Smooth and non-smooth aspects of Ricci curvature lower bounds

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- The Gaussian Curvature $K^G_\Sigma(p)$ of (Σ, g) at p is defined by

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- Examples:
  - $0 \equiv$ Gaussian curvature of the euclidean plane $\mathbb{R}^2$.
  - $\frac{1}{r^2} \equiv$ Gaussian curvature of a 2-dimensional round sphere in $\mathbb{R}^3$ of radius r.
  - $-1 \equiv$ Gaussian curvature of the Hyperbolic plane.

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- Let $\Sigma_\Pi = \mathrm{Exp}_p(\Pi \cap B_\varepsilon(0))$=surface obtained by considering all the geodesics starting at p tangent to Π up to length ε. For $\varepsilon > 0$ small enough $\Sigma_\Pi \subset M$ is a smooth 2-dim surface.

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- Define the Sectional Curvature of (M, g) at the 2-dim plane $\mathrm{span}(\vec{e}_1, \vec{e}_2) = \Pi \subset T_pM$ as

$$\mathrm{Sec}_p(\vec{e}_1, \vec{e}_2) = K^G_{\Sigma_\Pi}(p) = \text{Gaussian curvature of } \Sigma_\Pi \text{ at } p.$$

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- Define the Ricci Curvature of (M, g) at the vector $\vec{v} \in T_pM$, $\vec{v} \neq \vec{0}$ as

$$\mathrm{Ric}_p(\vec{v}, \vec{v}) = |\vec{v}|^2 \sum_{i=1}^{n-1} \mathrm{Sec}_p(\vec{v}, \vec{e}_i) \text{``} = \text{trace of the curvature}''$$

where $\{\vec{e}_1, \dots, \vec{e}_{n-1}, \vec{v}/|\vec{v}|\}$ is an orthonormal basis of (T_pM, g_p). Set $\mathrm{Ric}_p(\vec{0}, \vec{0}) = 0$.

Some notational remarks on the curvature bounds

- For $K \in \mathbb{R}$, we write $\mathrm{Sec} \geq K$ (resp. $\leq K$) if for every $p \in M$ and every 2-dim plane $\Pi \subset T_pM$ it holds $\mathrm{Sec}_p(\Pi) \geq K$ (resp. $\leq K$).

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- Examples:
  - n-dimensional Euclidean space: $\mathrm{Sec} \equiv 0$, $\mathrm{Ric} \equiv 0$.
  - n-dimensional round sphere of radius 1: $\mathrm{Sec} \equiv 1$, $\mathrm{Ric} \equiv n-1$.
  - n-dimensional hyperbolic space: $\mathrm{Sec} \equiv -1$, $\mathrm{Ric} \equiv -(n-1)$.

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- Upper bounds on the Ricci curvature are very (too) weak assumption for topological conclusions. E.g. Lokhamp Theorem (Gao-Yau, Brooks in dim 3): any compact manifold of dim≥ 3 carries a metric with negative Ricci curvature.

Some basics of comparison geometry: lower Ricci bounds

Lower bounds on the Ricci curvature: natural framework for comparison geometry

- Bishop-Gromov volume comparison: (not most general form) If (M^n, g) has $\mathrm{Ric} \geq 0$ then

$$\mathrm{vol_g}(B_R(x)) \leq \omega_n R^n, \quad \forall R > 0, \, \forall x \in M,$$

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  If (M^n, g) has no boundary and $\mathrm{Ric} \geq n - 1$ then
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- Laplacian comparison,
- Cheeger-Gromoll splitting,
- Li-Yau inequalities on heat flow,
- Lévy-Gromov isoperimetric inequality,
- ...

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•Hope: useful also to establish new results for smooth manifolds.

Semi-smooth setting

- Cheeger-Colding 1997-2000 three fundamental papers on JDG on the structure of Ricci limit spaces.
  - Collapsing: $\lim_k vol_{g_k}(B_1(\bar{x_k})) = 0 \rightsquigarrow$ loss of dimension in the limit. More difficult, nevertheless they proved that the limit space has a uniquely defined volume measure (up to scaling) and a.e. point has a euclidean tangent space (the dimension may vary from point to point). Such points are called regular points, the complementary is called singular set.

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- Colding-Naber, Annals of Math. 2012: the dimension of the tangent space does not change on the regular set, even in the collapsed case.
- Cheeger-Jiang-Naber, Annals of Math 2020: in non-collapsed case, the singular set is stratified into rectifiable strata.

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Analogy with:

- Sobolev functions (non smooth functions, differentiable in a weak sense, via integration by parts)
- Geometric Measure Theory (sets of finite perimeter, currents, varifolds, which can be seen as generalized submanifolds, extremely useful for studying the calculus of variations for the area functional)

Preliminary Observation

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- Ricci curvature is a property of lenghts and volumes: needs also a reference volume measure
  $\rightsquigarrow$ natural setting metric measure spaces $(X, \mathrm{d}, \mathfrak{m})$.

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- Given $\mu_1, \mu_2 \in \mathcal{P}(X)$, define the (Kantorovich-Wasserstein) quadratic transportation distance

$$W_2(\mu_1, \mu_2) := \inf \left\{ \sqrt{\int_{X \times X} \mathsf{d}^2(x, y) \, \gamma(\mathrm{d}x\mathrm{d}y)} \right\}$$

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- $(\mathcal{P}(X), W_2)$ is a metric space, geodesic if (X, d) is geodesic .

Non smooth setting 2: Entropy functionals

- From now on, for simplicity of presentation, consider only probability measures absolutely continuous w.r.t. $\mathfrak{m}$, i.e. $\mu = \rho\,\mathfrak{m} \ll \mathfrak{m}$ with $\rho \geq 0$, $\int_X \rho\,\mathrm{d}\mathfrak{m} = 1$;

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- On the metric space $(\mathcal{P}(X), W_2)$ consider the Entropy functionals

$$\mathcal{U}_{N,\mathfrak{m}}(\rho\mathfrak{m}) \quad := \quad -N \int \rho^{1-\frac{1}{N}} \,\mathrm{d}\mathfrak{m} \quad \text{if } 1 < N < \infty \quad \text{Rényi Entropy}$$

$$\mathcal{U}_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) \quad := \quad \int \rho \log \rho \,\mathrm{d}\mathfrak{m} \quad \text{Boltzmann-Shannon Entropy}$$

Non smooth setting: intrinsic-axiomatic definition

- THM (McCann, Otto-Villani, Cordero-Erausquin-McCann-Schmuckenschläger, Sturm-von Renesse) If $(X, \mathrm{d}, \mathfrak{m})$ is a smooth Riemannian manifold (M, g), then $\mathrm{Ric} \geq 0$ iff the entropy functional $\mathcal{U}_{\infty,\mathfrak{m}}$ is weakly convex along geodesics in $(\mathcal{P}(X), W_2)$,

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- DEF of CD(K, N) condition [Lott-Sturm-Villani '06]: For $N \in [1, +\infty]$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a CD$(0, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is convex along geodesics in $(\mathcal{P}(X), W_2)$.

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- DEF of CD(K, N) condition [Lott-Sturm-Villani '06]: For $N \in [1, +\infty]$, we say that $(X, \mathsf{d}, \mathfrak{m})$ is a CD$(0, N)$-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is convex along geodesics in $(\mathcal{P}(X), W_2)$. For general $K \in \mathbb{R}$, $N \in [1, +\infty]$, we say $(X, \mathsf{d}, \mathfrak{m})$ is a CD(K, N)-space if the Entropy $\mathcal{U}_{N,\mathfrak{m}}$ is "(K, N)-geod. conv."

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Definition Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, $(X, \mathrm{d}, \mathfrak{m})$ is an RCD(K, N) space if it is a CD(K, N) space & the Cheeger energy is quadratic.

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