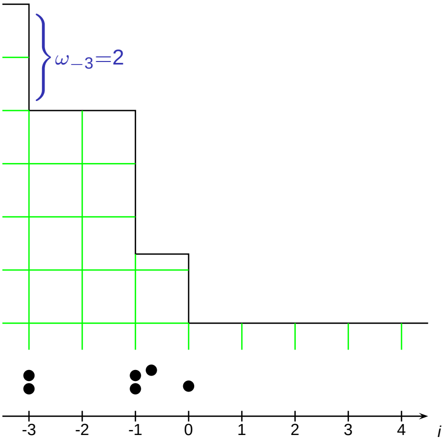

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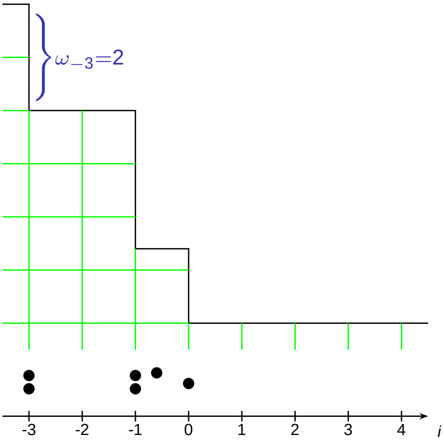

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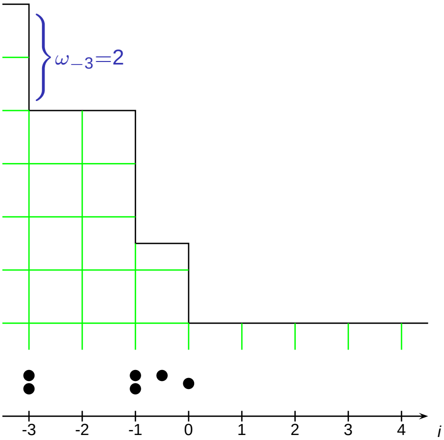

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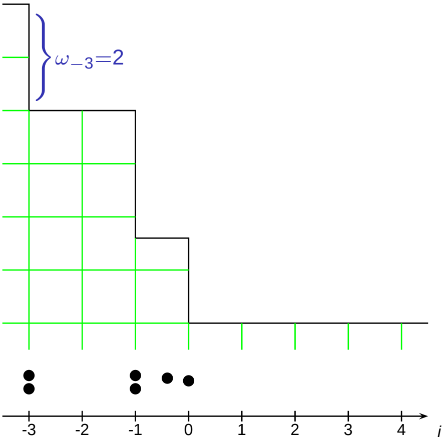

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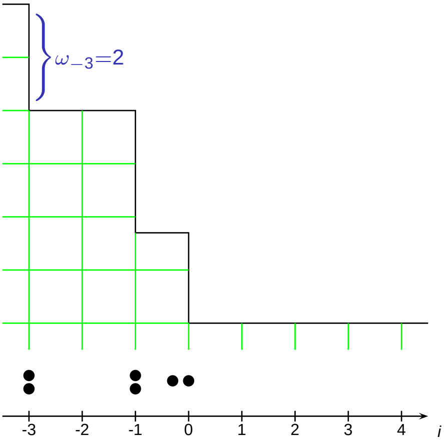

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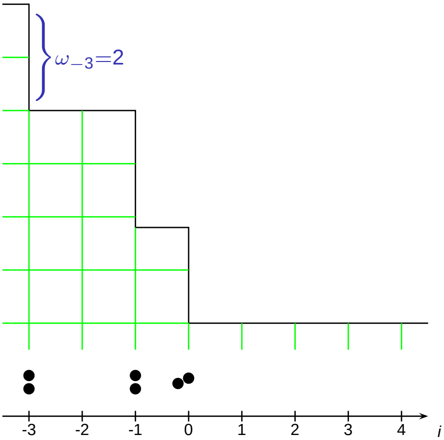

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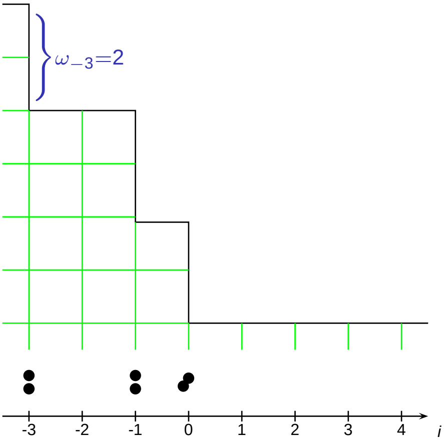

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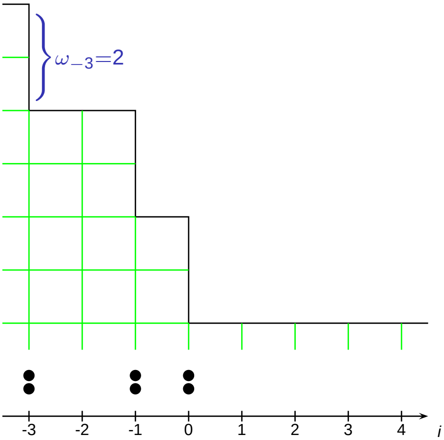

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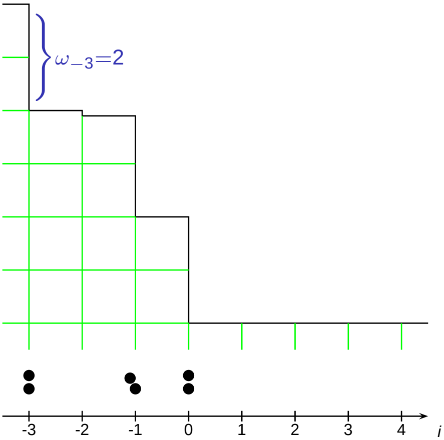

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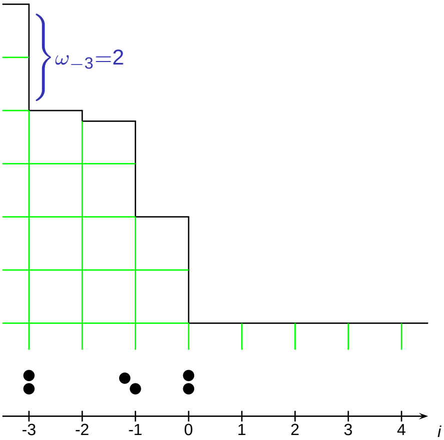

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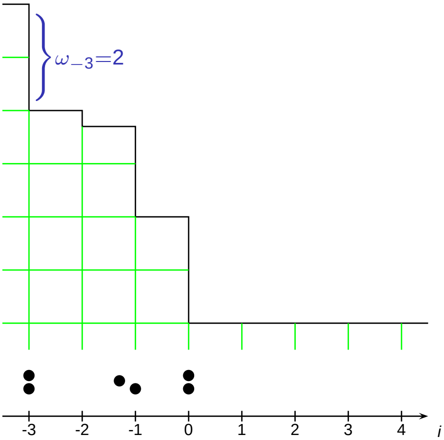

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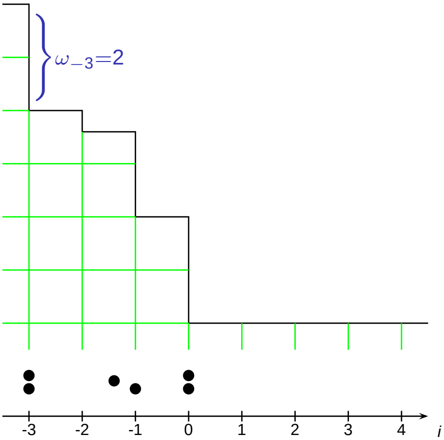

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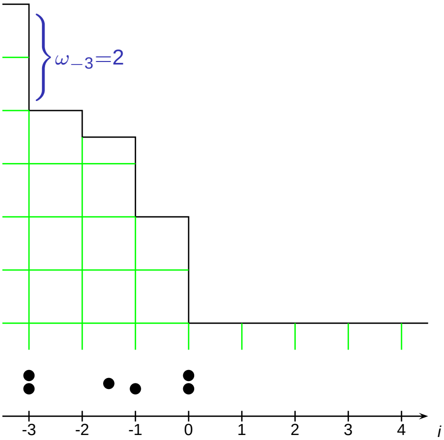

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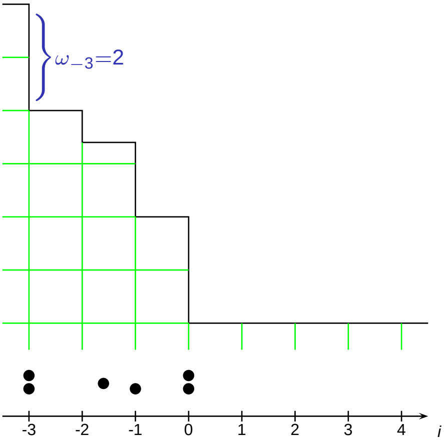

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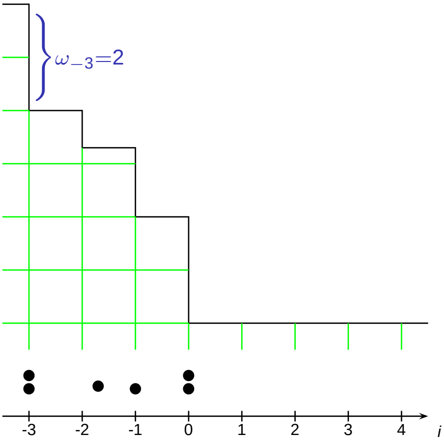

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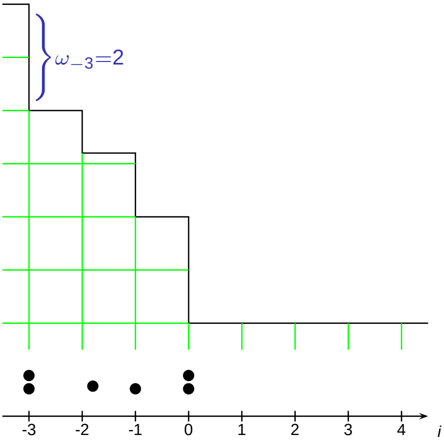

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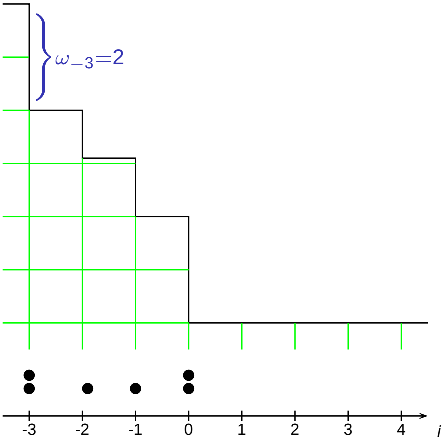

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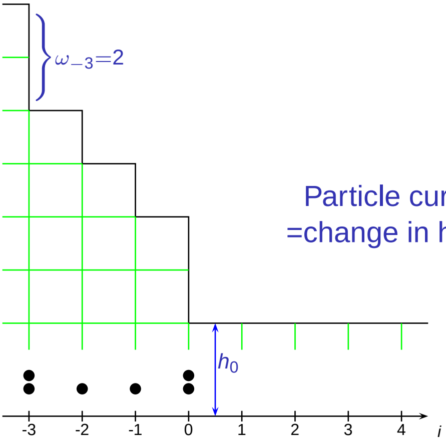

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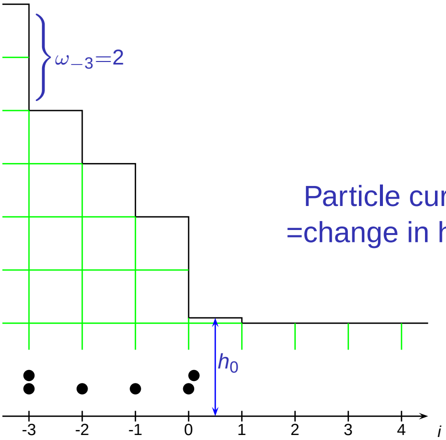

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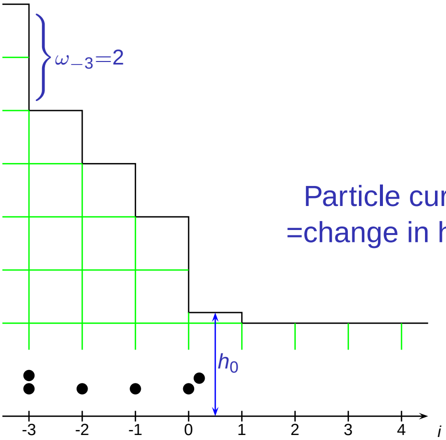

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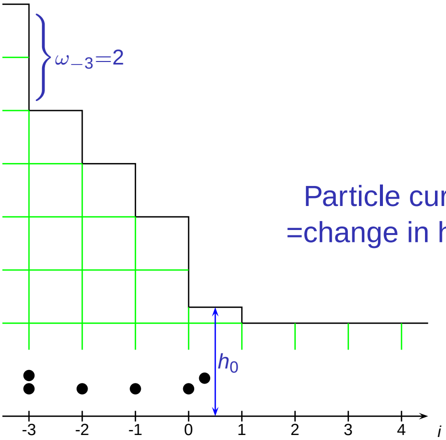

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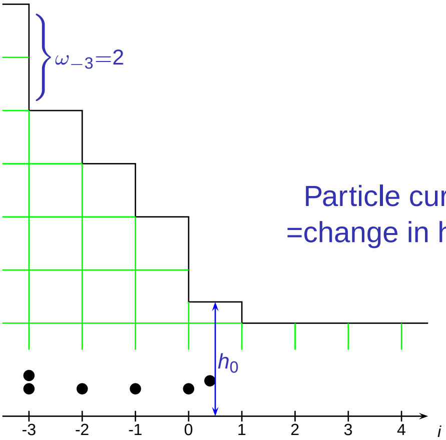

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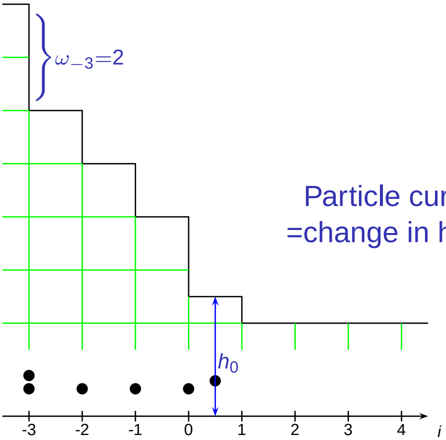

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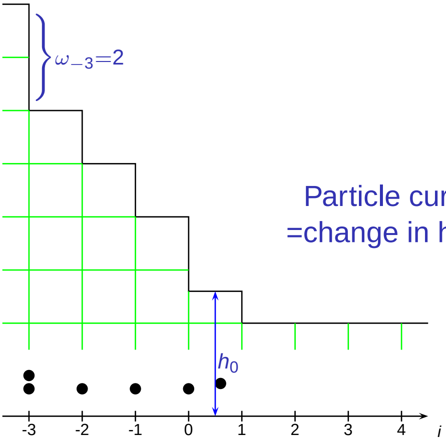

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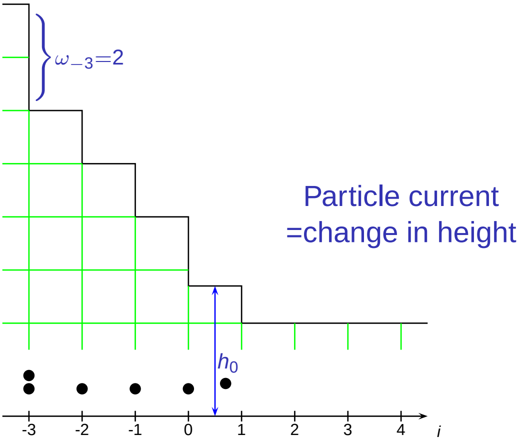

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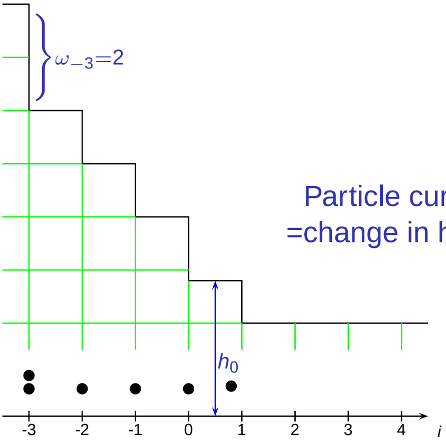

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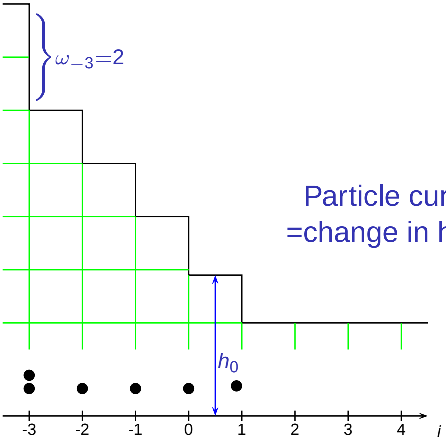

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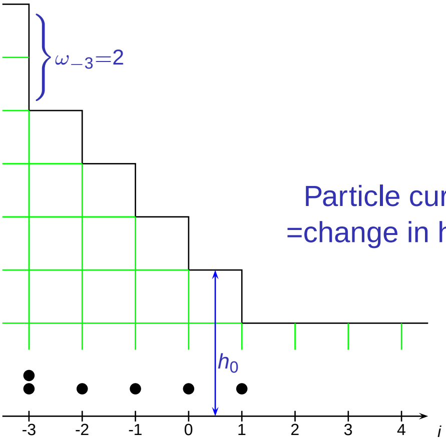

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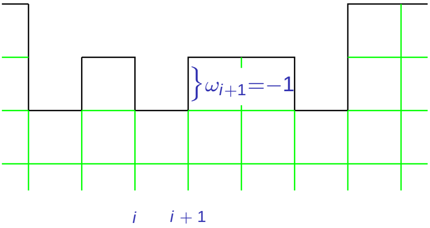
The asymmetric bricklayers process



$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

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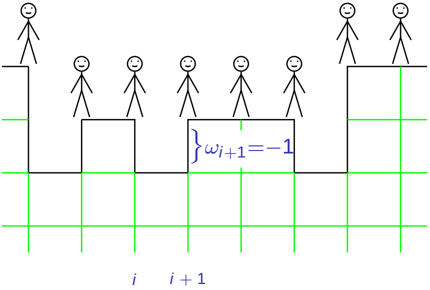

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a brick is added with rate $p \cdot [r(\omega_i) + r(-\omega_{i+1})]$
a brick is removed with rate $q \cdot [r(-\omega_i) + r(\omega_{i+1})]$.

$(r(\omega) \cdot r(1-\omega) = 1;$ r non-decreasing; $q = 1 - p < p)$.

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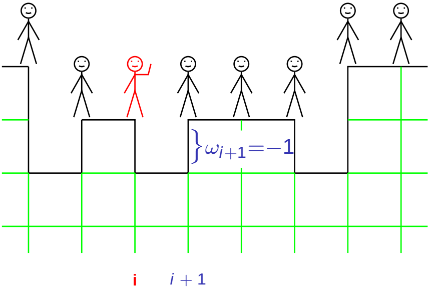

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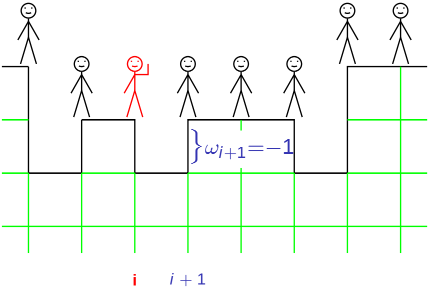
The asymmetric bricklayers process



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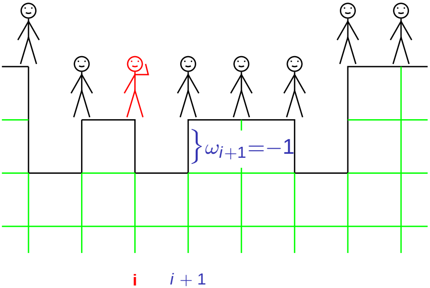

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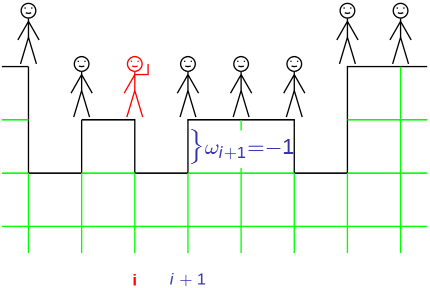

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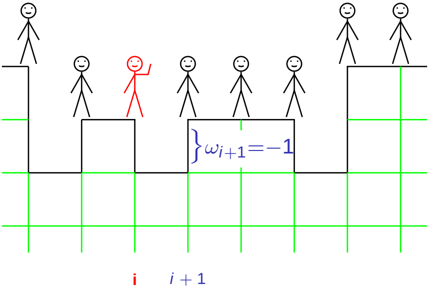

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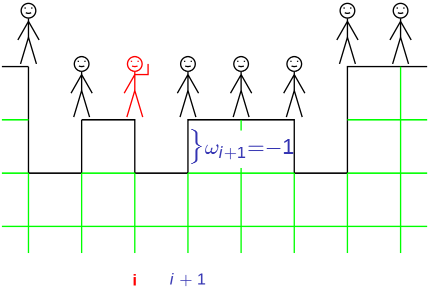

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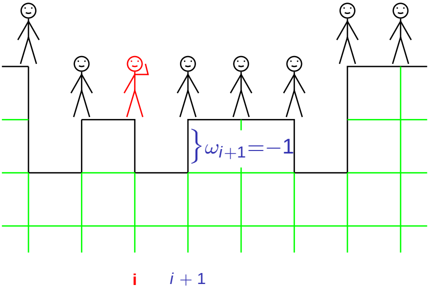

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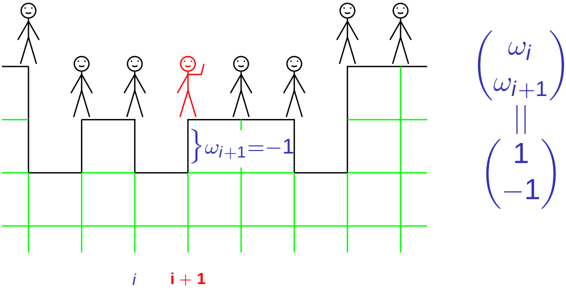

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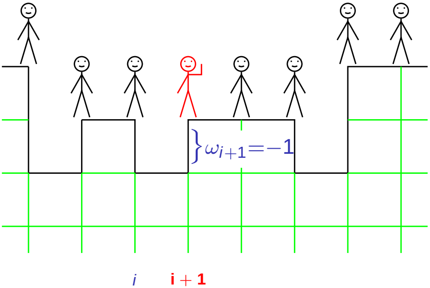

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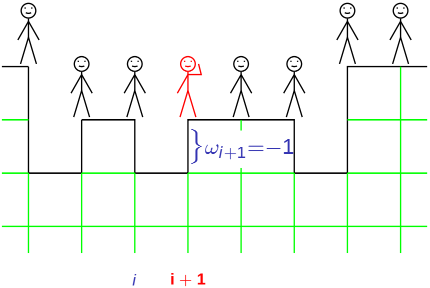

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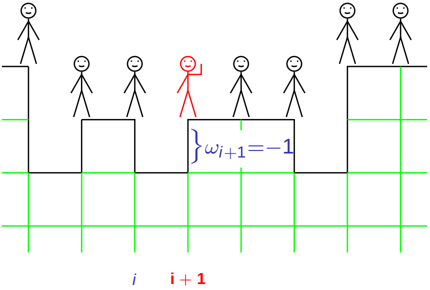

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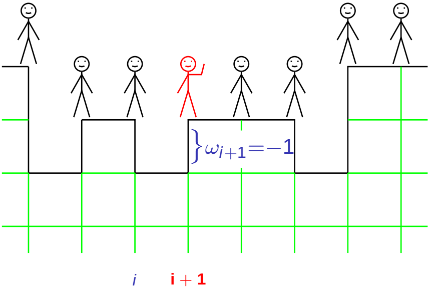

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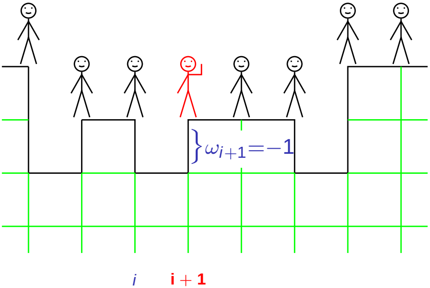

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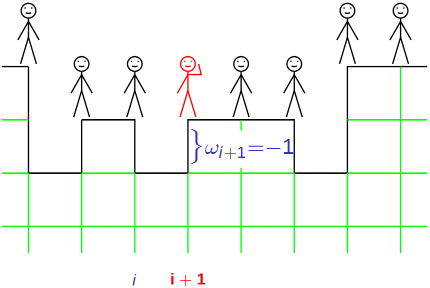

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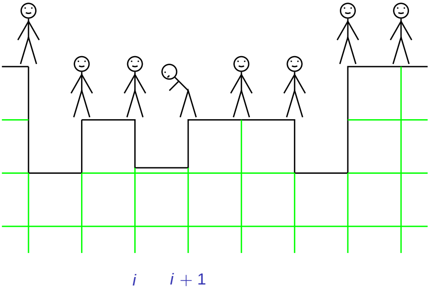

$$\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} \longrightarrow \begin{pmatrix}\omega_i - 1\\ \omega_{i+1}+1\end{pmatrix}$$

$$\begin{matrix} \| & & \| \\ \begin{pmatrix}1\\ -1\end{pmatrix} & \longrightarrow & \begin{pmatrix}0\\ 0\end{pmatrix}\end{matrix}$$

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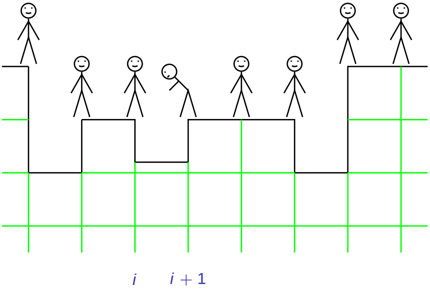

$$\begin{pmatrix} \omega_i \\ \omega_{i+1} \end{pmatrix} \to \begin{pmatrix} \omega_i - 1 \\ \omega_{i+1} + 1 \end{pmatrix}$$
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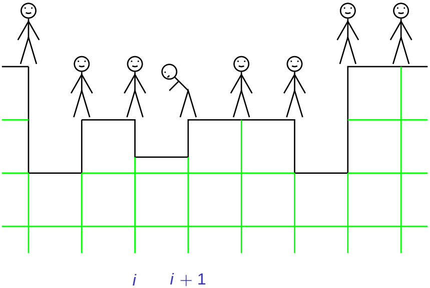

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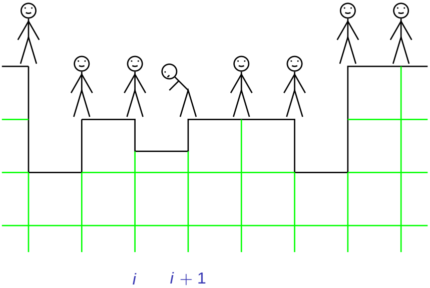

$$\begin{array}{ccc}\begin{pmatrix}\omega_i\\ \omega_{i+1}\end{pmatrix} & \longrightarrow & \begin{pmatrix}\omega_i-1\\ \omega_{i+1}+1\end{pmatrix}\\ \| & & \| \\ \begin{pmatrix}1\\ -1\end{pmatrix} & \longrightarrow & \begin{pmatrix}0\\ 0\end{pmatrix}\end{array}$$

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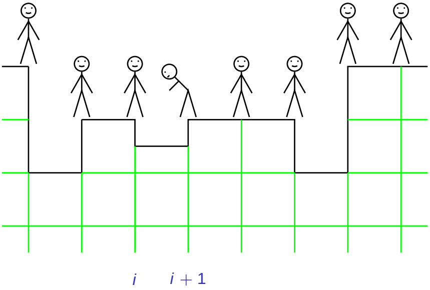

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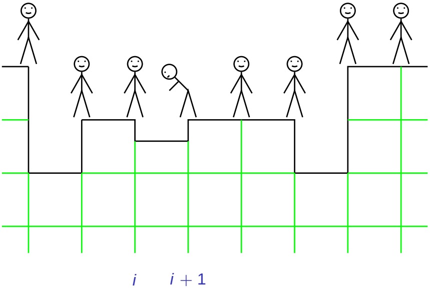

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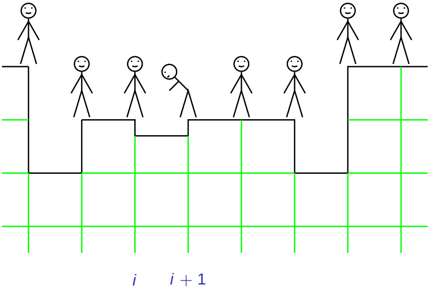

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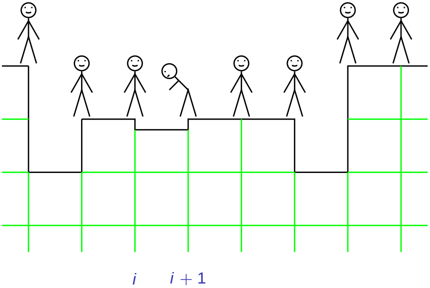

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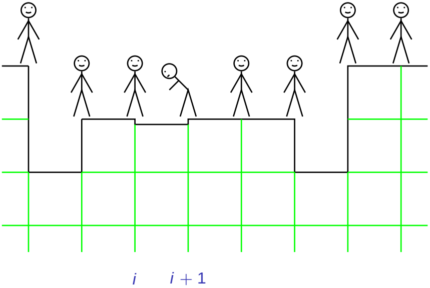

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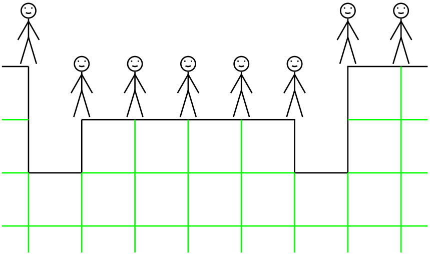

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$i \qquad i+1$

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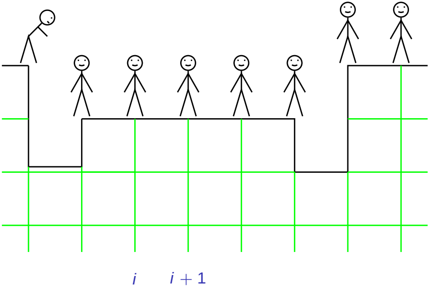

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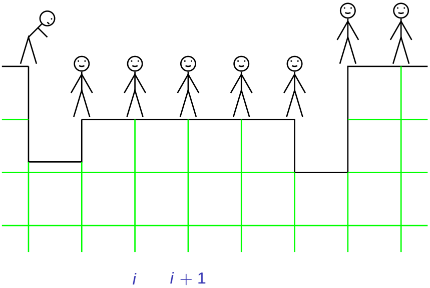

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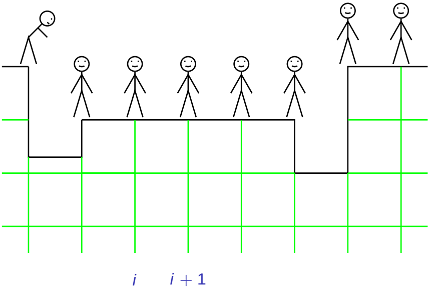

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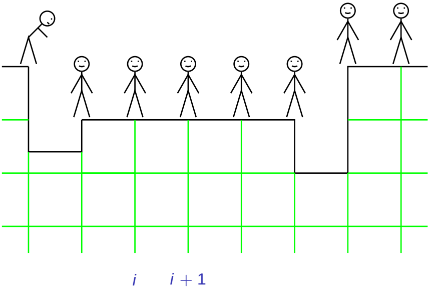

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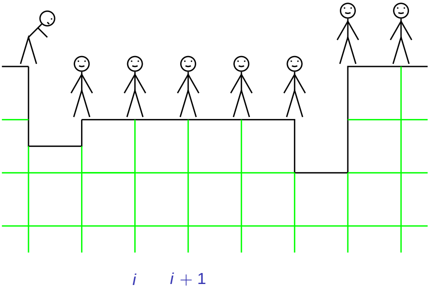

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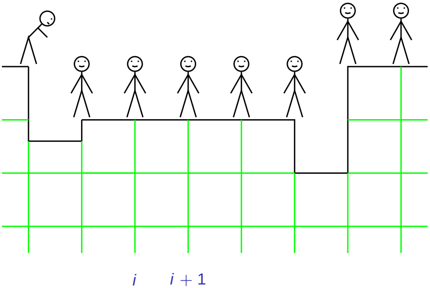

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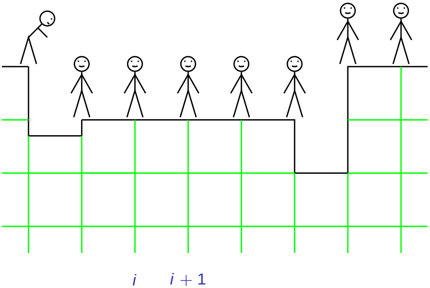

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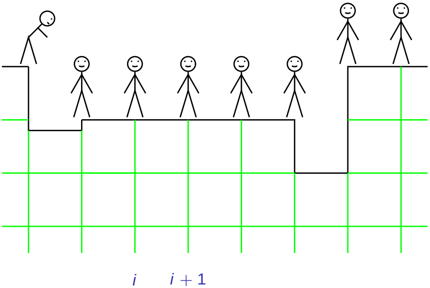

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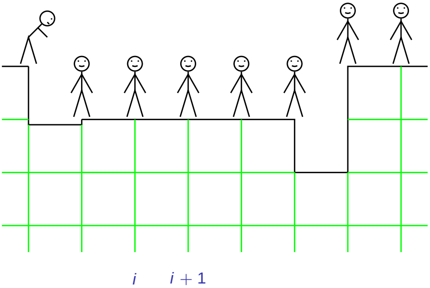

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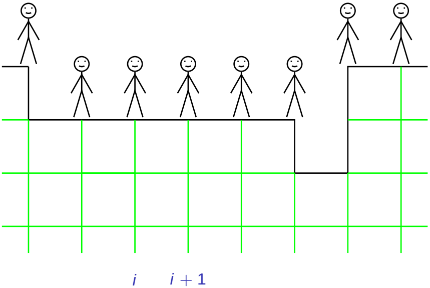

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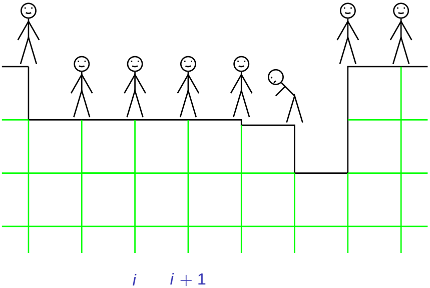

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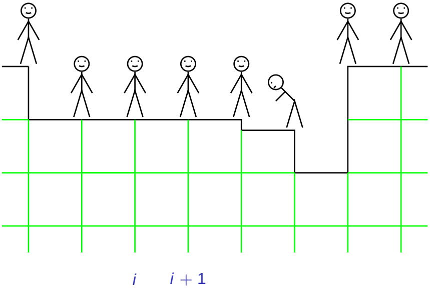

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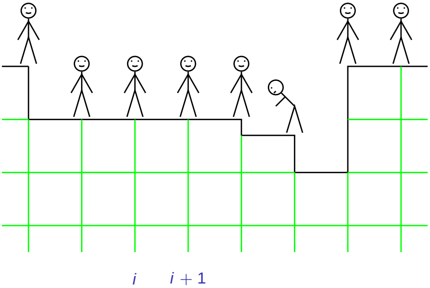

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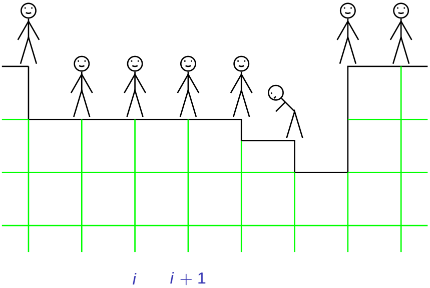

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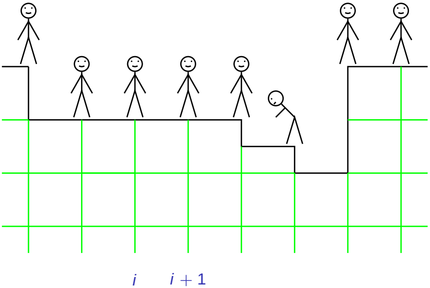

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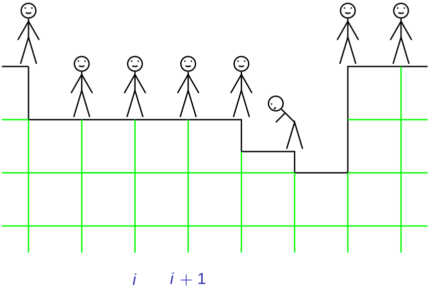

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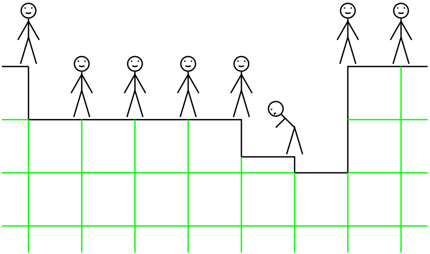

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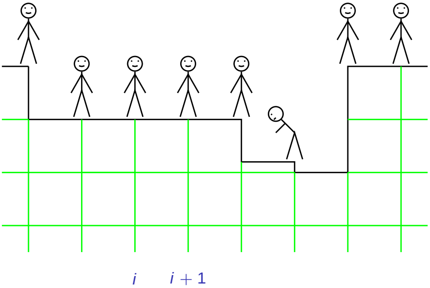

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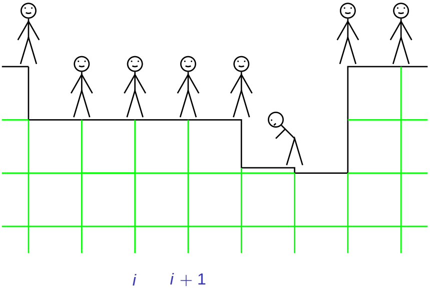

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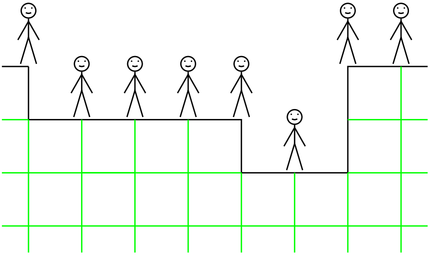

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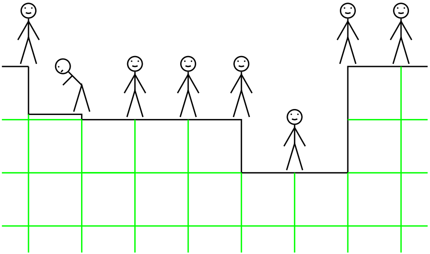

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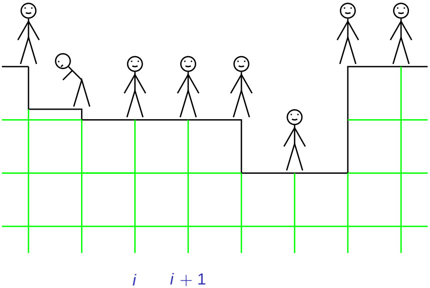

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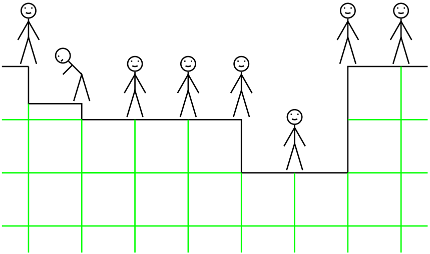

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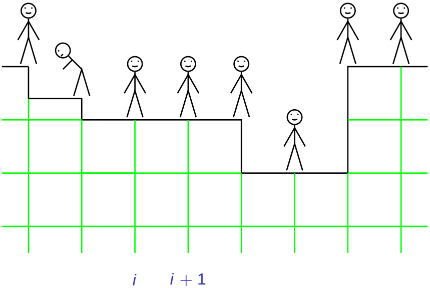

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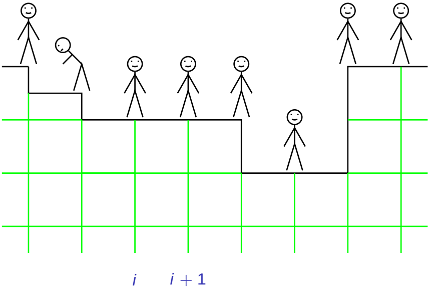

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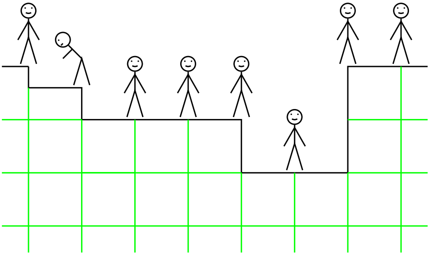

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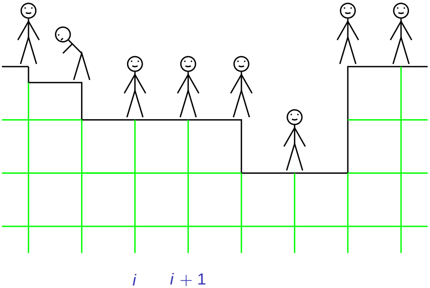

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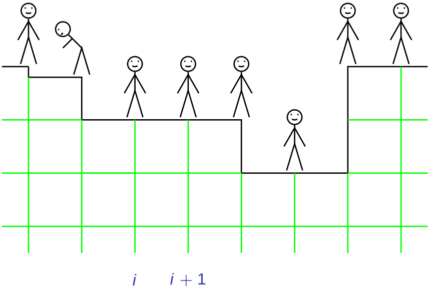

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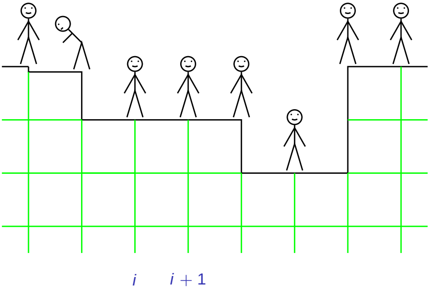

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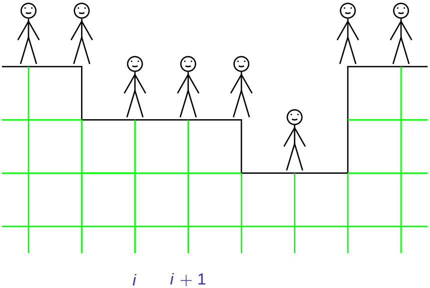

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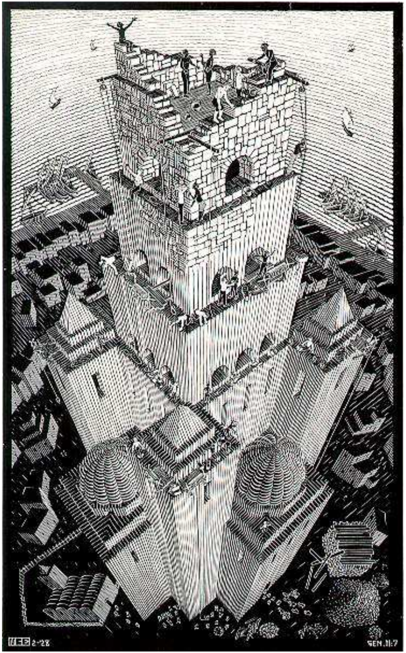

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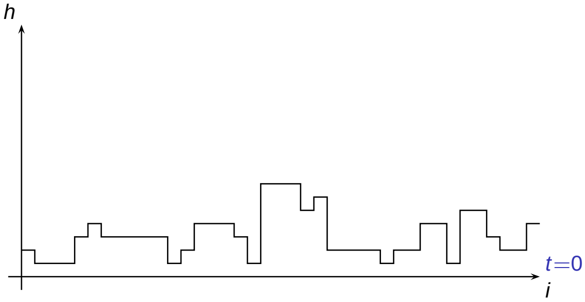

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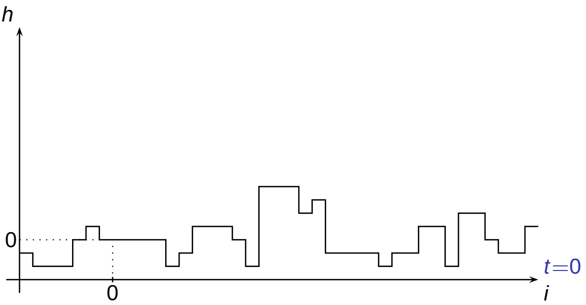

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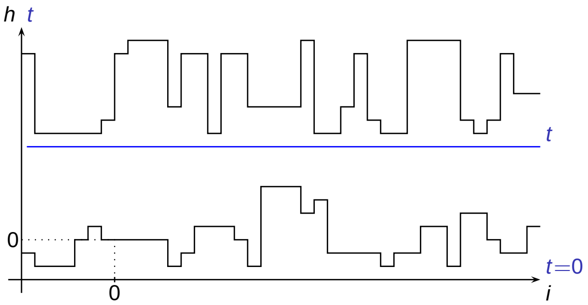

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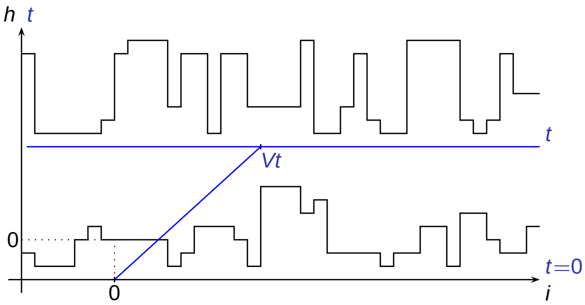

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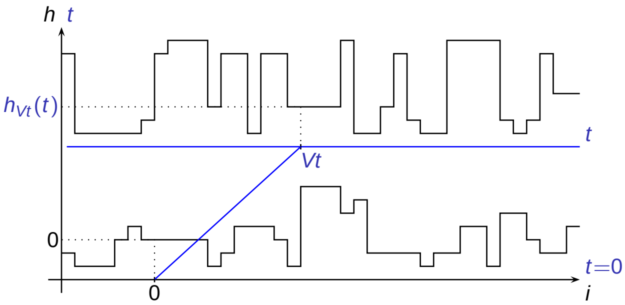

$h_{Vt}(t)$ = height as seen by a moving observer of velocity V.

= net number of particles passing the window $s \mapsto Vs$.

(Remember: particle current=change in height.)

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Space-time correlations

Theorem (Ideas originating from B. Tóth; proof coming later)

For any $V \in \mathbb{R}$ and $t > 0$ under the time-stationary evolution,

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- Both statements are already exact for finite times t.

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For any $V \in \mathbb{R}$ and $t > 0$ under the time-stationary evolution,

$$\mathbf{Var}(h_{Vt}(t)) = \sum_{i=-\infty}^{\infty} |Vt - i| \cdot \mathbf{Cov}(\omega_i(t),\, \omega_0(0)),$$

$$t \cdot \mathbf{Cov}(p(\omega_0, \omega_1) - q(\omega_0, \omega_1),\, (\omega_0 + \omega_1)) = \sum_{i=-\infty}^{\infty} i \cdot \mathbf{Cov}(\omega_i(t),\, \omega_0(0)).$$

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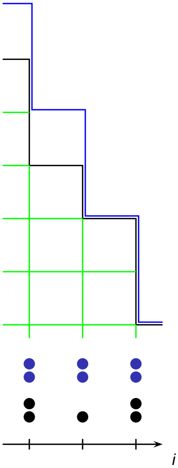
- Both statements are already exact for finite times t.
- Formulas of similar flavor for the ASEP have been derived by Prähofer and Spohn in 2001.
- To understand these formulas better, we need to introduce the *second class particle*.

The second class particle

States ω and ω only differ at one site.



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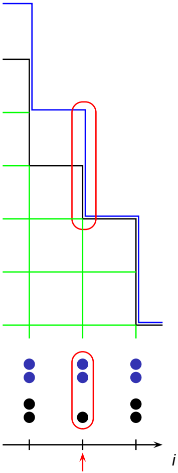

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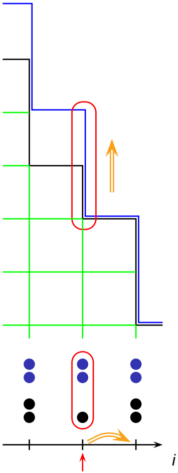

Growth on the right:

rate≤rate

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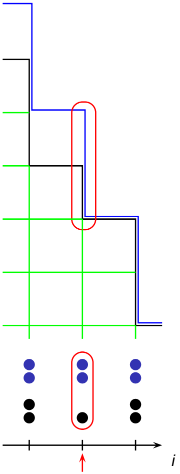

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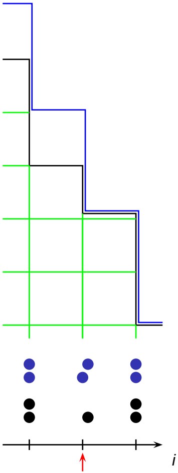

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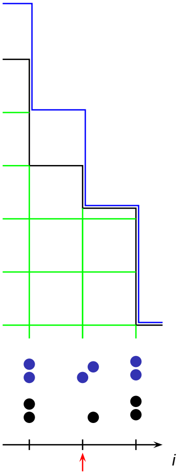

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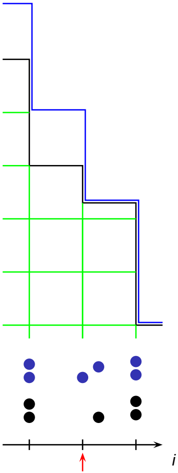

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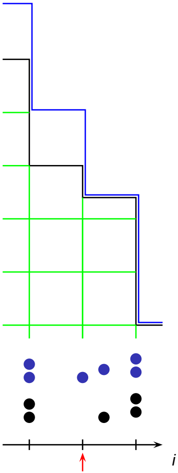

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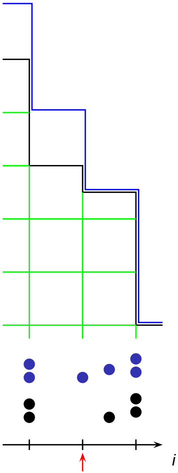

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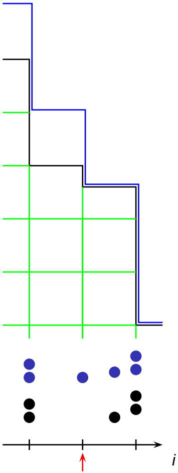

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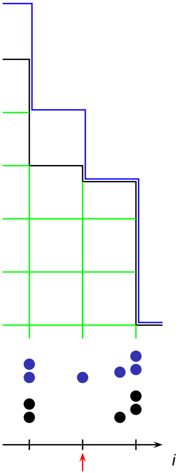

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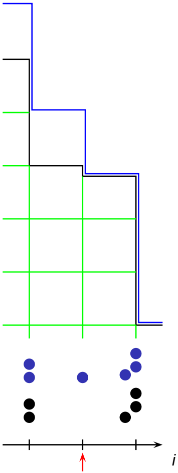

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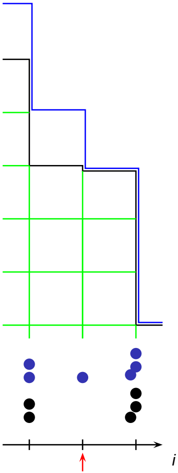

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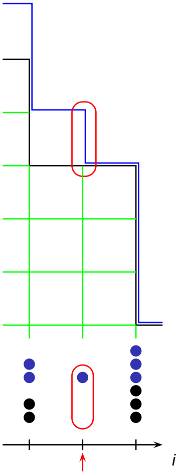

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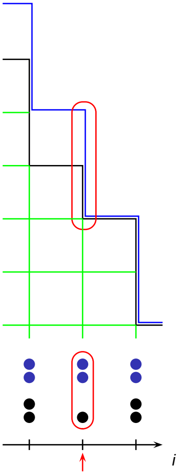

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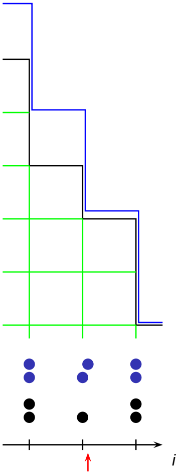

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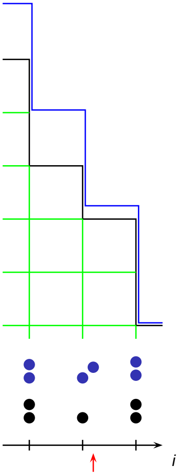

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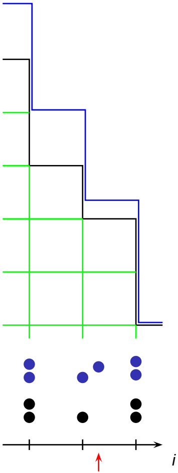

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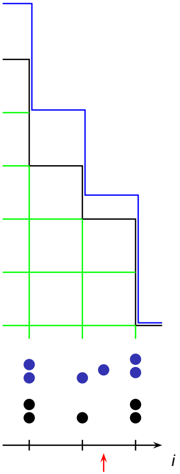

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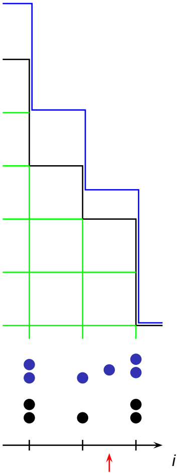

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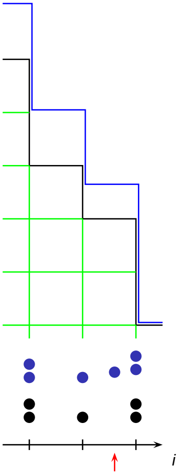

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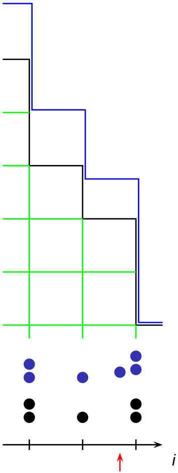

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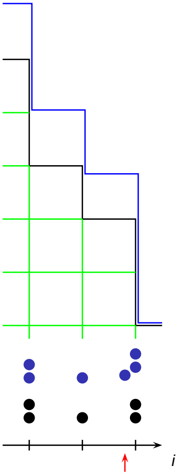

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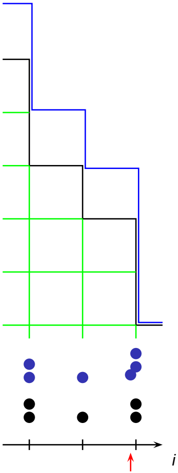

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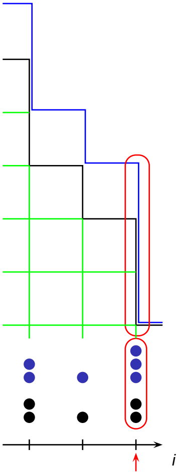

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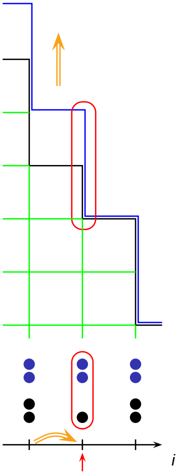

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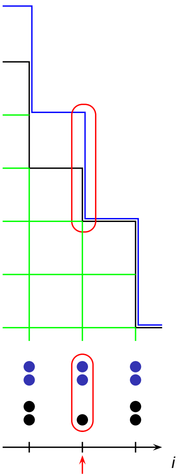

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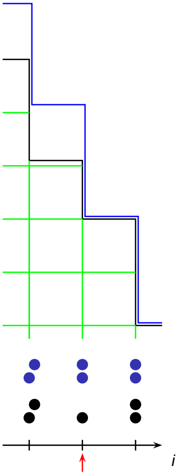

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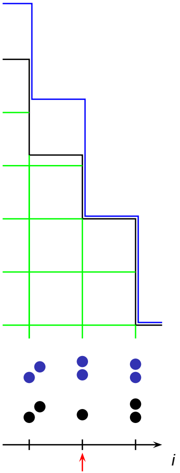

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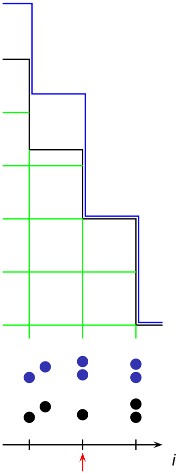

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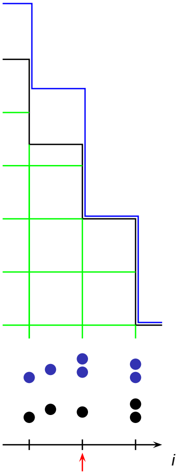

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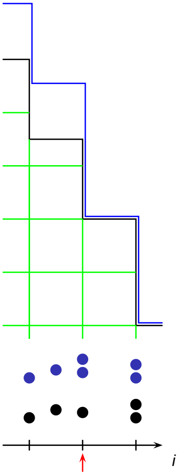

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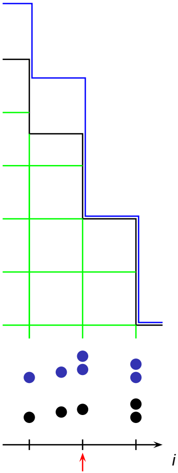

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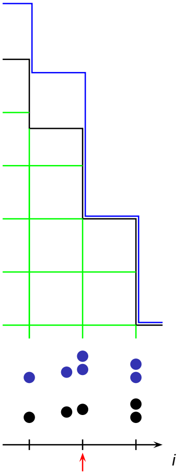

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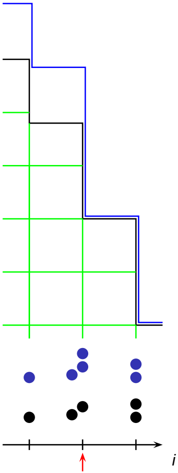

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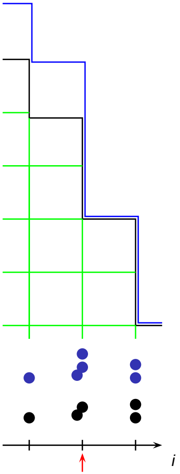

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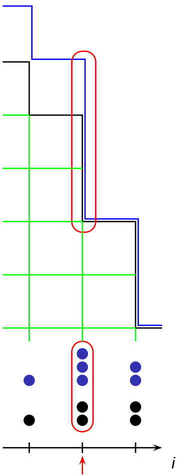

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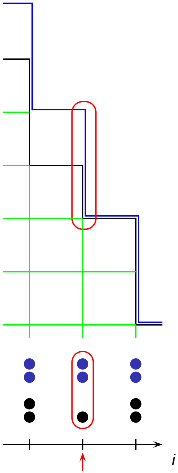

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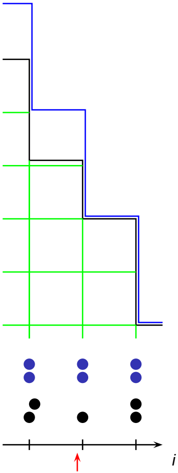

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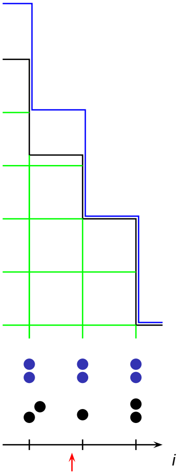

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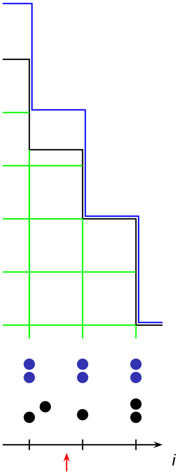

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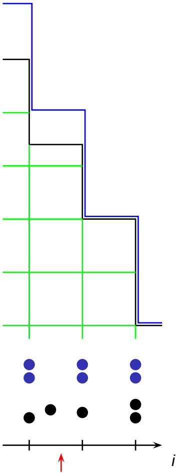

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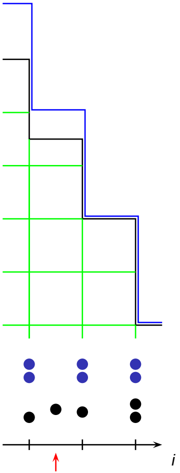

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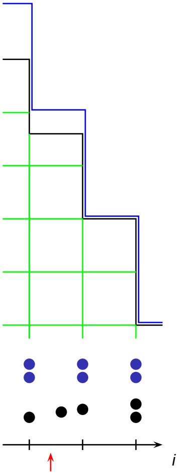

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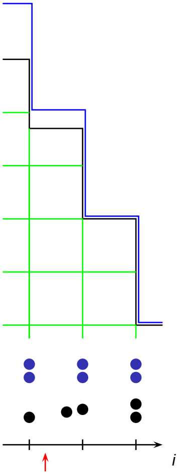

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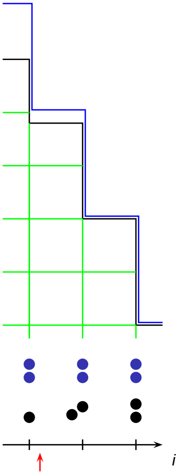

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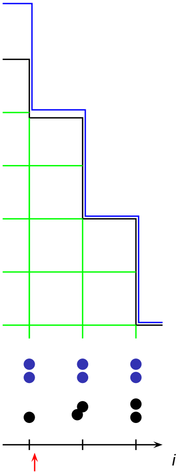

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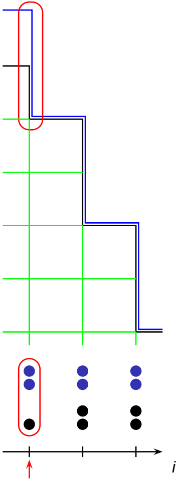

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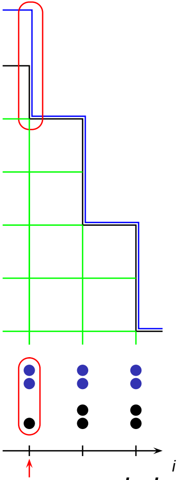

A single discrepancy↑, the *second class particle*, is conserved.
Its position at time t is $Q(t)$.

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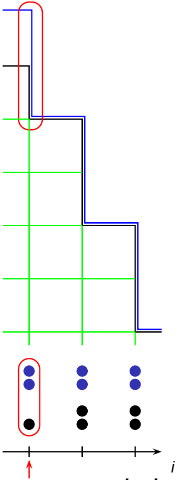

A single discrepancy↑, the *second class particle*, is conserved.
Its position at time t is $Q(t)$. So, $\omega_i(t) = \omega_i(t) + \mathbf{1}\{Q(t) = i\}$.

A way to use the second class particle

Set $Q(0) = 0$, that is, $\omega_i(0) = \omega_i(0) + \delta_{i0}$.

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A way to use the second class particle

Set $\textcolor{red}{Q(0)} = 0$, that is, $\textcolor{blue}{\omega_i(0)} = \omega_i(0) + \delta_{i0}$.

$$\begin{aligned}
\textcolor{blue}{\omega_i(t)} &= \omega_i(t) + \mathbf{1}\{\textcolor{red}{Q(t)} = i\} \\
\mathbf{E}[\textcolor{blue}{\omega_i(t)} \,|\, \omega_0(0) = z] &= \mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = z] \\
&\qquad + \mathbf{P}[\textcolor{red}{Q(t)} = i \,|\, \omega_0(0) = z]
\end{aligned}$$

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Set $Q(0) = 0$, that is, $\omega_i(0) = \omega_i(0) + \delta_{i0}$.

$$\omega_i(t) = \omega_i(t) + \mathbf{1}\{Q(t) = i\}$$

$$\mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = z] = \mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = z] + \mathbf{P}[Q(t) = i \,|\, \omega_0(0) = z]$$

$$\mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = z + 1] = \mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = z] + \mathbf{P}[Q(t) = i \,|\, \omega_0(0) = z]$$

A way to use the second class particle

Set ${\color{red}Q(0)} = 0$, that is, ${\color{blue}\omega_i(0)} = \omega_i(0) + \delta_{i0}$.

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$\rightsquigarrow$ Build up the space-time covariance $\mathbf{Cov}(\omega_i(t), \omega_0(0))$ of the stationary evolution.

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Example

For the ASEP, μ is the Bernoulli-distribution, and $\widehat{\mu}$ gives probability one on $\{\omega_0(0) = 0\}$.

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Theorem

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- Formulas of similar flavor for the ASEP have been derived by Prähofer and Spohn in 2001.
- To understand these formulas even better, let's take a look at the hydrodynamics.

Hydrodynamics (very briefly)

The *density* $u := \mathbf{E}(\omega)$ and the *hydrodynamic flux* $H := \mathbf{E}[p(\omega_i, \omega_{i+1}) - q(\omega_i, \omega_{i+1})]$ both depend on a parameter of the stationary distribution.

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- If the process is *locally* in equilibrium, but changes over some *large scale* (variables $X = \varepsilon i$ and $T = \varepsilon t$), then

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- The *characteristics* is a path $X(T)$ where $u(T, X(T))$ is constant.

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Thus, here is the final form of our theorem:

$$\frac{\mathbf{Var}(h_{Vt}(t))}{\mathbf{Var}(\omega)} = \widehat{\mathbf{E}}|Q(t) - Vt|, \qquad t \cdot C = \widehat{\mathbf{E}}(Q(t)).$$

Consequence 1: normal fluctuations

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Combine this, if available, with a (Weak) Law of Large Numbers for the second class particle: $\frac{Q(t)}{t} \xrightarrow{d} C$:

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Notice the vanishing right hand-side at $V = C$, from which the Central Limit Theorem also follows for all other cases:

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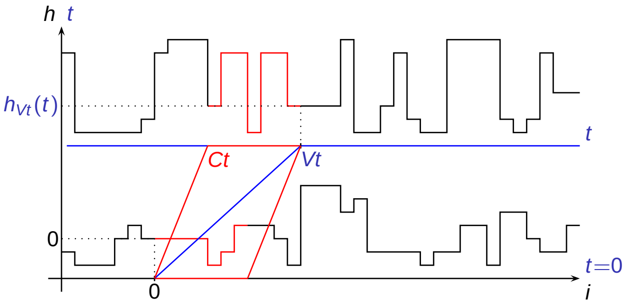

$$\lim_{t\to\infty}\frac{\mathbf{Var}(h_{Vt}(t))}{t}=\mathbf{Var}(\omega)\cdot|C-V|$$

Initial fluctuations are transported along the characteristics on this scale.

Consequence 2: $t^{2/3}$ scaling

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Theorem (B. - Seppäläinen)

For the stationary ASEP evolution,

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(h_{Ct}(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(h_{Ct}(t))}{t^{2/3}} < \infty.$$

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Important preliminaries were Cator and Groeneboom 2006, B., Cator and Seppäläinen 2006.

Other results

The hydrodynamic flux $H(u)$ of the ASEP is

$$H(u) = (p - q) \cdot u(1 - u),$$

strictly concave. It is expected that the above $t^{1/3}$ fluctuations come in for models with $H(u)'' \neq 0$.

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- The random average process (RAP),
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In their cases, we have

$$\lim_{t\to\infty} \frac{\textbf{Var}(h_{Ct}(t))}{t^{1/2}} = \dots,$$

even convergence of the finite-dimensional distributions of the $h_{Ct}(t)$ process to Gaussian limits is known (Seppäläinen 2005, Ferrari and Fontes 1998, B., Rassoul-Agha and Seppäläinen 2006).

Other results

As a contrary, $t^{1/3}$ scalings come with Tracy-Widom type limits of

$$\frac{h_i(t)}{t^{1/3}}$$

for i around the characteristics. Among distributional results are Baik, Deift and Johansson 1999, Johansson 2000, Prähofer and Spohn 2001, Ferrari and Spohn 2006. Their methods are completely different, relying on combinatorial tricks and asymptotic analysis of certain determinants.

A few words on the proof (Ideas originating from B. Tóth)

- Separate a martingale, and then a conditional variance martingale from $h_0(t)$ and, of course, reverse time. This leads to nontrivial terms like

$$\int_0^t \int_0^s \mathbf{Cov}\,(r(v),\, r^*(0))\,\mathrm{d}v\,\mathrm{d}s$$

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- Repeat similar tricks for $h_i(t)$.

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- For the ASEP, one can use the construction of Ferrari, Kipnis and Saada 1991.
- For the AZRP and ABLP processes with convex rates, LLN for $Q(t)$ could still be worked out with some coupling tricks (Rezakhanlou 1995, B. 2003). Not clear how to refine this for $t^{1/3}$ fluctuations.

Thank you.