Biased random walk on dynamical percolation

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Probability, Analysis and Dynamics, Bristol, 2025

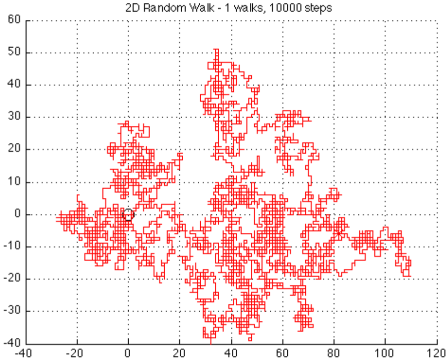
General goal: random motion in inhomogeneous/random environment

More and more realistic models:

- Random walks: classical but still active
- Random walks in random environment
- Random walks in dynamical random environment
- Random walks interacting with their environment (not in this talk)

Warm-up: simple random walks on the lattice

Consider first a simple random walk on the d-dimensional lattice, $d \geq 2$.



It starts from the origin and moves, with equal probabilities, to the nearest neighbours.

Warm-up: simple random walks on the lattice

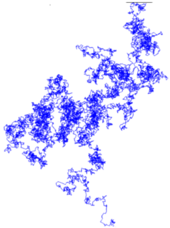
We can interprete the model in the following way: each bond is given "weight" or "conductance" 1 and the transition probabilities are proportional to the weights of the bonds. Hence, we have a random walk in a "homogeneous medium" or "constant environment".

Warm-up: simple random walks on the lattice

It is well-known that the scaling limit of simple random walk is a Brownian motion, a Gaussian process in continuous time on $\mathbb{R}^d$. More precisely, the law of (the linear interpolation of)

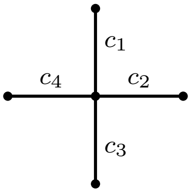
$$(X_m/\sqrt{n})_{m=0,1,\ldots,n}$$

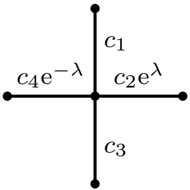
converges to the law of $(\sigma B_t)_{0\leq t\leq 1}$ where σ is a constant depending on the dimension d. This convergence is "universal" and holds (modifying σ) as well, for instance, for triangular lattices.



Warm-up: biased random walk on the lattice

Let's give a bias in a "favourite" direction. For simplicity, we assume that the favourite direction is e_1. We add a bias in direction e_1: choose a parameter $\lambda > 0$ for the strength of the bias and multiply the conductances with powers of e^{λ}.
Example: $d = 2, e_1 = (1, 0)$.

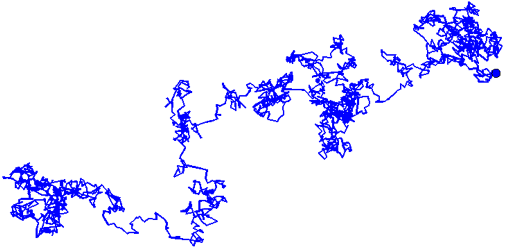




Here, $c_1 = c_2 = c_3 = c_4 = 1$. The parameter λ describes the *decisiveness* of the walker.

Warm-up: biased random walk on the lattice

The bias changes drastically the behaviour of the walk and it moves now with a constant linear speed.



Warm-up: biased random walk on the lattice

More precisely,

$$v(\lambda) := \lim_{n\to\infty} \frac{X_n}{n}$$

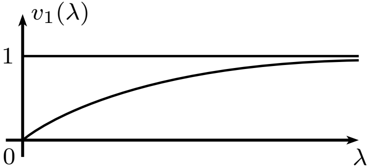
exists and $v(\lambda) = (v_1(\lambda), 0, 0, \dots, 0)$. Here, $v_1(\lambda)$ is the component of the speed in the favourite direction and $v_1(\lambda)$ can be computed with the law of large numbers. Namely,
$X_n = (X_1 - X_0) + (X_2 - X_1) + \dots (X_n - X_{n-1})$ and the increments $X_i - X_{i-1}$ are independent with the same law. This gives

$$v_1(\lambda) = \frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda} + (2d-2)}.$$

In particular, $v_1(\lambda) = v(\lambda) \cdot e_1$ is strictly positive.

Warm-up: biased random walk on the lattice

The function $\lambda \mapsto v_1(\lambda)$ looks as follows:



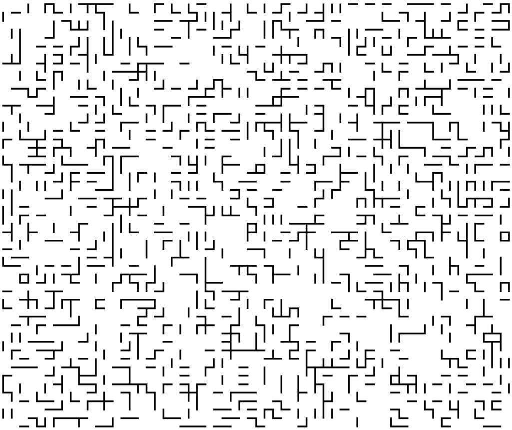
Not surprisingly, $\lambda \mapsto v_1(\lambda)$ is increasing in λ.

Bond percolation

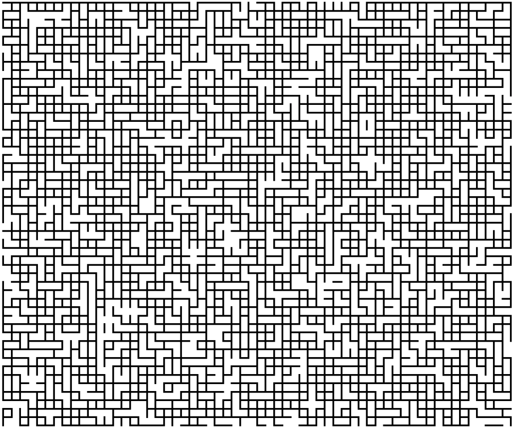
Now go to a random environment.
Consider bond percolation with parameter p on the d-dimensional lattice: all bonds are *open* with probability p and *closed* with probability $1 - p$, independently of each other. Hence the "weights" or "conductances" are not constant anymore but either 1 or 0.

Bond percolation p= 0.25

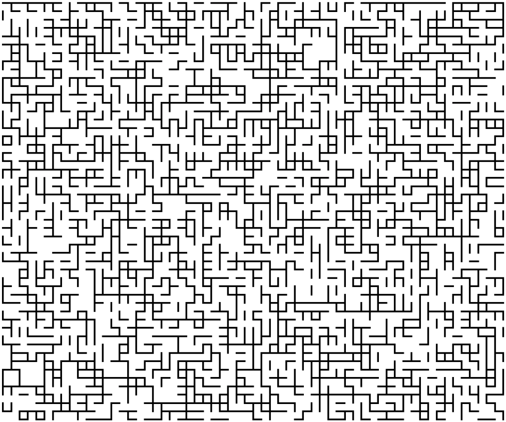
(copied from Geoffrey Grimmett's book)



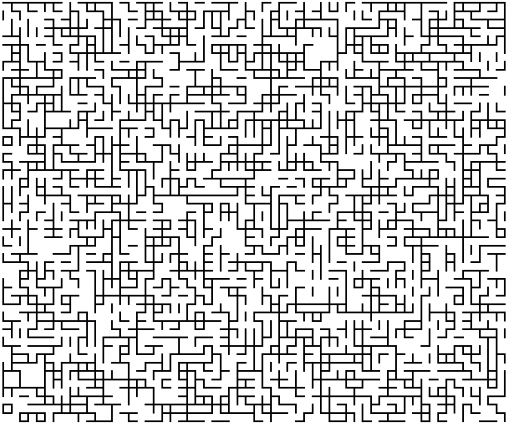
Bond percolation p= 0.75



Bond percolation p= 0.49



Bond percolation p= 0.51



Bond percolation

Note: this model shows a *phase transition* in p.
More precisely: there is a critical value $p_c = p_c(d) \in (0,1)$ such that the probability that the origin is in an infinite connected component is strictly positive for $p > p_c$ and zero for $p < p_c$.

For $d = 2$, we have that in the case $p = p_c$, the above probability is zero.
(Famous open problem: Is that still true for $d \in \{3, 4, \dots 10\}$?)

Bond percolation

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More precisely: there is a critical value $p_c = p_c(d) \in (0,1)$ such that the probability that the origin is in an infinite connected component is strictly positive for $p > p_c$ and zero for $p < p_c$.

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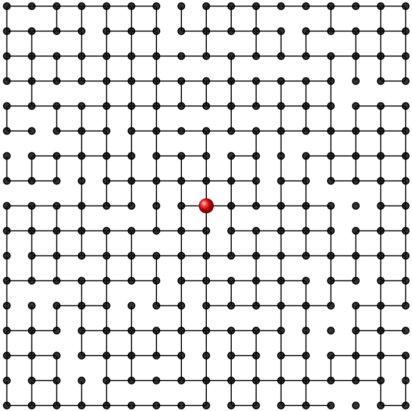
Moreover there is, for $p > p_c$, with probability 1, exactly one infinite connected component. It is called “infinite cluster”.

SRW on the infinite cluster of d-dim supercritical percolation

Take bond percolation on $\mathbb{Z}^d$, $d \geq 2$. Choose $p > p_c$.
Condition on the event that the origin is in the infinite cluster.
Start a random walk at the origin which can only walk on open bonds, and which goes with equal probabilities to all neighbours connected via open bonds. (In particular, this random walk never leaves the infinite cluster.)

SRW on the infinite cluster of 2-dim supercritical percolation (simulation due to Matthias Meiners)



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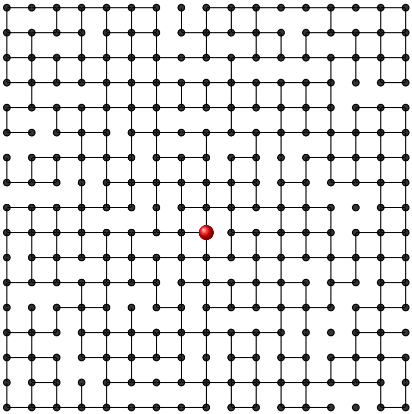

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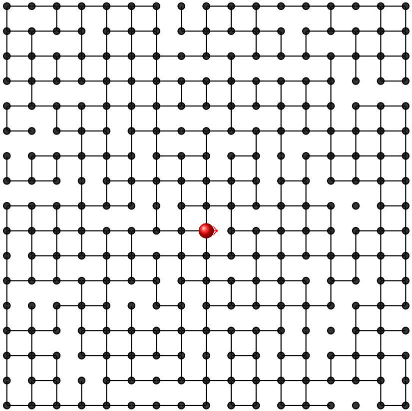

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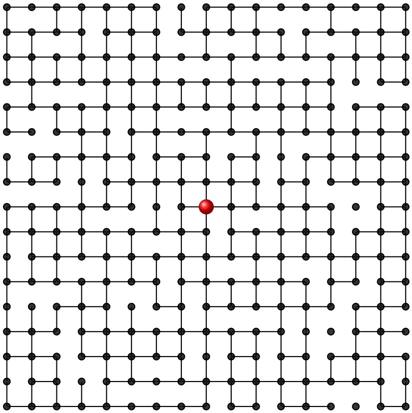

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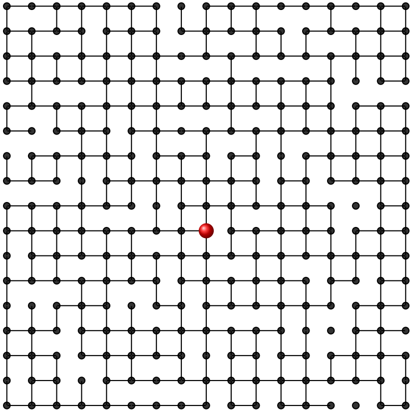

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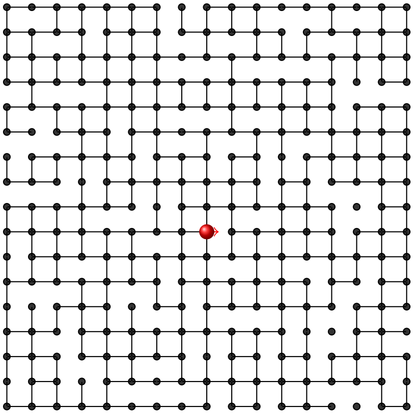

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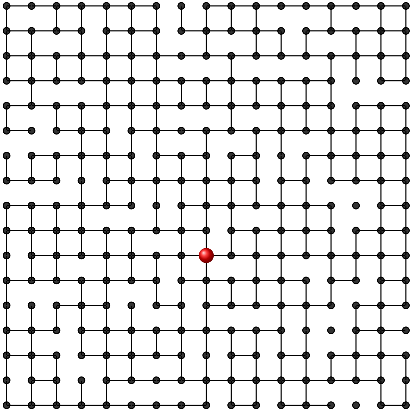

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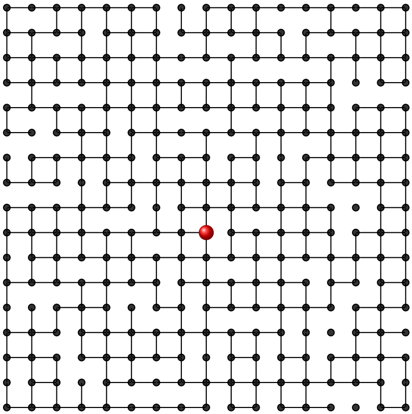

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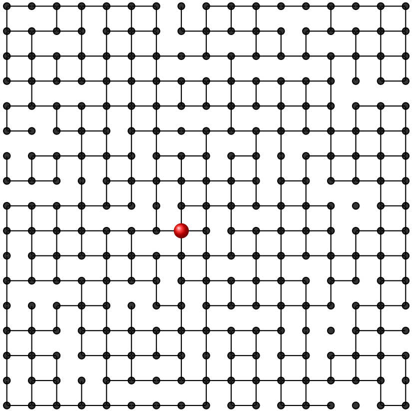

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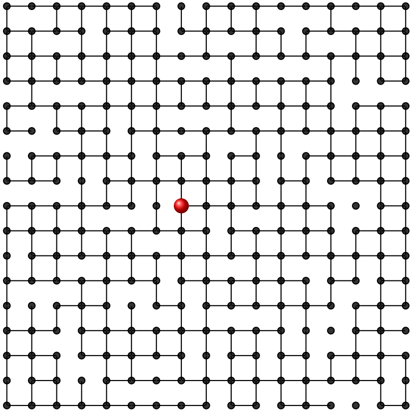

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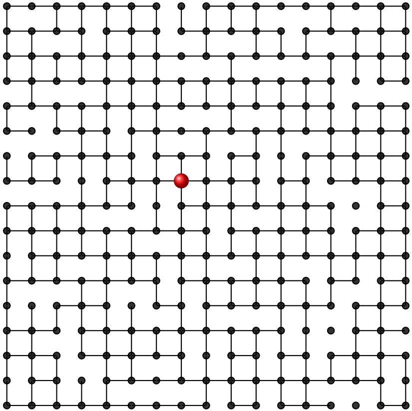

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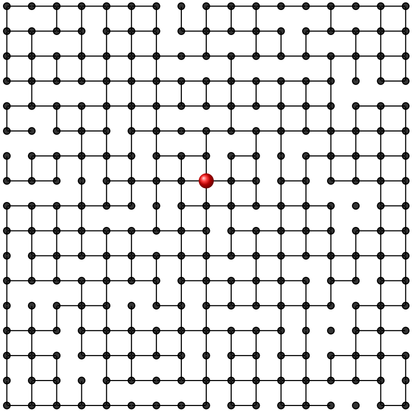

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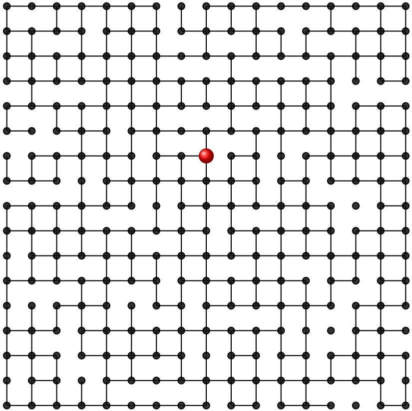

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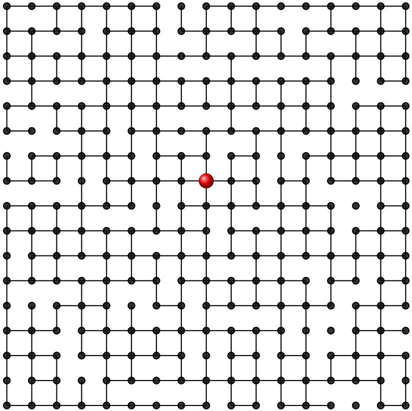

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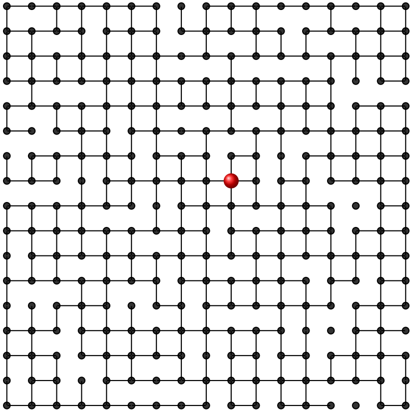

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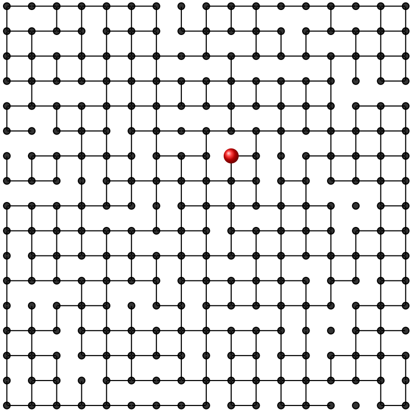

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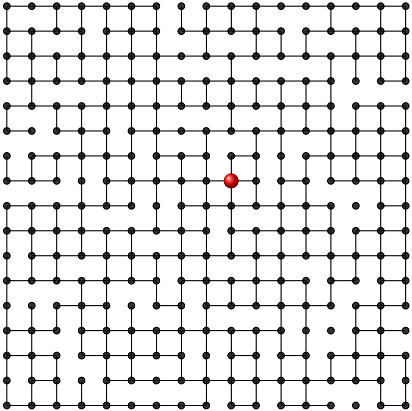

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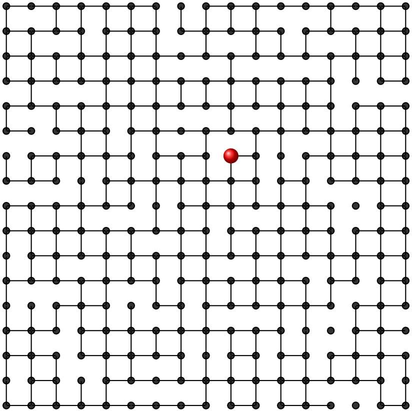

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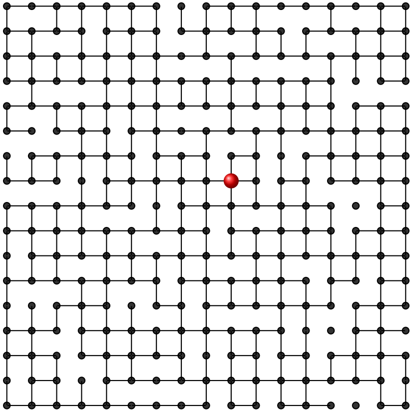

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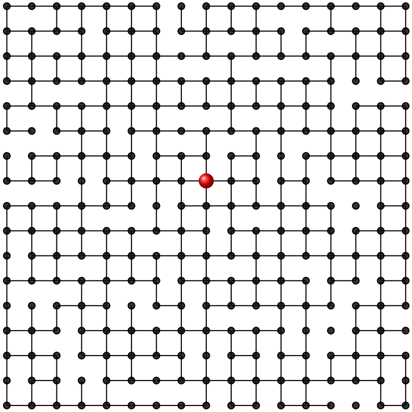

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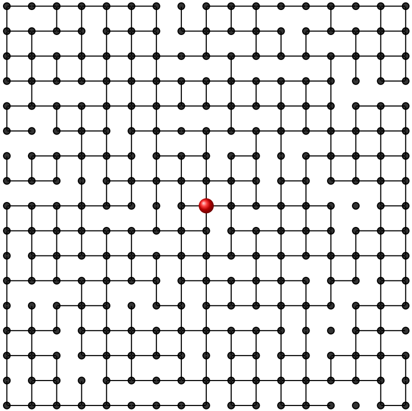

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Answer: Yes! (This was proved by Noam Berger/Marek Biskup, Pierre Mathieu/Andrey Piatnitski, Vladas Sidoravicius/Alain-Sol Sznitman). Method of proof: decompose the walk in a martingale part and a "corrector". Show that the corrector can be neglected and apply the CLT for martingales.

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Question

How does σ depend on p?

Biased random walk on percolation clusters

We add a bias in direction e_1 as before: choose a parameter $\lambda > 0$ for the strength of the bias and multiply the conductances with powers of e^{λ}. This model goes back to Mustansir Barma/Deephak Dhar, 1983. It has been proved that

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty\,.$$

Biased random walk on percolation clusters

Questions

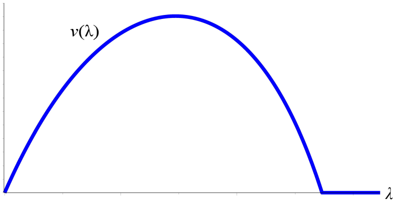
- *Does the random walk move with a constant linear speed, i.e.*

$$\text{does } v(\lambda, p) := \lim_{n\to\infty} \frac{X_n}{n} \text{ exist, and is it deterministic?}$$

- *If yes, is the component $v_1(\lambda, p) = v(\lambda, p) \cdot e_1$ in the favourite direction strictly positive?*
- *How does $v_1(\lambda, p)$ depend on λ and on p?*

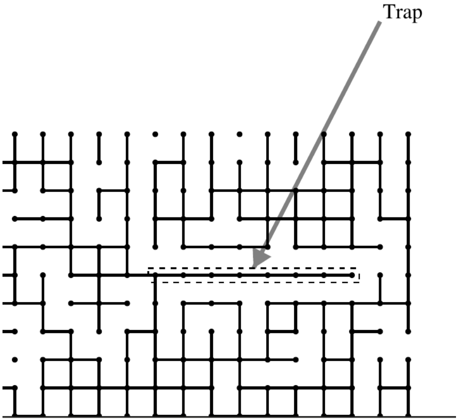
Biased random walk on percolation clusters

For the speed of the random walk on an infinite percolation cluster, the following picture is conjectured:
For each $p \in (p_c, 1)$ we have, with $v_1(\lambda) = v_1(\lambda, p)$:



Biased random walk on percolation clusters

Reason for the zero speed regime:



Biased random walk on percolation clusters

Alexander Fribergh and Alan Hammond showed that there is, for each $p \in (p_c, 1)$, a critical value λ_c such that $v_1(\lambda) > 0$ for $\lambda < \lambda_c$ und $v_1(\lambda) = 0$ for $\lambda > \lambda_c$.
Two important ingredients of the proof are:
– decomposition of the cluster in a backbone and traps
– renormalization

Biased random walk on percolation clusters

Quoting from their paper:

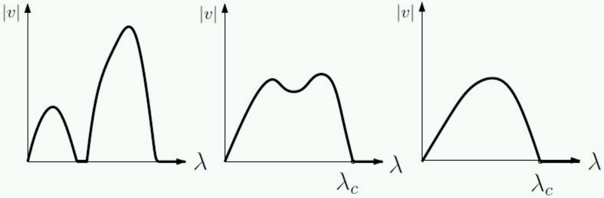
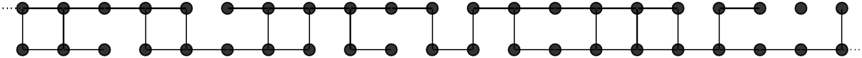
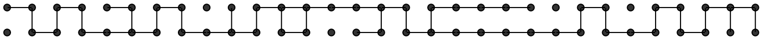


FIGURE 3. The speed as a function of the bias. Sznitman and Berger, Gantert and Peres established positive speed at low λ, but their works left open the possibility depicted in the first sketch. Our work rules this out, though the behavior of the speed in the ballistic regime depicted in the second sketch remains possible. The third sketch shows the unimodal form predicted physically.

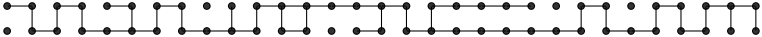
Biased random walk on percolation clusters, one-dimensional

Ladder percolation: this model goes back to Marina Axelson-Fisk/Olle Häggström.
Simulation due to Matthias Meiners.
Some results (2017, 2018) by NG/Matthias Meiners/Sebastian Müller.





Take-home message:
the speed should be the expectation of an increment – but the expectation is not taken with the original measure. It is the environment seen from the particle that matters!



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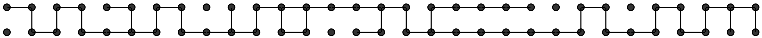

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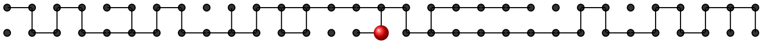

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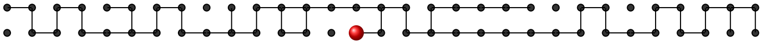

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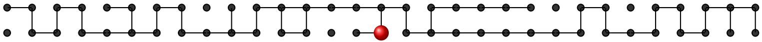

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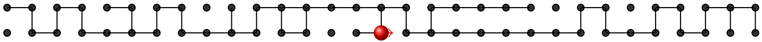

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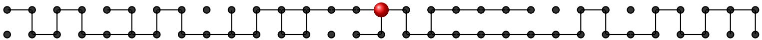

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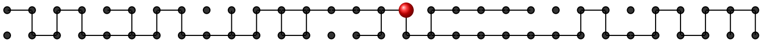

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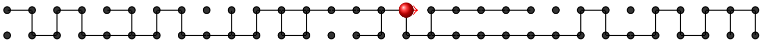

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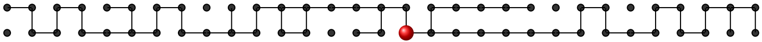

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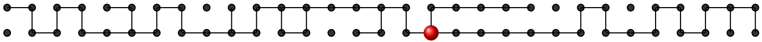

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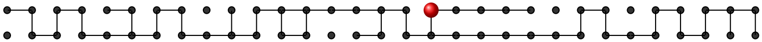

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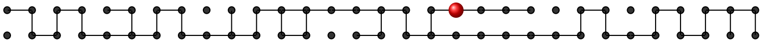

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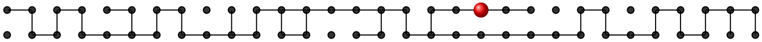

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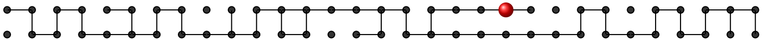

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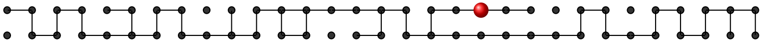

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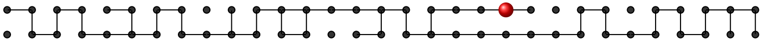

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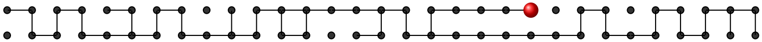

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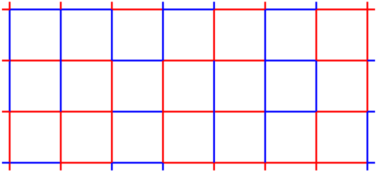
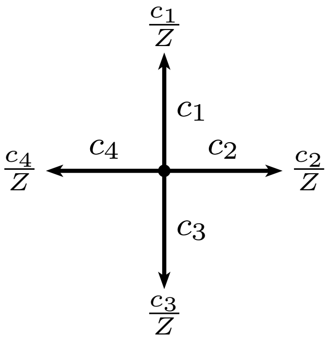
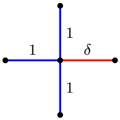
The random conductance model

Define a random medium by giving random weights - often called “conductances” - to the bonds of the lattice.
For simplicity, consider the case where the weights are independent, with the same law. Assume that they are bounded above and bounded away from zero.
The configurations of the weights is called “environment”. For a fixed environment, define the law of a random walk, where the transition probabilities from a point to its neighbours are proportional to the weights of the bonds.





where $Z := c_1 + c_2 + c_3 + c_4$.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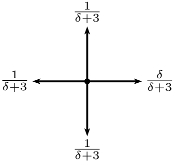
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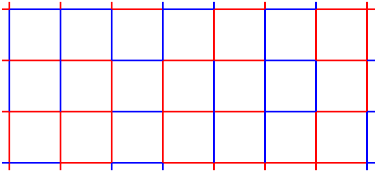
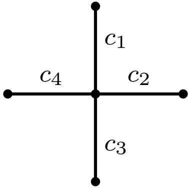
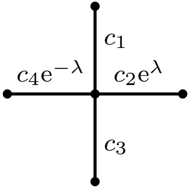


Biased random walks among random conductances

We add a bias in direction e_1: choose a parameter $\lambda > 0$ for the strength of the bias and multiply the conductances with powers of e^{λ}.
Example: $d = 2, e_1 = (1, 0)$.





It has been proved that

$$\lim_{n\to\infty} X_n \cdot e_1 = \infty \text{ almost surely}$$

Biased random walks among random conductances

Questions

- *Does the random walk move with a constant linear speed, i.e.*

$$\text{does } v(\lambda) := \lim_{n\to\infty} \frac{X_n}{n} \text{ exist, and is it deterministic?}$$

- *If yes, is the component of $v_1(\lambda) = v(\lambda, p) \cdot e_1$ in the favourite direction strictly positive?*
- *How does $v_1(\lambda)$ depend on λ and on the law of the conductances?*

Biased random walks among random conductances

The answer to the first and second question is “yes”.

Theorem (Lian Shen 2002)

For fixed bias, there is a law of large numbers:
For any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{X_n}{n} = v(\lambda),$$

where $v(\lambda)$ is deterministic and $v_1(\lambda) = v(\lambda) \cdot e_1 > 0$.

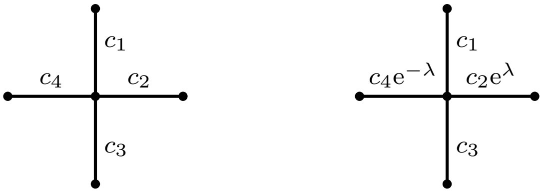
Note that the classical law of large numbers does not work anymore!
Key point in the proof: decomposition of walk AND environment in i.i.d. pieces. More precisely, there are *regeneration times* $\tau_1, \tau_2, \ldots$ such that

$$(\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n})_{n\geq 1} \text{ are i.i.d.} \qquad (1)$$

Biased random walks among random conductances

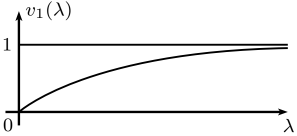
How does $v_1(\lambda)$ depend on λ? Note that since the conductances were assumed to be bounded above and bounded away from zero, a coupling argument gives immediately that

$$\lim_{\lambda \to \infty} v_1(\lambda) = 1\,.$$

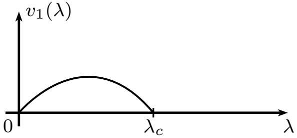


Biased random walks among random conductances

Recall that for the homogeneous medium, we have



For the infinite percolation cluster, the conjectured picture is



Biased random walks among random conductances

Question

For the random walk with bias among bounded random conductances, is the speed in the favourite direction increasing in λ?

Biased random walks among random conductances

Question

For the random walk with bias among bounded random conductances, is the speed in the favourite direction increasing in λ?

A bit surprisingly, the answer is “it depends”!

Theorem

(Noam Berger/NG/Jan Nagel, 2019)

(i) *There is a value* $\lambda_0 < \infty$ *such that*

$$\lambda \to v_1(\lambda) \text{ is increasing for } \lambda \geq \lambda_0 \,.$$

(ii) *There is* $\delta \in (0, 1)$ *such that if all conductances are in* $(1 - \delta, 1 + \delta)$, *then*

$$\lambda \to v_1(\lambda) \text{ is increasing for all } \lambda \geq 0 \,.$$

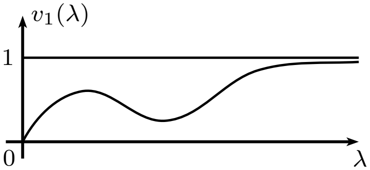
(iii) *If* $d = 2$ *and* δ_0 *is small enough and the conductances take the values* 1 *(with probability* $p > p_c$*) and* δ_0 *with probability* $1 - p$, *then*

$$\lambda \to v_1(\lambda) \text{ is NOT increasing,}$$

i.e. there are $\lambda, \bar{\lambda}$ *such that* $\lambda < \bar{\lambda}$ *but* $v_1(\lambda) > v_1(\bar{\lambda})$.

Biased random walks among random conductances

We believe that in case (iii), the picture is



(This is the simplest picture which is in agreement with our results!)

Random walks in dynamical random environments

- “Dynamical random environment” means that the environment/random graph changes in time.
- Application: spread of an epidemic in an evolving population.
- New results obtained by: Luca Avena, Antar Bandyopadhyay, Stein Andreas Bethuelsen, Marek Biskup, Oriane Blondel, Guillaume Conchon-Kerjan, Natalia Cardona-Tobón, Conrado da Costa, Alessandra Faggionato, Jonathan Hermon, Marcelo Hilario, Remco van der Hofstad, Daniel Kious, Milton Jara, Frank den Hollander, Marcel Ortgiese, Yuval Peres, Pierre-François Rodríguez, Marco Seiler, Alexandre Stauffer, Perla Sousi, Anja Sturm, Franco Tertuliano, Augusto Teixeira, Renato Soares, Jeffrey Steif, Daniel Valesin, Florian Völlering, Ofer Zeitouni... and many more!
- Hence: no survey but...
- An example

Dynamical percolation

Here: the environment is given by *dynamical percolation*. Consider $\mathbb{Z}^d$ and an initial state $\eta \in \{0,1\}^E$ of the edges, E = edges of $\mathbb{Z}^d$.
An edge e is **open** at time t if $\eta_t(e) = 1$, and **closed** otherwise.
Fix $\mu \geq 0$ and $p \in [0,1]$.
$(\eta_t)_{t\geq 0}$ with $\eta_0 = \eta$ defined as follows: each edge $e \in E$ has an independent Poisson process of rate μ. If there is a point of the Poisson process at time t, we refresh the state of e in η_t, i.e. we declare e open with probability p and closed with probability $1 - p$, independently of all other edges and previous states of e.

A simulation - thanks to Matt Roberts!

https://people.bath.ac.uk/mir20/programs/perco2/

Biased random walk on dynamical percolation

Define a continuous-time random walk $(X_t)_{t\geq 0}$ in the environment $(\eta_t)_{t\geq 0}$ with bias parameter $\lambda > 0$: set $X_0 = 0$ and assign a rate 1 Poisson clock to the particle. When the clock rings at time t and the random walker is currently at a site x, choose one of the neighbours y of x with probability

$$p(x, x + \mathrm{e}_1) = \frac{\mathrm{e}^{\lambda}}{Z(\lambda)},$$
$$p(x, x - \mathrm{e}_1) = \frac{\mathrm{e}^{-\lambda}}{Z(\lambda)},$$
$$p(x, x \pm \mathrm{e}_i) = \frac{1}{Z(\lambda)} \text{ for } i \in \{2, \dots, d\}$$

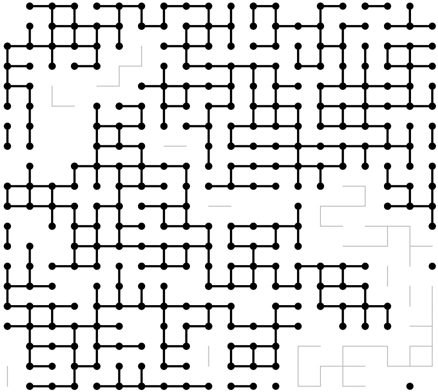
where $Z(\lambda) = \mathrm{e}^{\lambda} + e^{-\lambda} + 2d - 2$ is a normalizing factor.
If $\eta_t(\{x, y\}) = 1$, the random walker moves from x to y, and it stays at x, otherwise.

Continuation: biased random walk on dynamical percolation

$(X_t, \eta_t)_{t \geq 0}$ is a λ-biased random walk on dynamical percolation with parameters μ and p.
Note that we can also consider $p \leq p_c$.

Motivation: speed of a biased RW on a (fixed) supercritical percolation cluster

Can one take $\mu \to 0$? We do not know...



Invariance principle for SRW on dynamical percolation

In the unbiased case ($\lambda = 0$), already known:

Theorem (Yuval Peres/Alexandre Stauffer/Jeffrey Steif)

For $d \geq 1$, $\mu > 0$, $p \in (0,1)$ and $\lambda = 0$, there exists $\sigma = \sigma(d, \mu, p) \in (0, \infty)$ so that

$$\left(\frac{X_{kt}}{\sqrt{k}}\right)_{t \in [0,1]} \xrightarrow{(d)} (\sigma B_t)_{t \in [0,1]}$$

for $k \to \infty$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion.

Results

See Sebastian Andres/NG/Perla Sousi/Dominik Schmid AoP, 2024.

Theorem (Existence and positivity of the speed)

Let $d \geq 1$ and let $(X_t, \eta_t)_{t\geq 0}$ be a λ-biased random walk on dynamical percolation on $\mathbb{Z}^d$ with parameters $\mu > 0$ and $p \in (0,1)$. Then for all $\lambda > 0$, there exists $v_1(\lambda) = v_1(\lambda, \mu, p) > 0$ such that almost surely

$$\lim_{t\to\infty} \frac{X_t}{t} = (v_1(\lambda), 0, \ldots, 0) \,.$$

Moreover, the function $\lambda \mapsto v_1(\lambda)$ is continuously differentiable. and satisfies with σ from the previous theorem

$$\lim_{\lambda\to 0} v_1'(\lambda) = \sigma^2. \tag{2}$$

(2) is known as "Einstein relation".

Monotonicity of the speed for $d = 1$ or for “almost homogeneous” environment

When $d = 1$, one can couple two walks with different bias parameters to see that the speed is always increasing in the bias.
In fact, we show that in $d = 1$ the speed is strictly increasing as a function of the bias.

Can also show for $d \geq 2$ that if either p is close enough to 1 or μ is large enough, the speed is increasing.

Is the speed eventually increasing?

Is the speed increasing as a function of the bias for λ large enough?

Theorem (Monotonicity of the speed for $d \geq 2$, λ large enough)

Consider the biased random walk on dynamical percolation on $\mathbb{Z}^d$ for $d \geq 2$. For all $p \in (0,1)$ and $\mu > 0$ there exists some $\lambda_0 = \lambda_0(p,\mu)$ such that the following holds.

(i) *The speed $v_1(\lambda)$ is strictly increasing for all $\lambda \geq \lambda_0$ provided that $\mu^2 > p(1-p)$.*

(ii) *The speed $v_1(\lambda)$ is strictly decreasing for all $\lambda \geq \lambda_0$ provided that $\mu^2 < p(1-p)$.*

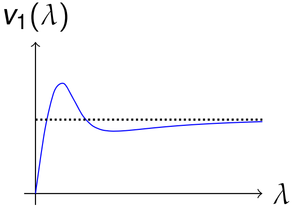
Continuation: Is the speed eventually increasing?

The critical case $\mu^2 = p(1-p)$ was solved in very recent work of Assylbek Olzhabayek/Dominik Schmid, 2025+.

Theorem

The speed $v_1(\lambda)$ is strictly increasing for all $\lambda \geq \lambda_0$ if $\mu^2 = p(1-p)$.

Possible shapes of $v_1(\lambda)$ as a function of λ



We do not know if the first picture can occur.

Formula for the speed

Proposition

There are regeneration times $(\tau_i)_{i\in\mathbb{N}}$ *(whose law does not depend on* λ*!) such that*

$$\lim_{t\to\infty}\frac{X_t}{t}=(v_1(\lambda),0,\dots,0)=\frac{\mathbb{E}_\lambda[X_{\tau_1}]}{\mathbb{E}[\tau_1]} \text{ a.s.}$$

$$\mathbb{E}_\lambda[X^1_{\tau_1}]=\mathbb{E}_0\left[(R-L)e^{\lambda(R_a-L_a)}\left(\frac{2d}{Z_\lambda}\right)^{U_a(\tau_1)}\right]$$

$U_a(\tau_1)$ *is the number of attempted jumps up to time* τ_1, R_a *the number of attempted jumps to the right and* L_a *the number of attempted jumps to the left,* R *the number of jumps to the right and* L *the number of jumps to the left up to time* τ_1.

Formula for the derivative of the speed

Proposition

In our model, have

$$v_1'(\lambda) = \frac{1}{\mathbb{E}[\tau_1]} \left(\mathbb{E}_\lambda [X^1_{\tau_1}(R_a - L_a)] - \frac{Z'(\lambda)}{Z(\lambda)} \mathbb{E}_\lambda \left[X^1_{\tau_1} \cdot U_a(\tau_1) \right] \right).$$

where $Z(\lambda) = e^\lambda + e^{-\lambda} + 2d - 2$ and $U_a(\tau_1)$ is the number of attempted jumps up to time τ_1, R_a the number of attempted jumps to the right and L_a the number of attempted jumps to the left.

Einstein relation

From the last formula, get

$$\lim_{\lambda\to 0} v_1'(\lambda) = \frac{1}{\mathbb{E}[\tau_1]}\mathbb{E}_0\left[X^1_{\tau_1}(R_a - L_a)\right] = \frac{1}{\mathbb{E}[\tau_1]}\mathbb{E}_0\left[(R-L)(R_a - L_a)\right]$$

where $Z(\lambda) = e^{\lambda} + e^{-\lambda} + 2d - 2$ and R_a is the number of attempted jumps to the right and L_a the number of attempted jumps to the left up to time τ_1, whereas R is the number of jumps carried out to the right and L the number of jumps carried out to the left up to time τ_1. But

$$\sigma^2 = \frac{1}{\mathbb{E}[\tau_1]}\mathbb{E}_0\left[\left(X^1_{\tau_1}\right)^2\right] = \mathbb{E}_0\left[(R-L)^2\right]$$

Hence, have to show

$$\mathbb{E}_0\left[(R-L)^2\right] = \mathbb{E}_0\left[(R-L)(R_a-L_a)\right]$$

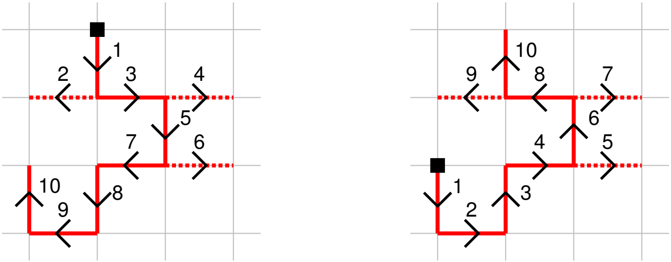
or in other words

$$\mathbb{E}_0\left[(R-L)(R_{\text{supp}}-L_{\text{supp}})\right] = 0 \tag{3}$$

where R_{supp} is the number of suppressed (attempted but not carried out) jumps to the right and L_{supp} the number of suppressed jumps to the left up to time τ_1.
Why should (3) be true?

Proof by picture

Consider time reversal: $(R - L)$ is an antisymmetric function while $(R_{\text{supp}} - L_{\text{supp}})$ is a symmetric function.



Expansion for the speed

Proposition

For $d \geq 1$, consider a λ-biased random walk on dynamical percolation on $\mathbb{Z}^d$ with parameters $\mu > 0$ and $p \in (0,1)$. There exists some $\lambda_0 = \lambda_0(\mu, d)$ such that for all $\lambda > \lambda_0$,

$$v_1(\lambda) = \frac{\mu p}{1 - p + \mu} - \frac{(2d-2)p}{(1-p+\mu)^2}(\mu^2 - p(1-p))Z_\lambda^{-1} + O(e^{-2\lambda}),$$

where the implicit constant in O depends on μ and d.

Outlook and open questions

- Dynamical conductances instead of dynamical percolation? (see forthcoming work of Eszter Couillard)
- Dependence of $v_1(\lambda, p)$ on p for fixed λ?
- Run several walkers in the same (dynamical) environment?
- Interacting particle systems on (dynamical) percolation?

... and many more! (which I am happy to discuss!)

Thanks for your attention!