Krzysztof Burdzy
University of Washington
Collaborators and advisers

Collaborators: Jayadev Athreya and Mauricio Duarte

Work in progress.

KB and Mauricio Duarte. “A lower bound for the number of elastic collisions.” arXiv:1803.00979

We are most grateful to (in chronological order): Jaime San Martin, Curt McMullen, Branko Grünbaum, Rekha Thomas.
Consider \( n \) billiard balls \((n < \infty)\) in a \( d \)-dimensional Euclidean space. The billiards table has no boundary.
Consider $n$ billiard balls ($n < \infty$) in a $d$-dimensional Euclidean space. The billiards table has no boundary.

The balls have equal radii and equal masses.
Consider $n$ billiard balls ($n < \infty$) in a $d$-dimensional Euclidean space. The billiards table has no boundary.

The balls have equal radii and equal masses.

The collisions between the balls are totally elastic. The total energy, total momentum and total angular momentum are preserved.
Billiard balls

Consider \( n \) billiard balls \( (n < \infty) \) in a \( d \)-dimensional Euclidean space. The billiards table has no boundary.

The balls have equal radii and equal masses.

The collisions between the balls are totally elastic. The total energy, total momentum and total angular momentum are preserved.
The number of intersections of $n$ half-lines can be any integer in the range from $0$ to $(n - 1)/2$ (= the number of pairs of distinct half-lines).

Let $K(n, d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

$K(3, 1) = 3$. Is $K(3, 2)$ greater than 3?
The number of intersections of $n$ half-lines can be any integer in the range from $0$ to $\binom{n}{2}$ (the number of pairs of distinct half-lines).

Let $K(n,d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

$K(3,1) = 3$. Is $K(3,2)$ greater than $3$?
The number of intersections of $n$ half-lines can be any integer in the range from 0 to $n(n - 1)/2 (= \text{the number of pairs of distinct half-lines})$. 

Let $K(n, d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

$K(3, 1) = 3$. Is $K(3, 2)$ greater than 3?
The number of intersections of $n$ half-lines can be any integer in the range from 0 to $n(n - 1)/2$ (the number of pairs of distinct half-lines).

Let $K(n, d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).
The number of intersections of $n$ half-lines can be any integer in the range from 0 to $n(n-1)/2$ (= the number of pairs of distinct half-lines).

Let $K(n, d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

$K(3, 1) = 3$. 
The number of intersections of $n$ half-lines can be any integer in the range from 0 to $n(n - 1)/2$ (=
the number of pairs of distinct half-lines).

Let $K(n, d)$ be the supremum of the number of collisions of $n$ balls in $d$-dimensional space. The supremum is taken over all initial conditions (positions and velocity vectors).

$K(3, 1) = 3$. Is $K(3, 2)$ greater than 3?
Murphy and Cohen (2000): In the 60’s, Uhlenbeck conjectured that the maximum number of collisions of three balls in any number of dimensions was 3. Intuition suggests that the balls are most “constrained” in one dimension.
Murphy and Cohen (2000): In the 60’s, Uhlenbeck conjectured that the maximum number of collisions of three balls in any number of dimensions was 3. Intuition suggests that the balls are most “constrained” in one dimension.

Sinai asked whether a finite family of balls can have an infinite number of collisions. (70’s ?)
EXAMPLE (Foch (1960’s, unpublished), Murphy and Cohen (2000))

\[ K(3, 2) \geq 4 > \frac{3(3 - 1)}{2} = 3 = K(3, 1). \]
Three balls — the answer

THEOREM (Cohen (1966), Murphy and Cohen (1993))

\[ K(3, d) \leq 4. \]

Therefore, for \( d \geq 2, \)

\[ K(3, d) = 4 > 3(3 - 1)/2 = 3 = K(3, 1). \]
THEOREM (Cohen (1966), Murphy and Cohen (1993))

\[ K(3, d) \leq 4. \]

Therefore, for \( d \geq 2, \)

\[ K(3, d) = 4 > \frac{3(3 - 1)}{2} = 3 = K(3, 1). \]

There are no theorems (examples) in the existing literature showing that \( K(n, d) > K(n, 1) \) for any \( n \geq 4, \ d \geq 2. \)
THEOREM (Vaserstein (1979))

For any number of balls, dimension of the space and initial conditions, the number of collisions is finite.
Upper bounds

THEOREM (Vaserstein (1979))
For any number of balls, dimension of the space and initial conditions, the number of collisions is finite.

THEOREM (Burago, Ferleger and Kononenko (1998))

\[ K(n, d) \leq \left(32n^{3/2}\right)^{n^2}, \]
\[ K(n, d) \leq (400n^2)^{2n^4}. \]
New lower bound

THEOREM (B and Duarte (2018))

\[ K(n, d) \geq K(n, 2) \geq f(n) > \frac{n^3}{27}, \quad d \geq 2. \]

\[ K(n, 1) = \frac{n(n - 1)}{2} \quad \text{for } n \geq 2. \]
Lower bound: example

\[ \begin{align*}
A & \quad B \quad C \\
A_1 & \quad B_1 \quad C_1 \\
A_2 & \quad A_1 \\
A_3 & \\
B_2 & \quad C_2
\end{align*} \]
Improvement of Foch’s example

THEOREM (B and Duarte (2018))

\[ K(n, d) \geq K(n, 2) \geq f(n) > \frac{n^3}{27}, \quad d \geq 2. \]
Improvement of Foch’s example

THEOREM (B and Duarte (2018))

\[ K(n, d) \geq K(n, 2) \geq f(n) > \frac{n^3}{27}, \quad d \geq 2. \]

\[ f(n) > \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } n \geq 7, \]

\[ f(n) = \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } n = 6, \]

\[ f(n) < \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } 3 \leq n \leq 5. \]
THEOREM (B and Duarte (2018))

\[ K(n, d) \geq K(n, 2) \geq f(n) > \frac{n^3}{27}, \quad d \geq 2. \]

\[ f(n) > \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } n \geq 7, \]
\[ f(n) = \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } n = 6, \]
\[ f(n) < \frac{n(n - 1)}{2} = K(n, 1) \quad \text{for } 3 \leq n \leq 5. \]
OPEN PROBLEM

Do there exist \( n \geq 2 \) and \( d_1 > d_2 \geq 2 \) such that

\[
K(n, d_1) > K(n, d_2)?
\]
Consider a family of $n$ balls in $d$ dimensional space.
The center of the $k$-th ball: $x_k(t)$; the velocity of the $k$-th ball: $v_k(t)$.
\[
\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad \mathbf{v}(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}
\]
Consider a family of \( n \) balls in \( d \) dimensional space. The center of the \( k \)-th ball: \( x_k(t) \); the velocity of the \( k \)-th ball: \( v_k(t) \). 

\[
\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad \mathbf{v}(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}
\]

Assume ("without loss of generality") that \( \sum_{k=1}^{n} v_k(0) = 0 \) and \( |\mathbf{v}(t)| = 1 \).
Distant collisions

Consider a family of \( n \) balls in \( d \) dimensional space. The center of the \( k \)-th ball: \( x_k(t) \); the velocity of the \( k \)-th ball: \( v_k(t) \). 

\[
x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad v(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}
\]

Assume ("without loss of generality") that \( \sum_{k=1}^{n} v_k(0) = 0 \) and \( |v(t)| = 1 \).

**PROPOSITION (B and Duarte)**

(i) The family of \( n \) balls can be partitioned into two non-empty subfamilies such that no ball from the first family will ever collide with a ball in the second family after time \( 100n^3|x(0)| \).
Consider a family of $n$ balls in $d$ dimensional space. The center of the $k$-th ball: $x_k(t)$; the velocity of the $k$-th ball: $v_k(t)$.

$x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}$, $v(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}$

Assume ("without loss of generality") that $\sum_{k=1}^{n} v_k(0) = 0$ and $|v(t)| = 1$.

**PROPOSITION (B and Duarte)**

(i) The family of $n$ balls can be partitioned into two non-empty subfamilies such that no ball from the first family will ever collide with a ball in the second family after time $100n^3|x(0)|$.

(ii) The family of $n$ balls can be partitioned into two non-empty subfamilies such that no ball from the first family will ever collide with a ball in the second family when $|x(t)| \geq (1 + 100n^3)|x(0)|$.

Vaserstein (1979), Illner (1990)
COROLLARY (B and Duarte)

Suppose that at the time of collision between any two balls, the distance between any other pair of balls is greater than $n^{-n}$. Then the total number of collisions is bounded by $n^{5n}$, for large $n$. 
Pinned billiard balls

Krzysztof Burdzy

COLLISIONS OF BILLIARD BALLS AND FOLDINGS
Large family of pinned balls
Large family of pinned balls
Consider \( n \) balls which may touch but not overlap. The balls are labeled 1 to \( n \). The balls are pinned (they cannot move). The \( k \)-th ball is associated with a pseudo-velocity vector \( v_k \).

Consider an infinite sequence of pairs of labels \( \{(i_k, j_k), \ k \geq 1\} \). We have \( 1 \leq i_k, j_k \leq n \). The pairs of balls undergo pseudo-collisions, in the order determined by the sequence. If \( i_k = j_k \) or the balls labeled \( i_k \) and \( j_k \) do not touch then \( v_{i_k} \) and \( v_{j_k} \) remain unchanged. If the balls touch then the pseudo-velocities change according to the law of elastic collision.
Consider $n$ balls which may touch but not overlap. The balls are labeled 1 to $n$. The balls are pinned (they cannot move). The $k$-th ball is associated with a pseudo-velocity vector $v_k$.

Consider an infinite sequence of pairs of labels $\{(i_k, j_k), k \geq 1\}$. We have $1 \leq i_k, j_k \leq n$. The pairs of balls undergo pseudo-collisions, in the order determined by the sequence. If $i_k = j_k$ or the balls labeled $i_k$ and $j_k$ do not touch then $v_{i_k}$ and $v_{j_k}$ remain unchanged. If the balls touch then the pseudo-velocities change according to the law of elastic collision.

**PROPOSITION (Athreya, B and Duarte)**

For any family of balls, pseudo-velocities $v_k$ and sequence $\{(i_k, j_k), k \geq 1\}$, the pseudo-velocities will freeze after a finite number of pseudo-collisions.
Finite evolution

Consider \( n \) balls which may touch but not overlap. The balls are labeled 1 to \( n \). The balls are pinned (they cannot move).

The \( k \)-th ball is associated with a pseudo-velocity vector \( v_k \).

Consider an infinite sequence of pairs of labels \( \{(i_k, j_k), k \geq 1\} \). We have \( 1 \leq i_k, j_k \leq n \). The pairs of balls undergo pseudo-collisions, in the order determined by the sequence. If \( i_k = j_k \) or the balls labeled \( i_k \) and \( j_k \) do not touch then \( v_{i_k} \) and \( v_{j_k} \) remain unchanged. If the balls touch then the pseudo-velocities change according to the law of elastic collision.

**PROPOSITION (Athreya, B and Duarte)**

For any family of balls, pseudo-velocities \( v_k \) and sequence \( \{(i_k, j_k), k \geq 1\} \), the pseudo-velocities will freeze after a finite number of pseudo-collisions.

**OPEN PROBLEM**

Can the above proposition be derived from the theorems of Vaserstein or Burago, Ferleger and Kononenko?
PROPOSITION (Athreya, B and Duarte)

Suppose that a family of open halfspaces passing through 0 has a non-empty intersection $M$. Consider a point $x$ and a sequence $F_1, F_2, \ldots$ of foldings relative to these halfspaces such that every halfspace is represented infinitely often. Then there exists $k$ such that

$$F_k \circ F_{k-1} \circ \cdots \circ F_2 \circ F_1(x) \in M.$$
THEOREM (Athreya, B and Duarte)

The maximum number of collisions for a system $F$ of $n$ pinned balls in $d$-dimensional space does not exceed

$$\left( \frac{1024dn^5}{\alpha(F)} \right)^{\tau d n/2 - 1}.$$

Here, $\tau_d$ is the kissing number for balls in $d$ dimensions, and $\alpha(F)$ is the "index of approximate rigidity," a totally explicit but complicated function of the positions of the balls.
Upper bound for the number of pseudo-collisions

**THEOREM (Athreya, B and Duarte)**

The maximum number of collisions for a system $F$ of $n$ pinned balls in $d$-dimensional space does not exceed

$$\left( \frac{1024dn^5}{\alpha(F)} \right)^{\tau_d n/2-1}.$$ 

$\tau_d$ - the kissing number for balls in $d$ dimensions
THEOREM (Athreya, B and Duarte)

The maximum number of collisions for a system $F$ of $n$ pinned balls in $d$-dimensional space does not exceed

$$\left( \frac{1024dn^5}{\alpha(F)} \right)^{\tau_d n/2-1}.$$ 

$\tau_d$ - the kissing number for balls in $d$ dimensions

$\alpha(F)$ - “index of approximate rigidity;” a totally explicit but complicated function of the positions of the balls
COROLLARY (Athreya, B and Duarte)

If the family of \( n \) pinned balls is a tree then the number of (pseudo-)collisions is not greater than

\[
(1024dn^6)^{\frac{\tau_d n}{2} - 1}.
\]
Suppose that the centers of pinned balls belong to the triangular lattice. Then the number of (pseudo-)collisions is not greater than

$$10^{24n} n^{18n} g^{18n^2}.$$
$K(n, 2) > \frac{n^3}{27}$. 
Evolution (1)

Krzysztof Burdzy

COLLISIONS OF BILLIARD BALLS AND FOLDINGS
Beads on a line
Evolution (2)

Krzysztof Burdzy

COLLISIONS OF BILLIARD BALLS AND FOLDINGS
Evolution (4)

Krzysztof Burdzy
COLLISIONS OF BILLIARD BALLS AND FOLDINGS
Evolution (5)

$B_2$

$B_1$

$A_1$

$C_1$

$C_2$
Krzysztof Burdzy

COLLISIONS OF BILLIARD BALLS AND FOLDINGS
Reflections near corner
Multiple collisions

Krzysztof Burdzy

COLLISIONS OF BILLIARD BALLS AND FOLDINGS
A monotone functional

Consider a family of \( n \) balls in \( d \) dimensional space. The center of the \( k \)-th ball: \( x_k(t) \); the velocity of the \( k \)-th ball: \( v_k(t) \).

\[
\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad \mathbf{v}(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}
\]
A monotone functional

Consider a family of $n$ balls in $d$ dimensional space. The center of the $k$-th ball: $x_k(t)$; the velocity of the $k$-th ball: $v_k(t)$. 

$x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad v(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}$

Let $\tilde{x}(t)$ and $\tilde{v}(t)$ have the analogous meaning for the system of non-interacting balls with the same initial conditions.
A monotone functional

Consider a family of $n$ balls in $d$ dimensional space. The center of the $k$-th ball: $x_k(t)$; the velocity of the $k$-th ball: $v_k(t)$. 

$$\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}, \quad \mathbf{v}(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}$$

Let $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{v}}(t)$ have the analogous meaning for the system of non-interacting balls with the same initial conditions.

**LEMMA (Vaserstein (1979), Illner (1990))**

(i) The following functional is non-decreasing for colliding billiard balls,

$$t \rightarrow \frac{\mathbf{x}(t)}{||\mathbf{x}(t)||} \cdot \mathbf{v}(t).$$
A monotone functional

Consider a family of $n$ balls in $d$ dimensional space. The center of the $k$-th ball: $x_k(t)$; the velocity of the $k$-th ball: $v_k(t)$. $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^{dn}$, $v(t) = (v_1(t), \ldots, v_n(t)) \in \mathbb{R}^{dn}$

Let $\tilde{x}(t)$ and $\tilde{v}(t)$ have the analogous meaning for the system of non-interacting balls with the same initial conditions.

**LEMMA (Vaserstein (1979), Illner (1990))**

(i) The following functional is non-decreasing for colliding billiard balls,

$$ t \rightarrow \frac{x(t)}{|x(t)|} \cdot v(t). $$

(ii)

$$ \frac{x(t)}{|x(t)|} \cdot v(t) \geq \frac{\tilde{x}(t)}{|\tilde{x}(t)|} \cdot \tilde{v}(t). $$
A certain function, called entropy, of the amount of heat transferred divided by the temperature, can only increase. On the other hand, mechanics, to which Boltzmann wanted to reduce thermodynamics, is strictly reversible, in the sense that for every motion there is another motion described by reversing the sign of the variable denoting time, which is equally possible. The opponents of Boltzmann’s atomism kept pointing out that you cannot expect to obtain irreversibility from a theory according to which all processes are essentially reversible.
Non-existence of atoms

Evolutions of billiard ball (atom) families are time reversible.

Entropy is monotone. Hence, the universe is not made of atoms.
Evolutions of billiard ball (atom) families are time reversible.

Therefore, there are no (non-constant) monotone functionals of billiard ball evolutions.
Non-existence of atoms

Evolutions of billiard ball (atom) families are time reversible.

Therefore, there are no (non-constant) monotone functionals of billiard ball evolutions.

Entropy is monotone.
Non-existence of atoms

Evolutions of billiard ball (atom) families are time reversible.

Therefore, there are no (non-constant) monotone functionals of billiard ball evolutions.

Entropy is monotone.

Hence, the universe is not made of atoms.
Velocities and speeds

The center of the $k$-th ball: $x_k(t)$.
The velocity of the $k$-th ball: $v_k(t)$.
The speed of the $k$-th ball: $|v_k(t)|$. 
Evolutions of billiard ball (atom) families are time reversible.

FALSE: Therefore, there are no (non-constant) monotone functions of billiard ball positions and velocities.

TRUE: Therefore, there are no (non-constant) increasing functions of billiard ball positions and speeds.

Is entropy an increasing function of billiard ball positions and speeds?

DEFINITIONS
(i) Physics: Entropy is the logarithm of the number of (equally probable) microscopic configurations corresponding to the given macroscopic state of the system.
(ii) Applied mathematics (KB): Physical entropy is the mathematical (probabilistic) entropy of the empirical distribution of positions and momenta for the coarse grained model of the system.
Evolutions of billiard ball (atom) families are time reversible.

FALSE: Therefore, there are no (non-constant) monotone functions of billiard ball positions and velocities.

TRUE: Therefore, there are no (non-constant) increasing functions of billiard ball positions and speeds.
Non-existence of atoms revisited

Evolutions of billiard ball (atom) families are time reversible.

FALSE: Therefore, there are no (non-constant) monotone functions of billiard ball positions and velocities.

TRUE: Therefore, there are no (non-constant) increasing functions of billiard ball positions and speeds.

Is entropy an increasing function of billiard ball positions and speeds?
Non-existence of atoms revisited

Evolutions of billiard ball (atom) families are time reversible.

FALSE: Therefore, there are no (non-constant) monotone functions of billiard ball positions and velocities.

TRUE: Therefore, there are no (non-constant) increasing functions of billiard ball positions and speeds.

Is entropy an increasing function of billiard ball positions and speeds?

DEFINITIONS
(i) Physics: Entropy is the logarithm of the number of (equally probable) microscopic configurations corresponding to the given macroscopic state of the system.
(ii) Applied mathematics (KB): Physical entropy is the mathematical (probabilistic) entropy of the empirical distribution of positions and momenta for the coarse grained model of the system.