Rigorous results for timelike Liouville field theory

Sourav Chatterjee

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Liouville field theory has found applications in various areas of theoretical physics, including string theory, 3D general relativity, string theory in anti-de Sitter space, and supersymmetric gauge theory.

Liouville field theory in mathematics

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- and many other pathbreaking works.

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Timelike Liouville theory has applications in quantum cosmology, tachyon condensation, and other areas of theoretical physics.

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Both conjectures were recently proved by Ang, Cai, Sun and Wu (2021).

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One of the main results I will present shows how 2D gravity emerges from this model in the semiclassical limit.

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While spacelike Liouville theory has been made rigorous using tools from probability theory, converting those proofs to the timelike case (or indeed, any ‘true’ model of quantum gravity) would require a theory of Gaussian random variables with negative variance.

I will talk about the development of such a theory at the end of the talk.

The action of timelike Liouville field theory

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$$I(\phi) = \frac{1}{4\pi} \int_{\mathbb{C}} (\phi(z)\Delta_g \phi(z) + 2Q\phi(z) + 4\pi\mu :e^{2b\phi(z)}:) g(z) d^2 z.$$

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$:e^{2\phi(z)}:$ is not well-defined because $G_g(z,z) = \infty$ for all z.

Using the action to define the theory

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Main question: Can we make sense of this measure?

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$$\int_{\mathbb{C}} \phi(z)\Delta_g\phi(z)g(z)d^2z = -\int_{\mathbb{C}} |\nabla_g\phi(z)|^2 g(z)d^2z,$$

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The appearance of the kinetic term with the 'wrong' (i.e., negative) sign in the action is a common feature of models in quantum gravity. Its most consequential appearance is in the Einstein–Hilbert action for Einstein gravity, which is one of the roadblocks to quantizing Einstein gravity.

What to compute after defining the measure

Once we have some kind of sense of a measure with density $e^{-I(\phi)}$ on the space of fields, we would then like to understand the behavior of a 'random' field ϕ 'drawn' from this measure, in the sense of drawing a random field from a probability distribution.

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For example, the critical points of the Einstein–Hilbert action are the metrics on $\mathbb{R}^4$ that satisfy Einstein's equation of general relativity.

Correlation functions

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More generally, we can take expectations of products of such observables.

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This normalization yields the so-called vertex operators.

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The k-point correlation function of timelike Liouville theory is heuristically

$$C(\alpha_1, \ldots, \alpha_k; z_1, \ldots, z_k; b; \mu) = \int \left(\prod_{j=1}^{k} V_\phi(\alpha_j, z_j) \right) e^{-I(\phi)} \mathcal{D}\phi.$$

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$$V_\phi(\alpha, z) = e^{\chi\alpha(b-\alpha)} g(z)^{-\Delta_\alpha} :e^{2\alpha\phi(z)}:$$

where $\Delta_\alpha = \alpha(Q - \alpha)$ and $\chi := \ln 4 - 1$.

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where $\Delta_\alpha = \alpha(Q - \alpha)$ and $\chi := \ln 4 - 1$. The number Δ_α is called the conformal weight of the vertex operator, for reasons related to conformal field theory.

Main result #1: A formula for k-point correlations

Theorem (C., 2025)
Suppose that $k \geq 3$, $\mathrm{Re}(\alpha_j) > -1/2b$ for each j, and $w = (Q - \sum_{j=1}^k \alpha_j)/b$ is a positive integer. Let $z_1, \ldots, z_k$ be distinct points in $\mathbb{C}$. Then

$$C(\alpha_1, \ldots, \alpha_k; z_1, \ldots, z_k; b; \mu)$$

$$= \frac{e^{-i\pi w}\mu^w}{w!}(4/e)^{1-1/b^2} \prod_{1 \leq j < j' \leq k} |z_j - z_{j'}|^{4\alpha_j\alpha_{j'}}$$

$$\cdot \int_{\mathbb{C}^w} \left(\prod_{j=1}^k \prod_{l=1}^w |z_j - t_l|^{4b\alpha_j} \right) \left(\prod_{1 \leq l < l' \leq w} |t_l - t_{l'}|^{4b^2} \right) d^2t_1 \cdots d^2t_w.$$

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Our formula is similar to the Coulomb gas expression for the 3-point function of spacelike Liouville theory derived in physics by Goulian and Li (1991) and for timelike Liouville theory by Kostov and Petkova (2006).

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Interestingly, Guillarmou, Kupiainen and Rhodes (2023) obtain the same expression for the k-point correlations in a compactified model they define (where the field ϕ takes value in a compact interval). It is not clear why the same formula arises.

Similarly, it is not clear why the same formula arises for the 3-point connectivity probability of 2D critical percolation, as proved rigorously by Ang, Cai, Sun and Wu (2021) using techniques from SLE and CLE.

Main result #2: The timelike DOZZ formula

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It is proved using the formula from the previous theorem and a series of calculations using the complex Selberg integral formula of Dotsenko and Fateev (1985) and Aomoto (1987), following ideas from Giribet (2012).

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It is proved using the formula from the previous theorem and a series of calculations using the complex Selberg integral formula of Dotsenko and Fateev (1985) and Aomoto (1987), following ideas from Giribet (2012).

To state this result, we need some preparation.

Preparation

The following special function was introduced by Dorn and Otto (1994):

$$\Upsilon_b(z) = \exp\Bigg(\int_0^\infty \frac{1}{\tau}\Bigg(\left(\frac{b}{2} + \frac{1}{2b} - z\right)^2 e^{-\tau} - \frac{\sinh^2((\frac{b}{2} + \frac{1}{2b} - z)\frac{\tau}{2})}{\sinh(\frac{b\tau}{2})\sinh(\frac{\tau}{2b})}\Bigg) d\tau\Bigg)$$

on the strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < b + \frac{1}{b}\}$ and continued analytically to the whole plane.

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on the strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < b + \frac{1}{b}\}$ and continued analytically to the whole plane.

Let $\gamma(z) = \Gamma(z)/\Gamma(1-z)$, where Γ is the classical Gamma function.

The timelike DOZZ formula

Theorem (C., 2025)
Let $\alpha_1, \alpha_2, \alpha_3$ be complex numbers such that $w = (Q - \sum_{j=1}^3 \alpha_j)/b$ is a positive integer less than $1 + (2b^2)^{-1}$, and $\Upsilon_b(2\alpha_j + 1/b) \neq 0$ for $j = 1, 2, 3$. Take any distinct $z_1, z_2, z_3 \in \mathbb{C}$. For $1 \leq j < k \leq 3$, define $z_{jk} := |z_j - z_k|$ and $\Delta_{jk} := 2\Delta_{\alpha_j} + 2\Delta_{\alpha_k} - \sum_{l=1}^3 \Delta_{\alpha_l}$. Then

$$\begin{aligned} &C(\alpha_1, \alpha_2, \alpha_3; z_1, z_2, z_3; b; \mu) \\ &= e^{-i\pi w}(-\pi\mu\gamma(-b^2))^w (4/e)^{1-1/b^2} b^{2b^2 w + 2w} \\ &\qquad \cdot \frac{\Upsilon_b(bw+b)}{\Upsilon_b(b)} \prod_{j=1}^3 \frac{\Upsilon_b(2\alpha_j + bw + 1/b)}{\Upsilon_b(2\alpha_j + 1/b)} \prod_{1 \leq j < k \leq 3} |z_{jk}|^{2\Delta_{jk}}. \end{aligned}$$

The formula also holds if w is any positive integer and $\alpha_1, \alpha_2, \alpha_3$ have real parts greater than $-1/2b$.

Remarks

This formula is the same (up to notational changes) as the one given in Harlow, Maltz and Witten (2011), as well the ones appearing in the original proposals of Schomerus (2003), Zamolodchikov (2005), and Kostov and Petkova (2006).

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At this time, this seems to be the widely accepted formula for the 3-point function of timelike Liouville field theory.

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This formula is the same (up to notational changes) as the one given in Harlow, Maltz and Witten (2011), as well the ones appearing in the original proposals of Schomerus (2003), Zamolodchikov (2005), and Kostov and Petkova (2006).

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It is unclear how to extend the arguments of my paper to the full parameter space.

Main result #3: Semiclassical limit

The semiclassical limit of timelike Liouville field theory under insertion of heavy operators is obtained by taking $b \to 0$, while simultaneously scaling the α_j's and μ as $\alpha_j = \widetilde{\alpha}_j / b$ and $\mu = \widetilde{\mu}/b^2$, where the $\widetilde{\alpha}_j$'s and $\widetilde{\mu}$ are fixed real numbers as $b \to 0$.

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Semiclassical limits are important for the following reason. Suppose one is able to construct a quantum theory of gravity. A valid theory should yield the equations of general relativity in the semiclassical limit. A toy version of this should hold for models of 2D gravity.

We need some preparation. First, for a function $f : \mathbb{C} \to \mathbb{C}$, let $G_g f$ denote the function

$$G_g f(z) := \int_{\mathbb{C}} G_g(z, z') f(z') g(z') d^2 z'.$$

This operator will appear several times in the following slide.

Preparation

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\begin{aligned}
H(\rho) &:= \int_{\mathbb{C}} \rho(z) \ln \rho(z) g(z) d^2 z, \\
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L(\rho) &:= \sum_{j=1}^{k} 4\widetilde{\alpha}_j \int_{\mathbb{C}} G_g(z_j, z) \rho(z) g(z) d^2 z.
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Let $\mathcal{P}'$ be the subset of $\mathcal{P}$ consisting of all ρ such that $H(\rho)$ is finite. It turns out that for $\rho \in \mathcal{P}'$, the functionals $R(\rho)$ and $L(\rho)$ are also finite.

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For $\rho \in \mathcal{P}'$, define $S(\rho) := L(\rho) + 2\beta R(\rho) + H(\rho)$.

The semiclassical limit

Theorem (C., 2025)

Suppose $\widetilde{\alpha}_j \in (-1/2, \infty)$ *for each* j, *and* $\beta = -1 - \sum_{j=1}^k \widetilde{\alpha}_j > 0$. *For each* $n \geq 1$, *let*

$$b_n := \sqrt{\frac{\beta}{n-1}}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log C(\widetilde{\alpha}_1/b_n, \ldots, \widetilde{\alpha}_k/b_n; z_1, \ldots, z_k; b_n; \widetilde{\mu}/b_n^2)$$

$$= 1 + \ln \widetilde{\mu} - \ln \beta - i\pi + (1 - \ln 4) \sum_{j=1}^k \frac{\widetilde{\alpha}_j^2}{\beta^2} + \sum_{j=1}^k \frac{\widetilde{\alpha}_j(1+\widetilde{\alpha}_j)}{\beta^2} \ln g(z_j)$$

$$- \frac{4}{\beta} \sum_{1\leq j<j'\leq k} \widetilde{\alpha}_j \widetilde{\alpha}_{j'} G_g(z_j, z_{j'}) - \inf_{\rho \in \mathcal{P}'} S(\rho).$$

Moreover, the infimum on the right is attained at a unique $\widehat{\rho} \in \mathcal{P}'$.

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The proof goes by analyzing the explicit formula for the k-point function, which is an n-fold Coulomb gas integral with $n \to \infty$ as $b \to 0$.

We will see shortly that the limit indeed yields the classical equations for 2D gravity.

Preparation for main result #4

It turns out that as $b \to 0$,

$$\widetilde{C}(\widetilde{\alpha}_1/b, \ldots, \widetilde{\alpha}_k/b; z_1, \ldots, z_k; b; \widetilde{\mu}/b^2) = \int e^{J(\psi)/b^2 + O(1)} \mathcal{D}\psi,$$

where

$$\begin{aligned} J(\psi) = & -\chi \sum_{j=1}^{k} \widetilde{\alpha}_j + \sum_{j=1}^{k} \widetilde{\alpha}_j (1 + \widetilde{\alpha}_j) \ln g(z_j) \\ & + \sum_{j=1}^{k} (2\widetilde{\alpha}_j \psi(z_j) + 2\widetilde{\alpha}_j^2 G_g(z_j, z_j)) \\ & + \frac{1}{2\pi} \int_{\mathbb{C}} \psi(z) g(z) d^2 z \\ & - \frac{1}{4\pi} \int_{\mathbb{C}} (\psi(z) \Delta_g \psi(z) + 4\pi \widetilde{\mu} e^{2\psi(z)}) g(z) d^2 z. \end{aligned}$$

Critical points of J

So, we may expect that as $b \to 0$, $C(\widetilde{\alpha}_1/b, \ldots, \widetilde{\alpha}_k/b; z_1, \ldots, z_k; b; \widetilde{\mu}/b^2)$ behaves like $e^{J(\widehat{\psi})/b^2}$ for some critical point $\widehat{\psi}$ of J.

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Formal computations show that a critical point $\widehat{\psi}$ must satisfy the functional equation

$$2\sum_{j=1}^{k} \widetilde{\alpha}_j g(z)^{-1} \delta_{z_j}(z) + \frac{1}{2\pi} - \frac{1}{2\pi} \Delta_g \widehat{\psi}(z) - 2\widetilde{\mu} e^{2\widehat{\psi}(z)} = 0.$$

Equation of 2D gravity

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This is the equation of motion in 2D JT gravity upon insertion of charges at $z_1, \ldots, z_k$.

Main result #4: Emergence of 2D gravity

Theorem (C., 2025)
The limit obtained in the previous theorem can be expressed as $J(\widehat{\psi})/\beta$ for some critical point $\widehat{\psi}$ of J. Moreover, this critical point is given by

$$\widehat{\psi}(z) = -2\beta G_g \widehat{\rho}(z) - \frac{\lambda}{2} + \frac{1}{2}\ln\beta + \frac{i\pi}{2} - \frac{1}{2}\ln\widetilde{\mu} - 2\sum_{j=1}^{k}\widetilde{\alpha}_j G_g(z, z_j),$$

where $\widehat{\rho}$ is the unique minimizer of the function S from before, and

$$\lambda = \ln\int_{\mathbb{C}} \exp\left(-4\beta G_g \widehat{\rho}(z) - 4\sum_{j=1}^{k}\widetilde{\alpha}_j G_g(z_j, z)\right) g(z) d^2 z.$$

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Note that $\widehat{\psi}$ has a constant imaginary component of $\frac{1}{2}i\pi$. This has to be the case, because J has no critical points among real-valued functions, as already observed by Harlow, Maltz and Witten (2011).

But since the metric induced by $\widehat{\psi}$ is $e^{2\widehat{\psi}(z)}g(z)$, it is real-valued even though $\widehat{\psi}$ is not.

Gaussian random variables with negative variance

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We will now define a notion of a 'wrong sign' Gaussian distribution, where the variance is allowed to be negative. More generally, we will define an $(m+n)$-dimensional random vector $Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ where the coordinates are independent, $X_1, \ldots, X_m$ are $N(0,1)$ random variables, and $Y_1, \ldots, Y_n$ are $N(0,-1)$ random variables.

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To be precise, we will define $\mathbb{E}(f(Z))$ for f belonging to a class of complex-valued functions $\mathcal{F}_{m,n}$ on $\mathbb{R}^m \times \mathbb{R}^n$.

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2. Since expectation must be linear, $\mathcal{F}_{m,n}$ should be a vector space over $\mathbb{C}$ and $f \mapsto \mathbb{E}(f(Z))$ should be linear. Moreover, if f is identically equal to a constant c, then $\mathbb{E}(f(Z))$ should be equal to c.

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3. If f is real-valued, then $\mathbb{E}(f(Z))$ should be real. This comes from physical considerations, because the expected value of a real-valued observable should not have a nonzero imaginary component.

Ideas that do not work

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And lastly, it cannot be defined as an integration with respect to a complex measure.

The arguments are given in the preprint.

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This works well in many situations, for example when f is polynomial or exponential. However, there is no mathematical theory around this, and therefore we do not know precise conditions under which this approach does not lead to contradictions or violations of the conditions listed before.

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The example in the next slide illustrates the kind of problem that leads to this failure.

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Let $h(s) = \mathbb{E}(f(sY))$ for $s > 0$, where $Y \sim N(0,1)$. Then

$$h(s) = \frac{1}{\sqrt{2\pi}s} \int_0^\infty \frac{1}{u} \exp\left(-u - \frac{1}{u} - \frac{1}{2s^2}(\ln u)^2\right) du.$$

This function continues analytically to $\mathbb{C} \setminus \{0\}$ by the same formula.

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This function continues analytically to $\mathbb{C} \setminus \{0\}$ by the same formula.

Thus, we should define, for $X \sim N(0, -1)$,

$$\mathbb{E}(f(X)) = h(i) = \frac{1}{\sqrt{2\pi}i} \int_0^\infty \frac{1}{u} \exp\left(-u - \frac{1}{u} + \frac{1}{2}(\ln u)^2\right) du.$$

But this is not a real number, thus violating our condition that the expected value of a real-valued function should be real.

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This small adjustment guarantees that expected values of real-valued functions are real.

Fundamentally, this is a consequence of the Schwarz reflection principle.

General definition

Take any $m \geq 0$ and $n > 0$. We define $\mathcal{F}_{m,n}$ to be the class of functions $f : \mathbb{R}^{m+n} \to \mathbb{C}$ such that f has an analytic continuation in the last n coordinates to a function $\widetilde{f} : \mathbb{R}^m \times \Omega \to \mathbb{C}$, where Ω is an open subset of $\mathbb{C}^n$ that contains $(\mathbb{R} \cup i\mathbb{R})^n$, such that

$$\mathbb{E}|\widetilde{f}(W_1, \ldots, W_m, iW_{m+1}, \ldots, iW_{m+n})| < \infty,$$

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If such an $\widetilde{f}$ exists, we define

$$\mathbb{E}(f(Z)) = \mathbb{E}(\widetilde{f}(W_1, \ldots, W_m, iW_{m+1}, \ldots, iW_{m+n})),$$

for a vector $Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ has independent coordinates, $X_1, \ldots, X_m$ are $N(0,1)$ random variables, and $Y_1, \ldots, Y_n$ are $N(0,-1)$ random variables. Let us denote this by $Z \sim N_{m,n}$.

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where $W_1, \ldots, W_{m+n}$ are i.i.d. $N(0,1)$ random variables.

If such an $\widetilde{f}$ exists, we define

$$\mathbb{E}(f(Z)) = \mathbb{E}(\widetilde{f}(W_1, \ldots, W_m, iW_{m+1}, \ldots, iW_{m+n})),$$

for a vector $Z = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ has independent coordinates, $X_1, \ldots, X_m$ are $N(0,1)$ random variables, and $Y_1, \ldots, Y_n$ are $N(0,-1)$ random variables. Let us denote this by $Z \sim N_{m,n}$.

In the preprint, it is shown that this is well-defined.

Validity of the revised approach

Theorem (C., 2025)
Suppose that $f \in \mathcal{F}_{m,n}$ *is real-valued and* $Z \sim N_{m,n}$. *Then* $\mathbb{E}(f(Z))$ *is real.*

Sketch of the proof

Consider the simplest case, where $f : \mathbb{R} \to \mathbb{R}$ has an analytic continuation to the whole of $\mathbb{C}$, and $\mathbb{E}|f(iZ)| < \infty$ for $Z \sim N(0,1)$.

Sketch of the proof

Consider the simplest case, where $f : \mathbb{R} \to \mathbb{R}$ has an analytic continuation to the whole of $\mathbb{C}$, and $\mathbb{E}|f(iZ)| < \infty$ for $Z \sim N(0, 1)$.

We have to show that $\mathbb{E}(f(iZ)) \in \mathbb{R}$.

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By the Schwarz reflection principle and the fact that f is real-valued on $\mathbb{R}$, we deduce that $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

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Since the distribution of Z is symmetric around zero, this implies that

$$\mathbb{E}(f(iZ)) = \frac{1}{2}[\mathbb{E}(f(iZ)) + \mathbb{E}(f(-iZ))] = \frac{1}{2}[\mathbb{E}(f(iZ)) + \overline{\mathbb{E}(f(iZ))}] \in \mathbb{R}.$$

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A preprint is available on arXiv.