Random walks on Lie groups

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Random walks

Let $G$ be a group and let $X_1, X_2, \ldots$ be a sequence of i.i.d. random elements of $G$.

**Example:** fix two elements $a, b \in G$ and let

$$P(X_i = a) = P(X_i = b) = \frac{1}{2}.$$ 

The random walk is the sequence of random elements:

$$Y_1 = X_1, \quad Y_2 = X_2 X_1, \ldots, \quad Y_l = X_l \cdots X_1, \ldots$$

When $G$ is the additive group of $\mathbb{R}$, $Y_l$ is the sum of i.i.d. random variables.
For the first part of the talk, $G$ is a compact semisimple Lie group, e.g. $\text{SO}(3)$.

We want to understand how fast the distribution of $Y_l$ converges to the Haar measure. To formulate this more precisely, we introduce some notation.

For a function $f : L^2(G) \to \mathbb{R}$ and $g \in G$ write

$$Tf(g) = \mathbb{E}[f(X_1g)].$$

This defines a linear operator on $L^2(G)$.

If the law of $X_1$ is as in the example above, then:

$$Tf(g) = \frac{1}{2}f(ag) + \frac{1}{2}f(bg).$$

Observe that

$$\mathbb{E}[f(Y_l)] = T^l f(1),$$

where 1 is the unit element of $G$. 
Fourier coefficients

Denote by $\mu$ the common law of $X_i$. Let $\rho$ be a unitary representation of $G$ and write

$$\rho(\mu) = \mathbb{E}[\rho(X_1^{-1})],$$

which is an operator on the representation space of $\rho$.

**Example:** $\mathcal{L}$, the left regular representation of $G$ acting on $L^2(G)$ is defined by

$$\mathcal{L}(h)f(g) = f(h^{-1}g).$$

We can express our previously defined operator as

$$T = \mathcal{L}(\mu).$$

The collection of irreducible unitary representations of $G$ is denoted by $\hat{G}$. The operators $\rho(\mu)$ for $\rho \in \hat{G}$ plays to role of the Fourier coefficients.
Spectral gap

To simplify our discussion, we assume for the rest of the talk that $\mu$, the common law of $X_i$ is symmetric, i.e. $X_i$ and $X_i^{-1}$ has the same law. This implies that $T$ and all the operators $\rho(\mu)$ are selfadjoint.

We say that $T$ has uniform spectral gap, if there is a number $c > 0$ such that

$$\|\rho(\mu)\| < 1 - c$$

for all nontrivial $\rho \in \hat{G}$.

Since $L$ can be decomposed as the sum of irreducible components, uniform spectral gap implies that

$$\|T^l f - \int f\|_{L^2} \leq (1 - c)^l \|f\|_{L^2}.$$
The a.c. case

If $\mu$ is absolutely continuous with $L^2$ density, then $T$ is a Hilbert-Schmidt operator, hence has spectral gap.

Even more, we have Plancherel’s formula:

$$\left\| \frac{d\mu}{dt} \right\|_{L^2}^2 = \sum_{\pi \in \hat{G}} \dim \pi \cdot \left\| \pi(\mu) \right\|_{HS}^2.$$ 

This implies that

$$\left\| \pi(\mu) \right\| \leq \left\| \pi(\mu) \right\|_{HS} \leq \frac{\left\| \frac{d\mu}{dt} \right\|_{L^2}}{\sqrt{\dim \pi}}.$$ 

Note that semi-simple Lie groups have only finitely many irreducible representations of dimension less than a constant.

A variant of the above observation has been made by various authors.
Is it possible that $\mu$ is finitely supported and $T$ has uniform spectral gap?

Yes, examples were given by Margulis, Sullivan and Drinfeld. Lubotzky, Philips and Sarnak gave an example with very good quantitative bounds using the solution of a case of the Ramanujan conjecture. Further examples by Gamburd, Jakobson and Sarnak.
Theorem (Bourgain, Gamburd)

Suppose that $\mu$ is supported on finitely many elements of $\text{SU}(d)$, which
- generate a dense subgroup, and
- are given by matrices with algebraic entries.

Then the $T$ has uniform spectral gap.

The theorem has been generalized very recently to arbitrary simple Lie groups by Benoist and Saxcé.

The first condition is clearly necessary, whether or not the second could be removed is a major open problem. Even the following is wide open:

Open problem: Let $\mu = \frac{1}{4} (\delta_a + \delta_b + \delta_a^{-1} + \delta_b^{-1})$. Does $T$ have uniform spectral gap for almost every pair $\langle a, b \rangle \in G$?
For any compact, semi-simple Lie group $G$, there is a number $c > 0$ and a finite set of non-trivial irreducible representations $A \subset \hat{G}$ such that the following holds.

For any symmetric probability measure $\mu$ on $G$, we have

$$1 - \|\pi(\mu)\| > c \min_{\rho \in X} (1 - \|\rho(\mu)\|) \frac{1}{(\log \dim \pi)^2}.$$

for any $\pi \in \hat{G}$.

For simple Lie groups other than $SO(3)$, the proof gives better exponents than 2. The exponent converges to 1 as the dimension of the group grows.

Similar results with worse exponents can be obtained from the Solovay-Kitaev algorithm.
Heuristic outline of the proof

For each positive integer $n$, we choose a suitable set of representations $A_{2n} \subset \hat{G}$ and prove the theorem by induction on $2^n$. Vaguely: $A_{2n}$ consists of functions “oscillating on scale up to $2^{-n}$”.

Induction hypothesis: The claim holds for $\pi \in A_{2n}$. We can deduce from this that the $l \approx n^3$ step of the random walk is “equidistributed on scale $2^{-n}$”. That is:

$$P(\text{dist}(Y_l, g) \leq 2^{-n}) \approx 2^{-n} \dim G$$

for any $g \in G$.

Then we choose a number $r$ smaller but not much smaller than $2^{-n}$ such that the representations $\pi \in A_{n+1}$ “are not sensitive” to perturbations on scale $r$, and we let $\nu_r$ be the uniform probability in the $r$-neighborhood of $1 \in G$.

We use Plancherel’s formula to estimate $\|\pi(\mu^*(l) \ast \nu_r)\|$ for $\pi \in A_{2n+1}$. 
Random walks in Euclidean space

In the second part of the talk: \( X_1, X_2, \ldots \) are i.i.d. random isometries of Euclidean space \( \mathbb{R}^d \). We also fix a point \( x_0 \in \mathbb{R}^d \).

The random walk in Euclidean space is the sequence of random points:

\[
Y_0 = x_0, \quad Y_1 = X_1(x_0), \ldots \quad Y_l = X_l \cdots X_1(x_0).
\]

We want to understand the distribution of \( Y_l \) as \( l \to \infty \).

**Theorem (CLT, Tutubalin, \ldots)**

Suppose that \( \text{supp}(X_1) \) generates a dense subgroup in \( \text{Isom}(\mathbb{R}^d) \) and \( Y_1 \) has finite second moment. Then \( Y_l/\sqrt{l} \) converges in law to a centrally symmetric Gaussian.
Theorem (RLT, Kazhdan, Guivarc’h)

Suppose that $d = 2$ and that $\text{supp}(X_1)$ is finite and generate a dense subgroup in $\text{Isom}(\mathbb{R}^d)$. Then

$$\frac{\mathbb{E}[f(Y_t)]}{\mathbb{E}[g(Y_t)]} \rightarrow \int f \int g$$

for any $f, g \in C_c(\mathbb{R}^d)$ and $\int g \neq 0$. 
Theorem (V)

Suppose that $d \geq 3$, $\text{supp}(X_1)$ is finite and generate a dense subgroup in $\text{Isom}(\mathbb{R}^d)$. Then

$$
E[f(Y_l)] = E[f(\sqrt{l}Z)] + O(l^{-d/2-1})\|f\|_{L_1} + O(e^{-cl^{1/4}})\|f\|_{W^{2,(d+1)/2}},
$$

where $f$ is a smooth function of compact support, $Z$ is a random variable of the limiting Gaussian distribution in the CLT, $c > 0$ is a number and $W^{2,(d+1)/2}$ is a Sobolev norm.

If $B_l$ is a ball of radius $\geq e^{-cl^{1/4}}$, of distance $\leq \sqrt{l}$ from the origin, then

$$
P(Y_l \in B_l) \approx P(\sqrt{l}Z \in B_l).
$$
Denote by $\theta : \text{Isom}(\mathbb{R}^d) \to \text{SO}(d)$ the natural homomorphism. And let

$$Tf(g) = \mathbb{E}[f(\theta(X)g)]$$

be the operator on $L^2(\text{SO}(d))$ introduced earlier.

**Theorem (Lindenstrauss, V)**

Suppose that $d \geq 3$, supp($X_1$) is finite and generate a dense subgroup in $\text{Isom}(\mathbb{R}^d)$. Suppose further that $T$ has spectral gap. Then

$$\mathbb{E}[f(Y_l)] = \mathbb{E}[f(\sqrt{l}Z)] + O(l^{-d/2-1})\|f\|_{L^1} + O(e^{-cl})\|f\|_{W^{2,(d+1)/2}},$$

where $f$ is a smooth function of compact support, $Z$ is a random variable of the limiting Gaussian distribution in the CLT, $c > 0$ is a number and $W^{2,(d+1)/2}$ is a Sobolev norm.
Bernoulli convolutions

Let $\lambda < 1$ be a number and let $\nu_\lambda$ be the law of the random variable

$$\sum A_n \lambda^n,$$

where $A_n$ are i.i.d. with $P(A_n = 1) = P(A_n = -1) = 1/2$. This measure $\nu_\lambda$ is called a Bernoulli convolution.

- When $\lambda < 1/2$, $\nu_\lambda$ is a fractal measure.
- When $\lambda = 1/2$, $\nu_\lambda$ is the normalized Lebesgue measure on the interval $[-2, 2]$.
- When $\lambda > 1/2$, $\nu_\lambda$ can be a.c. or singular.

Open problem: Is there a number $\lambda_0 < 1$ such that $\nu_\lambda$ is a.c. for all $\lambda_0 < \lambda < 1$?
Selfsimilar measures

Let $\kappa_1, \ldots, \kappa_n$ be contractive similarities on $\mathbb{R}^d$ and let $p_1, \ldots, p_n$ be a probability vector. Then there is a unique probability measure $\nu$ on $\mathbb{R}^d$ such that

$$\nu = \sum p_i \kappa_i(\nu).$$

This is called a selfsimilar measure.

Example: $\nu_\lambda$ is selfsimilar for $\kappa_1(x) = \lambda x + 1$ and $\kappa_2(x) = \lambda x - 1$. 
Theorem (Lindenstrauss, V)

Let $g_1, g_2, \ldots, g_n \in \text{SO}(d)$ and let $p_1, \ldots, p_n$ be a probability vector. Suppose that the operator

$$Tf(g) = \sum p_i f(g_i g)$$

has uniform spectral gap. Then there is a number $\lambda_0$ such that for any $\lambda_0 < \lambda_1, \ldots, \lambda_n < 1$ and $v_1, \ldots, v_n \in \mathbb{R}^d$, the measure selfsimilar for

$$\kappa_i(x) = \lambda_i g_i(x) + v_i$$

and $p_i$ is absolutely continuous or concentrated in a single point. The latter case arises precisely when the similarities have a common fixed point.
Idea of proofs:

Recall notation: $X_1, X_2, \ldots$ are i.i.d. random isometries.

\[
Y_0 = x_0, \quad Y_1 = X_1(x_0), \ldots \quad Y_l = X_l \cdots X_1(x_0)
\]

$\theta : \text{Isom}(\mathbb{R}^d) \to \text{SO}(d), \quad v : \text{Isom}(\mathbb{R}^d) \to \mathbb{R}^d$

We introduce the following operators acting on $L^2(S^{d-1})$ for a parameter $0 \leq r < \infty$:

\[
S_r f(\xi) = \mathbb{E}[e^{-2\pi ir \langle \xi, v(X_1) \rangle} f(\theta(X_1)^{-1} \xi)]
\]

We prove the estimate

\[
\|S_r\| \leq 1 - c \min(1, r^2).
\]

Both theorems can be deduced from this.
The proof relies on the method of Bourgain and Gamburd.

The most important new ingredient is the estimate

$$\mathbb{P}(|Y_l| < e^{-c_1 l}) \leq e^{-c_2 l}$$

for some numbers $c_1, c_2$.

This is proved without any Diophantine assumption only relying on the spectral gap for the rotations.

There are three steps:

- There are numbers $\varepsilon_1, \varepsilon_2 > 0$ such that $\min(\|S_r\|, \|S_{r+\varepsilon_1}\|) \leq 1 - \varepsilon_2$ for any $r$,
- $\min(\|\hat{\nu}_r\|_{L^2(r \cdot S^{d-1})}, \|\hat{\nu}_r\|_{L^2((r+\varepsilon_1) \cdot S^{d-1})}) \leq e^{-\varepsilon_2 l}$, where $\nu_l$ is the law of $Y_l$,
- $|\|\hat{\nu}\|_{L^2(r \cdot S^{d-1})} - \|\hat{\nu}\|_{L^2((r+\varepsilon_1) \cdot S^{d-1})}| \leq r^{-\varepsilon_3}$ for any probability measure $\nu$ on $\mathbb{R}^d$. 
There are numbers $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\min(\|S_r\|, \|S_{r+\varepsilon_1}\|) \leq 1 - \varepsilon_2$$

for any $r$.

Introduce the unitary representation of $\text{Isom}(\mathbb{R}_d)$ acting on $L^2(S^{d-1})$:

$$\rho_r(g) f(\xi) = e^{-2\pi ir\langle v(g), \xi \rangle} f(\theta(g)^{-1} \xi).$$

With this notation: $S_r = \rho_r(\mu)$.

If $\|S_r \varphi_1\|$ and $\|S_{r+\varepsilon_1} \varphi_2\|$ are both large, then $\varphi_1$ is almost invariant for $\rho_r(g)$ and $\varphi_2$ is almost invariant for $\rho_{r+\varepsilon_2}(g)$ for $g \in \text{supp}\mu$. Then $\varphi_1 \varphi_2$ is almost invariant for $\rho_{\varepsilon_2}(g)$ for $g \in \text{supp}\mu$, hence $\|S_{\varepsilon_1} \varphi_1 \varphi_2\|$ is also large.

$S_{\varepsilon_1}$ is a perturbation of $S_0$ that we understand.
Thank You!