Finite Codimension Stability of Invariant Surfaces

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Billiards in Polygons

A polygonal billiard is the dynamical system given by the motion of a light ray in a polygonal (planar, Euclidean) mirror chamber P: Straight linear motion inside the polygon and reflection at the boundary angle of incidence = angle of reflection.

Its (3-dimensional) phase space is $P \times S^1/E$ with E expressing the reflection equivalence relation at ∂P. The flow is defined almost everywhere (on the complement of countably many hypersurfaces).

Mechanical systems

The motion of two points masses m_1, m_2 on the interval $[0, 1]$ with elastic collisions between them and at the endpoints is equivalent to a billiard in a right triangle with angle $\arctan(\sqrt{m_1/m_2})$.

The motion of three point masses m_1, m_2, m_3 on a circle with elastic collisions between them is equivalent to a billiard in an acute triangle with angles $\arctan(m_i\sqrt{(m_1+m_2+m_3)/m_1 m_2 m_3})$.

Motivations

Semiclassical methods for weakly chaotic systems:
pseudo-integrable systems (Richens and Berry, 1981).
Dynamics of parabolic systems or elliptic with singularities (Fox and Kershner, 1936; Zemlyakov and A. Katok, 1975,)

Billiards in planar domains provide examples of several important classes of dynamical behavior:
Birkhoff smooth convex billiards, Sinai's billiards, Bunimovich stadium, polygonal billiards (see A. Katok, "Billiard table as a playground ..." or R. Schwartz ICM Survey Lecture on Billiards)

More general problem: geodesic flow on a flat compact orientable surface with conical singularities. (The double of the polygon is a flat sphere).

Basic dichotomy between a rational / a non-rational polygon.

Rational: the group G_P generated by reflection with respect to edges is finite $\Rightarrow$ the phase space is foliated by invariant surfaces.

The invariant surfaces can be constructed by considering branched covers (Fox and Kershner, 1936) of the double of the polygon or its unfolding (Zemlyakov and Katok, 1975). They are translation surfaces and the flow on invariant surfaces is a translation flow.

For a polygon with angles $\pi m_i/n_i$ (in lowest terms) the invariant surfaces have genus

$$g = 1 + \frac{l.c.m.(n_1, \ldots, n_k)}{2}(k - 2 - \sum_{i=1}^{k} \frac{1}{n_i})$$

Classification of completely integrable billiards (phase space foliated by invariant tori).

Dynamical properties:

For all ALL Billiards in Polygons: Zero Entropy (A. Katok, CMP 1987) (Billiards in Polygons are not hyperbolic, rather they may be classified as Parabolic/Elliptic with singularities)

Rational Case:

- Uniquely Ergodic on almost all invariant surfaces (Kerckhoff, Masur and Smillie, Ann. Math. 1986)
  (refined to results on HD, see Chaika-Masur, Inv. Math. 2020, speed of ergodicity, see Athreya-F., Duke M. J. 2008),
- Never mixing (A. Katok, Israel J. 1980)
  The above results are proved for ALL translation surfaces.
- Weak mixing for almost all translation surfaces: with respect to Masur-Veech measures, Avila-F, Ann. Math. 2007 (also for almost all IET's).

Weakly Mixing Rational Polygons ? (*on almost all invariant surfaces*)

Theorem (F. Arana-Herrera, J. Chaika, G. F. 2024)

A billiard in a rational polygon is weakly mixing on almost all invariant surfaces (in almost all directions), unless the polygon is almost integrable (in the sense of E. Gutkin) or it is the rotation of a polygon with only vertical and horizontal sides such that the horizontal sides or the vertical sides have commensurable lengths.

Remarks

A. Katok, 1980: The flow of a rational polygonal billiard is never mixing.

E. Gutkin, 1986: conjectured a weak (generic) form of the above theorem.

A. Katok and E. Gutkin, 1988: Topological genericity of the weak mixing property in a given direction for certain classes of polygons.

Avila and Delecroix, JAMS 2016: Any non-arithmetic Veech translation surface, including Regular Polygons with at least 5 edges and other special examples

Aulicino, Avila and Delecroix (announced): Any translation surface in a rank one suborbifold, including countably many L-shaped billiards with a barrier (Aulicino, Avila and Delecroix);

For the more general class of translation surfaces:

Theorem (F. Arana-Herrera, J. Chaika, G. F. 2024)

Let (X, ω) be a translation surface. The following are equivalent:

1. *The directional flow of (X, ω) is weakly mixing in almost all directions;*
2. *The directional flow of (X, ω) is weakly mixing in some direction;*
3. *The surface (X, ω) does not have an affine circle factor;*
4. *The plane $T(\omega) = \mathbb{R}Re(\omega) + \mathbb{R}Im(\omega) \subset H^1(X, \mathbb{R})$ has trivial intersection with the lattice $H^1(X, \mathbb{Z})$.*

The core of the theorem is (4) $\Rightarrow$ (1).

KAM stability

The (flat) billiard flow on a completely integrable polygon is a completely integrable Hamiltonian flow.

By the KAM theorem, for Hamiltonian perturbations of the billiard flow, sufficiently small in a smooth topology, a positive measure set of invariant tori persist (in fact these are the tori with Diophantine frequency vectors).

An example of a smooth Hamiltonian perturbation could be a perturbation of the flat metric in the interior of the polygon, or of an edge away from the corners.

Question Does KAM stability holds when the invariant surfaces have higher genus (and the flat metric on them has conical singularities) ?

Theorem (F. 2025 following T. Alazard and C. Shao)

The typical surface persists with finite codimension, in the following sense. For any regularity class there is a local submanifold of the space of Hamiltonians which coincide with the unperturbed Hamiltonian near the corners, such that for Hamiltonians on such a submanifold the Hamiltonian flow has an invariant surface in the given regularity class (the codimension increases linearly with the degree of smoothness) on which the flows is conjugated to the unperturbed flow on the invariant surface.

Question Does there exist Hamiltonian perturbations supported away from corners with no invariant surfaces ? with ergodic flow ?

The linearized problem: the cohomological equation

Theorem (F97, Marmi, Moussa, Yoccoz MMY05)

There exists $s_0 > 0$ such that, for all $s > s_0$ and for almost all $\xi \in \mathbb{R}^2$ the cohomological equation $X_\xi u = f$ has a solution for all $f \in H^s(M)$ which belongs to the kernel $Ker\Big(\mathcal{I}^s_\xi(M)\Big)$ of the space of invariant distributions in $H^{-s}(M)$ and for all $t < s - s_0$ there exists a constant $C_{s,t}(\xi) > 0$ such that

$$\|u\|_{H^t(M)} \leq C_{s,t}(\xi) \|f\|_{H^s(M)} ;$$

The space $\mathcal{I}^s_\xi(M)$ is finite dimensional but its dimension grows linearly with respect to the regularity parameter $s > 0$ (and with respect to the genus).

A model problem: smooth conjugacies

Theorem (Marmi, Moussa, Yoccoz, MMY12)

The typical translation flows is smoothy stable with finite codimension, in the following sense. For any regularity class there is a local submanifold of the space of flows which coincide with the unperturbed flow near the cone points, such that flows on such a submanifold are smoothly conjugate to the translation flow by a conjugacy in the given regularity class (the codimension increases linearly with the degree of smoothness).

No KAM (Nash-Moser) proof of this theorem is available.

MMY12 proof is based on M. Herman's *Schwartzian derivative trick* to compensate for the loss of regularity of the linearized problem, and to reduce the conjugacy problem to a fixed point problem in a Banach space (contraction mapping principle).

This method seems limited to the smooth conjugacy problem one-dimensional maps (2-dimensional flows).

Para-differential linearization Para-differential calculus gives a technique to reduce KAM type problems to fixed point problems in Banach space.

Proposition (Continuity of para-product operators)

If $a \in L^\infty(M)$, then the para-product T_a or $Op(a)$ is a bounded linear operator from $H^s(M)$ to itself, and in fact there exists a constant $C_s > 0$ such that

$$\|T_a\|_{\mathcal{L}(H^s(M),H^s(M))} \le C_s\|a\|_{L^\infty(M)} .$$

Proposition (Composition of para-product operators)

If $a, b \in C^r(M)$, then $T_{ab} - T_a T_b$ is a bounded linear operator from $H^s(M)$ to $H^{s+r}(M)$, and in fact there exists a constant $C_{r,s} > 0$ such that

$$\|T_{ab} - T_a T_b\|_{\mathcal{L}(H^s(M),H^{s+r}(M))} \le C_{r,s}\|a\|_{C^r}\|b\|_{C^r} .$$

Theorem (Para-linearization)

Let $s > 1$ and let $N_s \in \mathbb{N}$ denote the smallest integer such that $N_s > 2s - 1$. For any functions $u \in H^s(M, \mathbb{R}^2)$ and $F := F(x,u) \in C^{N_s+3}(M \times \mathbb{R}^2)$, the following para-linearization formula holds:

$$F(x,u) - F(x,0) = Op(\frac{\partial F(x,u)}{\partial u})u + \mathcal{R}_{PL}(F(x,\,),u)u \in H^s(M) + H^{2s-1}(M)\,,$$

where $\mathcal{R}_{PL}(F(x,\cdot),u)u$ is a bounded linear operator from $H^s(M)$ to $H^{2s-1}(M)$ such that for a constant $C'_s > 0$

$$\|\mathcal{R}_{PL}(F(x,\cdot),u)\|_{\mathcal{L}(H^s(M),H^{2s-1}(M))} \le C'_s \|F\|_{C^{N_s+3}(M\times\mathbb{R}^2)}(1+\|u\|_{H^s(M)})\,.$$

Moreover, the operators $Op(\frac{\partial F(x,u)}{\partial u}) \in \mathcal{L}(H^s(M), H^s(M))$ and $\mathcal{R}_{PL}(F(x,\cdot),u) \in \mathcal{L}(H^s(M), H^{2s-1}(M))$ are continuously differentiable in $u \in H^s(M)$ with respect to the operator norms.

The simplest example: the conjugacy problem for circle rotations (after T. Alazard and C. Shao)

Let $R_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation (translation) by α and let $F = R_\alpha + f$ a small perturbation.

Problem: Find $H = id + h$ and $\lambda \in \mathbb{R}$ such that $H \circ R_\alpha = F \circ H - \lambda$ or

$$\Delta_\alpha h := h \circ R_\alpha - h = f \circ H - \lambda \,.$$

Let $u = (h, \lambda)$ and $\mathcal{F}(f, u) = \Delta_\alpha(h) - f \circ H + \lambda$.

Find solutions of the equation $\mathcal{F}(f, u) = 0$ for f small enough in a smooth topology. By para-linearization of $f \circ H = f \circ (Id + h)$

$$\mathcal{F}(f, u) = \Delta_\alpha h - f - T_{f' \circ H}(h) + \lambda + \mathcal{R}_0(f, h)(h)$$

with remainder $\mathcal{R}_0(f, h)(h) \in H^{2s-1/2}(\mathbb{T})$ (for $h \in H^s(\mathbb{T})$).

By differentiating $\mathcal{F}(f,u)$:

$$f' \circ H = \frac{\Delta_\alpha(h')}{1+h'} - \frac{[\mathcal{F}(f,h)]'}{1+h'} \,,$$

hence, by rewriting $T_{f'\circ H}$ according to the above identity,

$$\begin{aligned}
\mathcal{F}(f,u) &= \Delta_\alpha h - f - OP\Big(\frac{\Delta_\alpha(h')}{1+h'}\Big)(h) \\
&- OP\Big(\frac{[\mathcal{F}(f,h)]'}{1+h'}\Big)(h) + \lambda + \mathcal{R}_1(f,h)(h) \\
&= T_{1+h'\circ R_\alpha} \Delta_\alpha T_{1/(1+h')}(h) - f \\
&+ T_{[\mathcal{F}(f,h)]'/(1+h')}(h) + \lambda + \mathcal{R}_2(f,h)(h)
\end{aligned}$$

The para-cohomological equation is obtained by dropping the term $T_{[\mathcal{F}(f,h)]'/(1+h')}(h)$ (linear in $\mathcal{F}(f,h)$):

$$T_{1+h'\circ R_\alpha}\Delta_\alpha T_{1/(1+h')}(h) - f + \lambda + \mathcal{R}_2(f,h)(h) = 0\,.$$

Fixed point problem in $H^s(\mathbb{R})$:

$$h = T^{-1}_{1/(1+h')}\Delta_\alpha^{-1} T^{-1}_{1+h'\circ R_\alpha}\Big[f - \lambda - \mathcal{R}_2(f,h)(h)\Big]\,.$$

(Δ_α^{-1} exists by Fourier series if α is Diophantine) to get

$$\mathcal{F}(f,u) = T_{[\mathcal{F}(f,h)]'/(1+h')}(h)$$

which implies $\mathcal{F}(f,h) = 0$ by a contraction argument.

THANKS FOR YOUR ATTENTION!