Connectivity of spatial networks

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Outline

- Random geometric graphs (RGG) and applications
- Connectivity in random connection models (RCM)
  Phys. Rev. E **93**, 032313 (2016)
- K-connectivity: How RCM differs from RGG.
  EPL **103**, 28006 (2013)
- Anisotropy: Realism and optimisation.
  Trans. Wireless Commun. **13**, 4534 (2014)
- Line of sight with fractal boundaries
  ISWCS 2015, 636-640.
- Inhomogeneous networks (in progress)
- Outlook: Some current and future generalisations.

Poisson point processes

A point process is a random set of points Φ. We denote the number of points in a set A as

$$\Phi(A) = \sharp\{\Phi \cap A\}$$

Then the Poisson point process (PPP) with intensity measure Λ is defined by

1. $\Phi(A) \sim \mathrm{Poi}(\Lambda(A))$, the discrete Poisson distribution with mean $\Lambda(A)$.

2. If A and B are disjoint, $\Phi(A)$ and $\Phi(B)$ are independent.

A uniform PPP has $\Lambda = \rho \times \mathrm{Leb}$ in some domain.

Random geometric graphs

Introduced in 1961 by E. N. Gilbert:

> *Recently random graphs have been studied as models of communications networks. Points (vertices) of a graph represent stations; lines of a graph represent two-way channels. . . . To construct a random plane network, first pick points from the infinite plane by a Poisson process with density D points per unit area. Next join each pair of points by a line if the pair is separated by distance less than R.*

Then:

Communications networks Many authors, since 1980s

Connectivity threshold Penrose (1997), Gupta & Kumar (1999)

Books/reviews:

- **Meester & Roy (1996)** Continuum percolation
- **Penrose (2003)** Random geometric graphs
- **Franceschetti & Meester (2008)** Random networks for communication
- **Walters (2011)** Random geometric graphs
- **Barthélemy (2011)** Spatial networks
- **Haenggi (2012)** Stochastic geometry for wireless networks

Wireless network considerations

Mesh architectures Multihop connections rather than direct to a base station: Reduces power requirements, interference, single points of failure.

Random node locations In many applications (sensor, vehicular, swarm robotics, disaster recovery, ...) device locations are unplanned and/or mobile.

Network characteristics Full connectivity, k-connectivity (resilience), algebraic connectivity (synchronisation), betweenness centrality (importance, overload).

Useful extensions:

Random connection models Extra randomness: Form a link with (iid) probability $H(r) \in [0, 1]$, a function of mutual distance r.

Nonuniform Choose points using a PPP with a nonuniform measure; realistic for mobility and complex geometries.

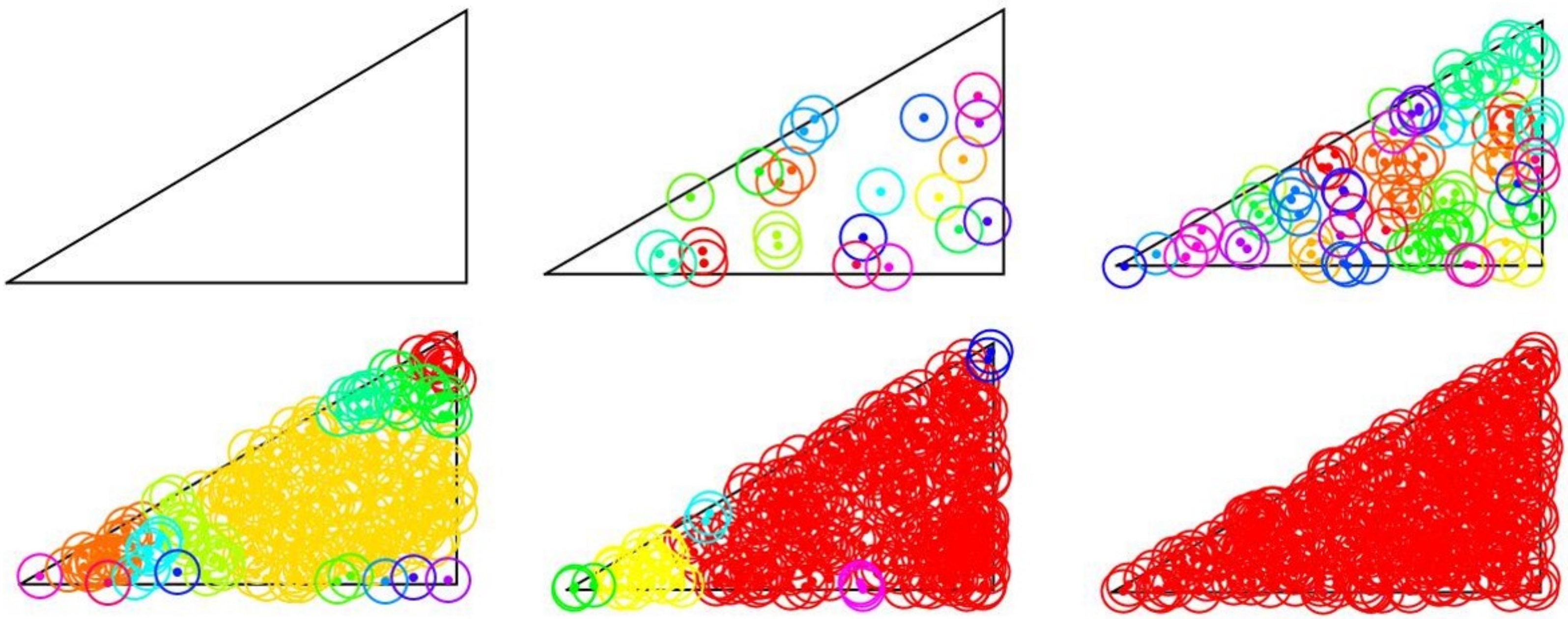
Anisotropy Orientations as well as positions.

Line of sight condition Impenetrable and/or reflecting boundaries: Particular relevance to millimetre waves for 5G.

Directed graphs Ability to transmit/receive need not be symmetric.

Remark: The last three extensions violate the metric space axioms.

Example: A triangle



Isolated nodes occur mostly near the corners...

Dependence on density and geometry

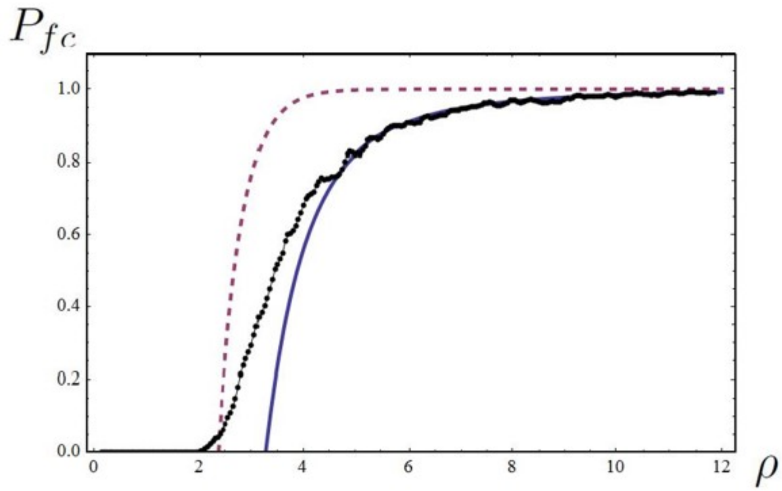
Notation: Mean degree $\mathcal{K}$, (full) connection probability P_{fc}.

We see two main transitions as density increases:

Percolation Formation of a cluster comparable to system size:
Largely independent of geometry. $\mathcal{K} = 4.5122\ldots$ in 2D

Connectivity All nodes connected in multi-hop fashion:
Strongly dependent on geometry. $\mathcal{K} \approx \ln N$.

P_{fc} as a function of density and geometry?



Previous results

Rigorous results are for $N \to \infty$, scaling at least two of r_0, ρ and L.

For the random geometric graph in dimension $d \geq 2$, it was shown by Penrose, and by Gupta & Kumar, that the r_0 threshold for **connectivity** is almost always the same as for **isolated nodes**.

In turn, isolated nodes are local events, so described by a limiting Poisson process: The probability of a node having degree k is given by

$$P(k) = \frac{\mathcal{K}^k}{k!} e^{-\mathcal{K}}$$

where $\mathcal{K}$ is the mean degree, equal to $\rho\pi r_0^2$ for the 2D RGG. This leads to

$$P_{fc} \approx \exp\left[-\rho V e^{-\rho\pi r_0^2}\right]$$

where V is the "volume" (ie area) of the domain.

Remarks: At fixed probability and connection range, V increases exponentially with ρ; also most isolated nodes are in the bulk when $d = 2$. The number of isolated nodes at corners cannot be Poisson.

Random connection model

Penrose (2016) showed that for connection functions that are symmetric, positive at the origin and stretched exponentially decaying (also radially symmetric and monotonic for $d > 2$), the number of isolated nodes is asymptotically Poisson distributed. Further, if its support is sufficiently small, the (full) connection probability is asymptotically that of there being no isolated nodes. (See also Mao & Anderson 2013, Iyer arxiv 2015).

Here we assume the resulting formula is approximately valid for finitely many nodes, including for connection functions with unbounded support:

$$P_{fc} \approx \exp\left[-\int \rho e^{-\int \rho H(r_{12}) d\mathbf{r}_1} d\mathbf{r}_2\right]$$

where ρ is the density, $H(r)$ is the iid probability of connection between nodes with mutual distance r and the integrals are over the domain $\mathcal{V} \subset \mathbb{R}^d$.

We want to approximate P_{fc} for finite ρ, taking into account boundaries.

In progress: $d = 1$, eg vehicular networks!

Specific random connection models

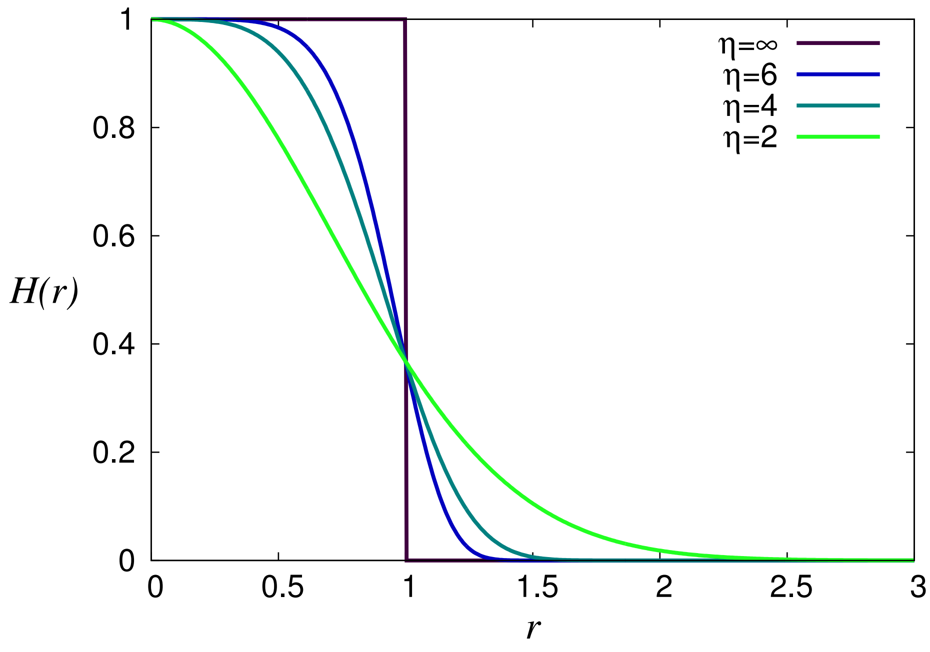
The connection function is the complement of the outage probability,

$$H(r) = \mathbb{P}(SNR > q)$$

neglecting interference, with the signal to noise ratio (SNR) proportional to $r^{-\eta}|h|^2$, path loss exponent $2 \lesssim \eta \lesssim 6$. Simplest is Rayleigh fading (diffuse signal), for which the channel gain $|h|^2$ is exponentially distributed, giving

$$H(r) = \exp[-(r/r_0)^\eta]$$

Similar, though more involved: MIMO, Rician (specular plus diffuse), ...



Connectivity and boundaries

For large ρ, P_{fc} is dominated by the regions of small connectivity mass

$$M(\mathbf{r}_2) = \int H(r_{12}) d\mathbf{r}_1$$

Exactly on the boundary, this is given by

$$M_B = H_{d-1}\omega_B$$

where

$$H_m = \int_0^\infty H(r) r^m dr$$

is the mth moment, and ω_B is the (solid) angle associated with the boundary component B, eg $\pi/2$ for a right angled corner, π for an edge.

Analysing the vicinity of boundaries more carefully...

General formula

$$P_{fc} = \exp\left[-\sum_B \rho^{1-i_B} G_B V_B e^{-\rho\omega_B H_{d-1}}\right]$$

where i_B is the boundary codimension, V_B is its $d-i$ dimensional volume, and G_B is the geometrical factor

G_B	$i=0$	$i=1$	$i=2$	$i=3$
$d=2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d=3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

where the 3D corner has a right angle.

Curved boundaries? To leading order, modification of the exponential but not the geometrical factor:

$$P_{2,1} = \ldots e^{-\rho(\pi H_1 - \kappa H_2)}$$

$$P_{3,1} = \ldots e^{-\pi\rho(2H_2 - \kappa H_3)}$$

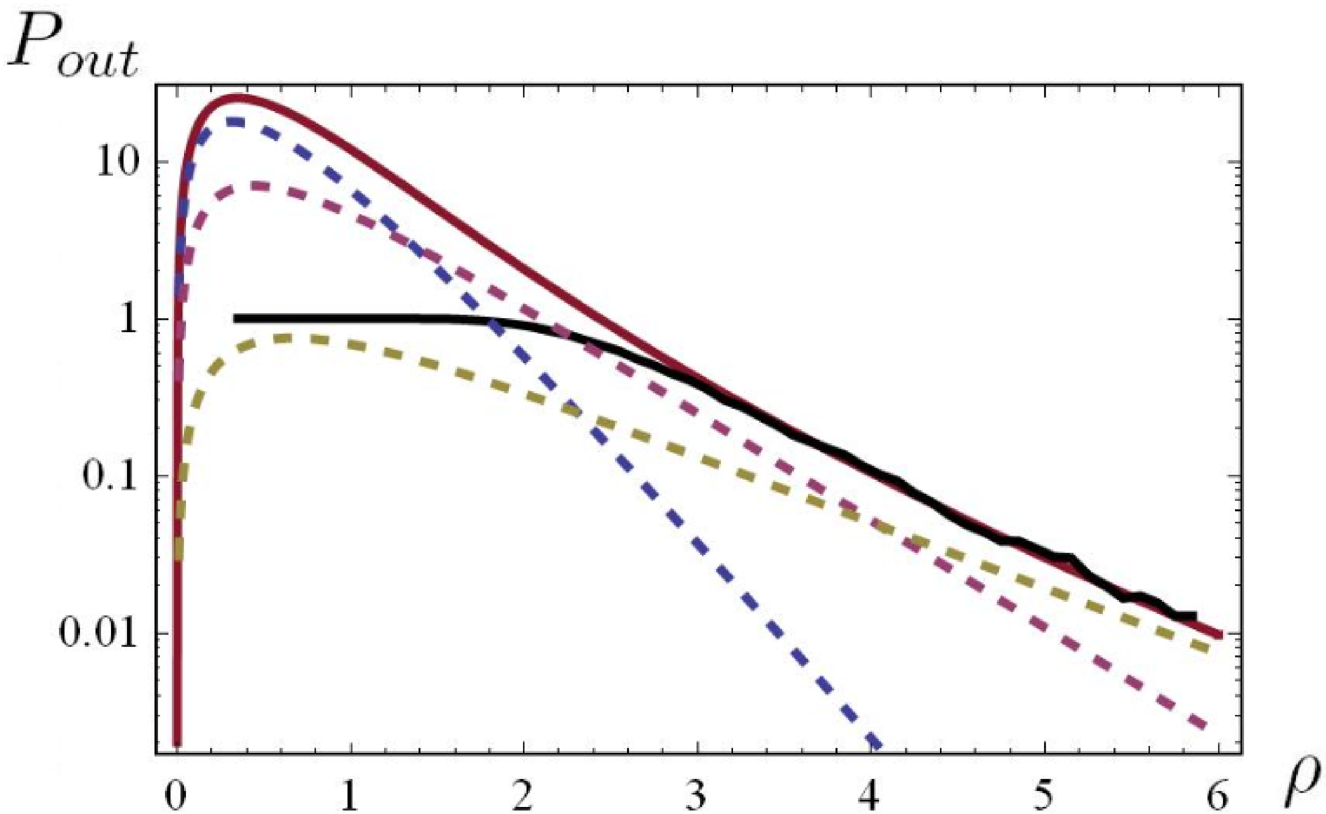
where κ is (mean) curvature.

Summary: We can do arbitrary convex geometries with piecewise smooth boundaries; $H(r)$ appears only via a few moments.

Example: A square

The previous formula gives

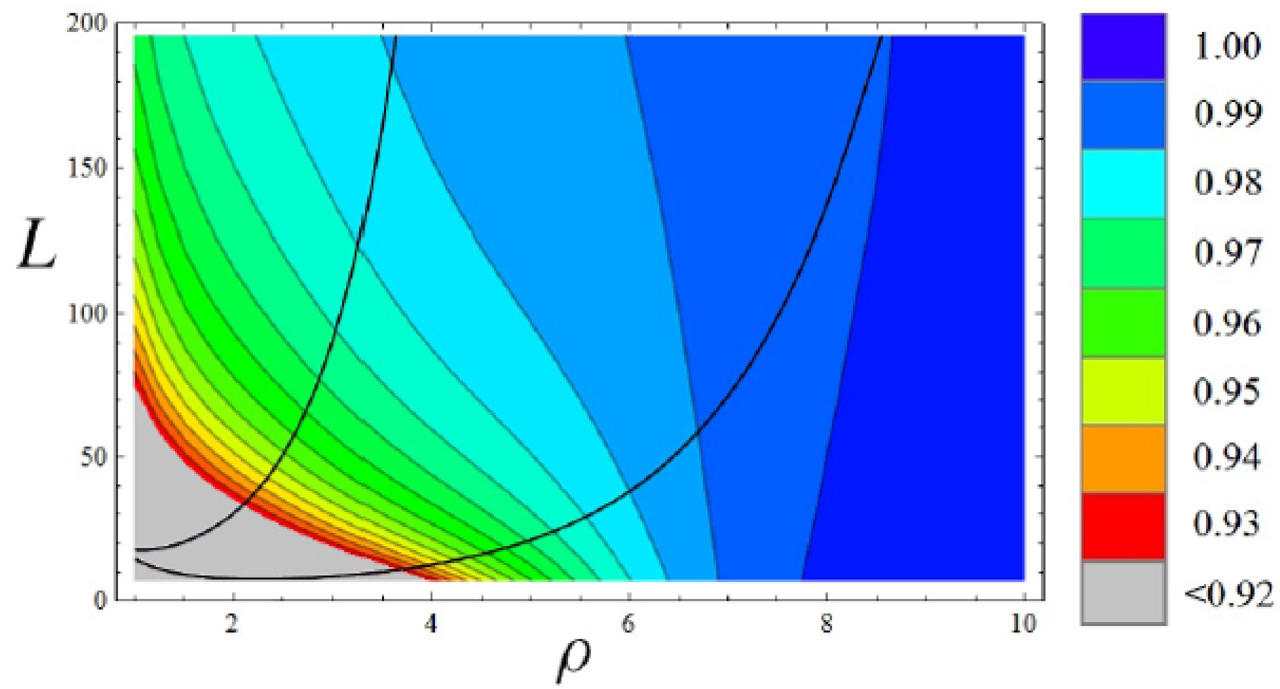
$$1 - P_{fc} \approx L^2 \rho e^{-\pi\rho} + \frac{4L}{\sqrt{\pi}} e^{-\frac{\pi\rho}{2}} + \frac{16}{\pi\rho} e^{-\frac{\pi\rho}{4}}$$



Phase diagram

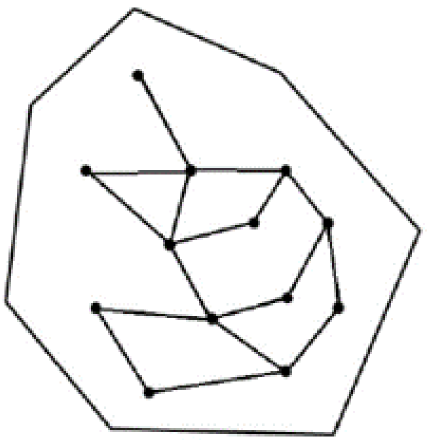
Testing convergence of

$$\frac{1 - P_{fc}}{\sum_B \cdots}$$

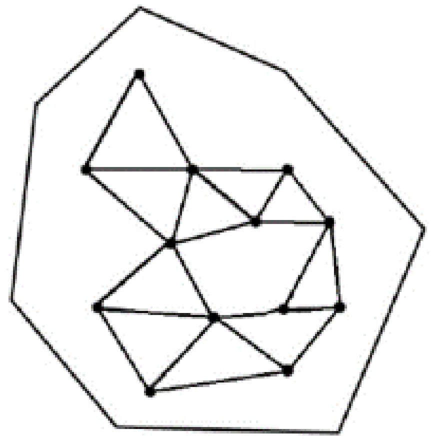


K-connectivity

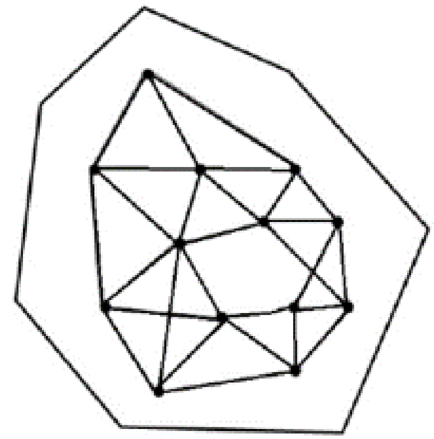
A network is (vertex) k-connected if any $k-1$ nodes can be removed and it remains connected. It is a useful measure of network resilience.



1-connected



2-connected



3-connected

Vertex connectivity $\leq$ Edge connectivity $\leq$ Minimum degree

Minimum degree

Assume independence ...

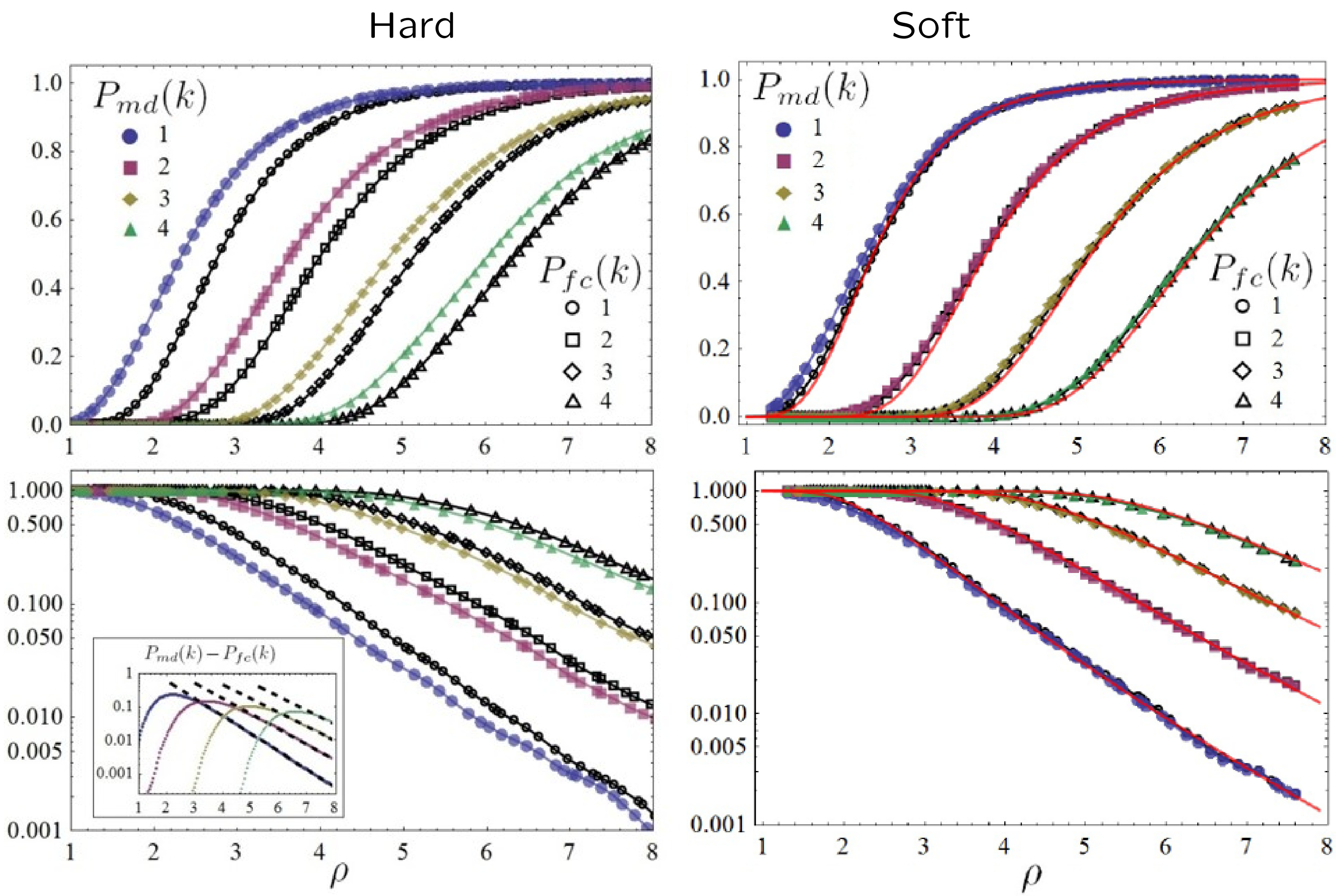
- For each node, degree is Poisson:

$$P_i(k) \approx \frac{\mathcal{K}_i^k}{k!} e^{-\mathcal{K}_i}$$

- Node degrees are independent:

$$P_{md}(k) \approx \left[1 - \sum_{m=0}^{k-1} \frac{\rho^m}{m!} \frac{1}{V} \int_{\mathcal{V}} M_H^m(\mathbf{r}_i) e^{-\rho M_H(\mathbf{r}_i)} d\mathbf{r}_i \right]^N$$

Numerical results



Random connections: Minimum degree is a better proxy for k-connectivity.

Why? Connections are less correlated in the random model.

Anisotropic connections

Now, nodes have random locations and orientations.

- Angle-dependent transmit and receive gains:

$$H(r, \phi, \theta_T, \theta_R) = \exp\left(-\frac{\beta r^\eta}{G_T(\phi - \theta_T)G_R(\phi + \pi - \theta_R)}\right)$$

- Fix total power per node

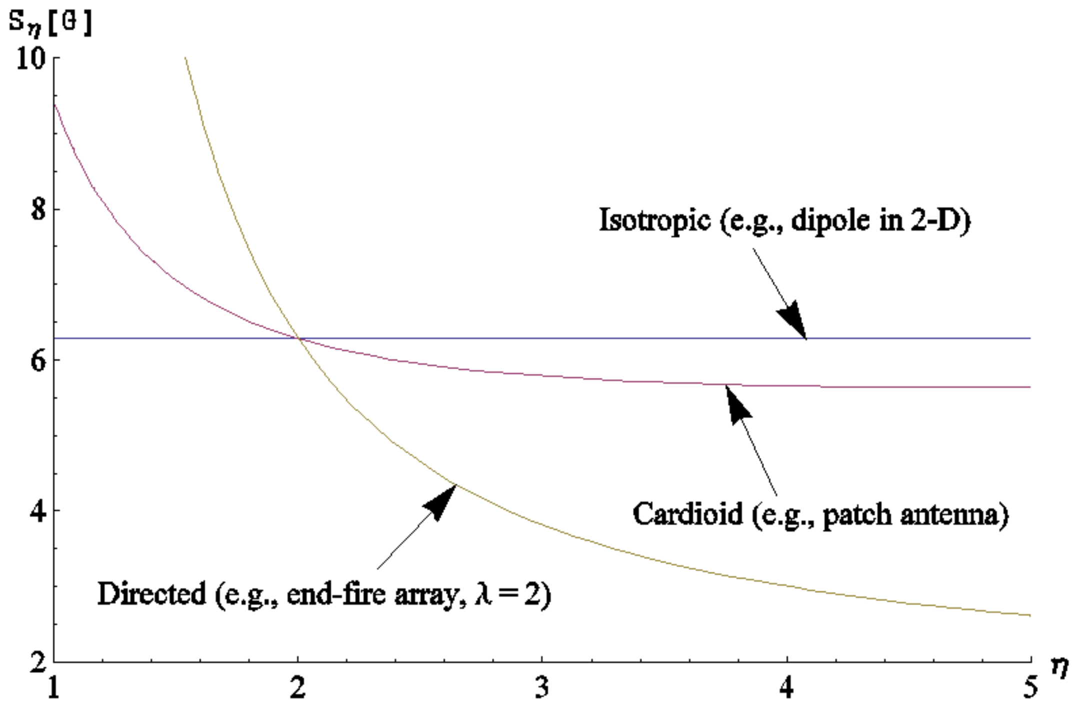
$$\int_0^{2\pi} G_T(\phi)d\phi = \int_0^{2\pi} G_R(\phi)d\phi = 2\pi$$

- Connectivity mass is now

$$M = \frac{1}{2\pi}\int rH(r, \phi, \theta_T, \theta_R)drd\phi d\theta_R$$

Path loss exponent $2 \lesssim \eta \lesssim 6$ as before.

Transition at $\eta = d$

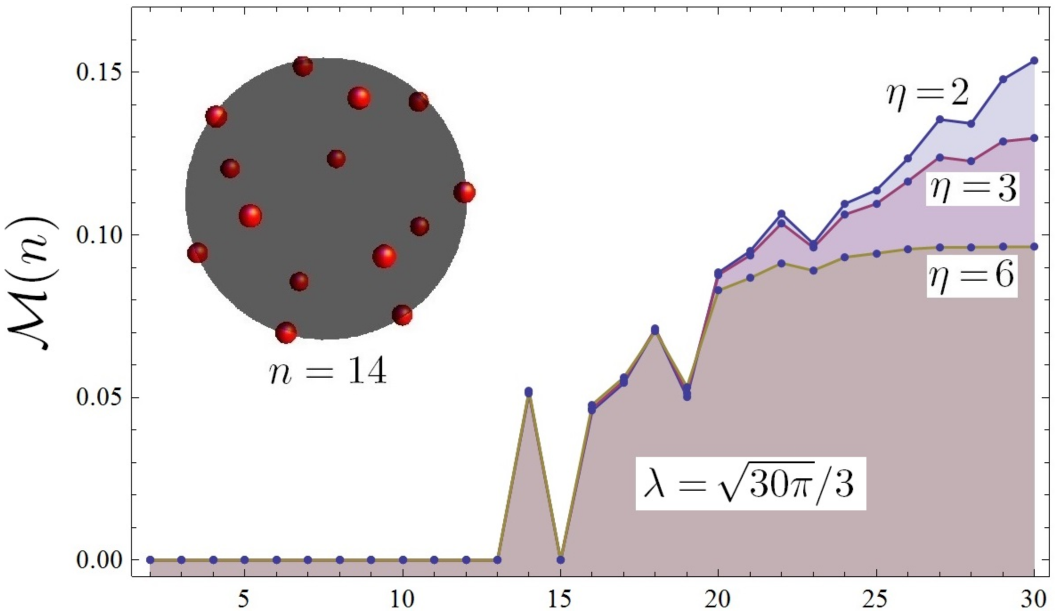


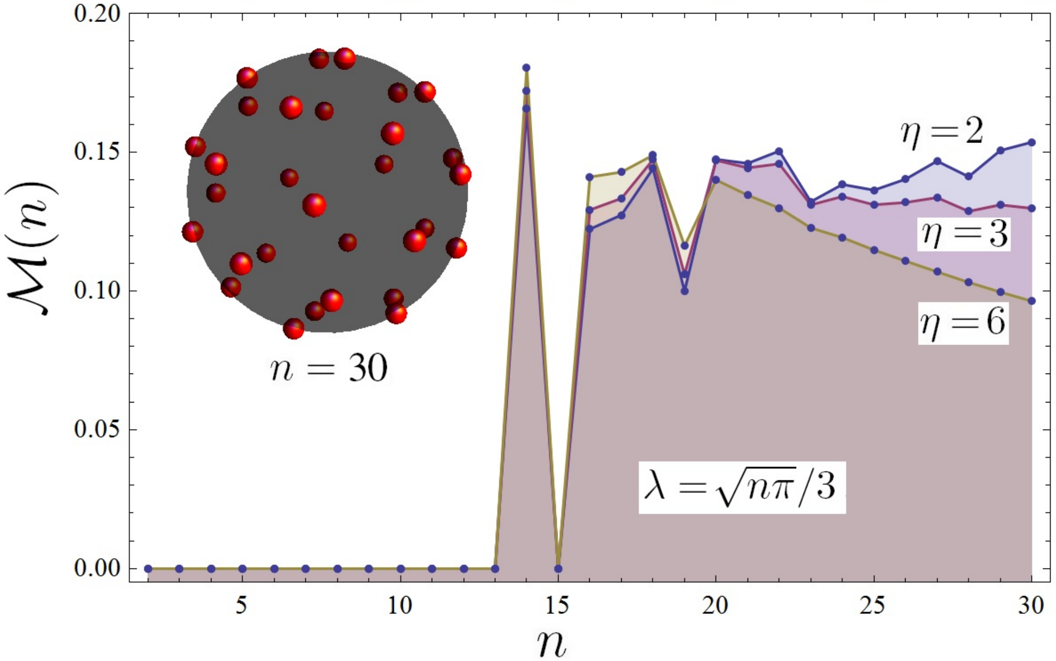
Anisotropy and boundaries

- Homogeneous case:
  - Path loss exponent $\eta > d$: Isotropic optimal
  - Path loss exponent $\eta < d$: Delta spike(s) optimal
- With boundaries, for $\eta < d$, trade-off between system size/shape and number/width of spikes. Examples:
  - Square, best to have a multiple of 4 spikes.
  - Cube . . .

Cube optimal pattern

14 spikes: Gyroelongated hexagonal bipyramid!





Fractals: Theory and applications

- Mathematical analysis: Zoo for topologists
- Dynamical systems: Attractors, Repellers, basin boundaries
- Biology: Trees, lungs, ...
- Natural environment: Coastlines, rivers, mountains, clouds
- Built environment: Land use, transport networks
- Realistic graphics for the above, art creation and analysis
- Fractal antennas and arrays: High ratio of length to volume, wide band

Biological and artificial fractals both solve optimisation problems.

Fractal dimensions

Box dimension Cover a set F with $N(\epsilon)$ boxes of size ϵ. Then take limit

$$D_B(F) = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{-\ln \epsilon}$$

if it exists. Finite range of ϵ for real objects. Popular for numerics.

Hausdorff dimension Cover with boxes of size $r_i \leq \epsilon$. Then the s-dimensional Hausdorff measure

$$C_H^s(F) = \inf_{\{r_i\}} \sum_i r_i^s$$

switches from ∞ to 0 at a single value of s, the Hausdorff dimension $D_H(F)$. Popular for rigorous mathematics. $D_H(F) \leq D_B(F)$

Self-similar fractals

Similarity transformation S: Combinations of dilations, rotations, reflections, translations. Choose some similarity transformations S_i with scale factors $0 < r_i < 1$. Then, there is a unique compact set F:

$$F = \cup_i S_i F$$

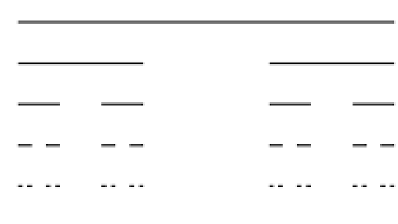
Open set condition (OSC): There is an open set U so that $\cup_i S_i U \subset U$ disjointly. Under the OSC,

$$D_B(F) = D_H(F) = D_S(F)$$

where the similarity dimension satisfies

$$\sum_i r_i^{D_S(F)} = 1$$

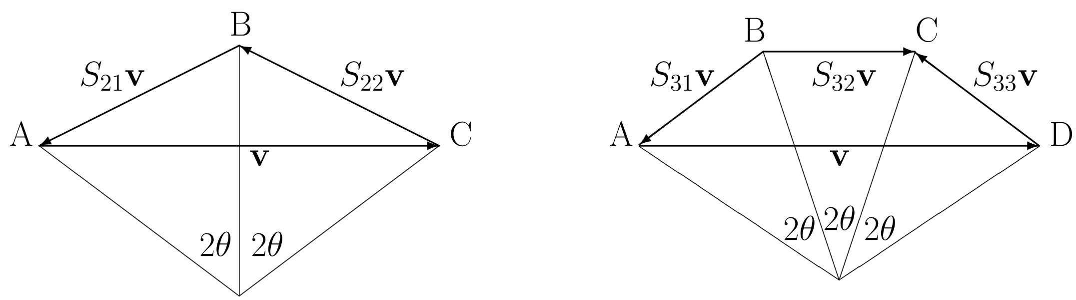
Example: Middle third Cantor set, $D = \frac{\ln 2}{\ln 3}$.

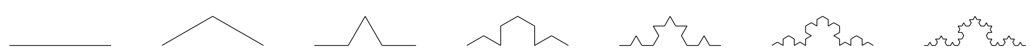


RGG with fractal boundaries I

Now, we return to the fixed connection model, but with a line of sight (LOS) condition.

Construct two classes of fractals, $F_2(\theta)$ with $\theta \in [0, \pi/4]$ and $F_3(\theta)$ with $\theta \in [0, \pi/6]$.

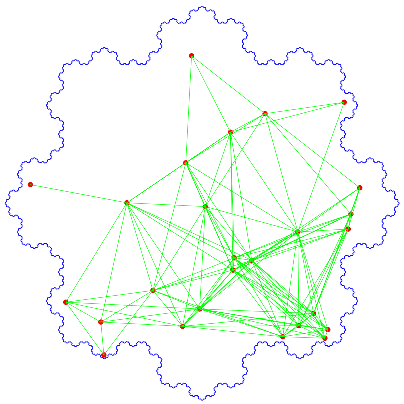




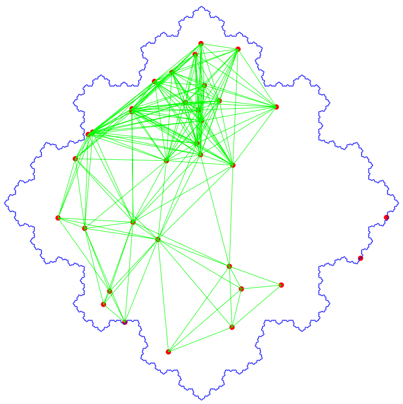
Then, make four copies and rotate to enclose a region.

RGG with fractal boundaries II

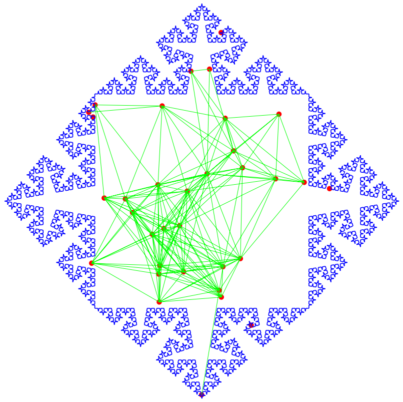
Distribute nodes randomly, connect with LOS condition.



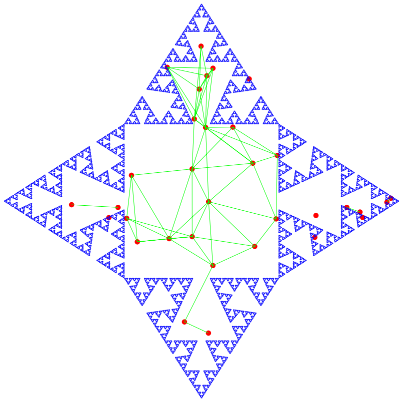
$F_2(0.4)$



$F_3(0.3)$



$F_2(0.7)$



$F_3(0.5)$

Connectivity paradox

For large values of θ, there are nodes near the boundary that cannot make a LOS connection. These increase with density.

Assuming we are at high enough density for the interior to be connected, the self-similarity gives a scaling relation for P_{fc}:

$$P_{fc}(r^{-d}\rho) = P_{fc}(\rho)^n$$

where $n \in \{2, 3\}$ is the number of transformations defining the fractal. Try

$$P_{fc} = \text{exp}\left[-a(\rho)\rho^{\beta}\right]$$

giving

$$a(r^{-d}\rho)r^{-d\beta} = na(\rho)$$

Choosing $\beta = D/d$ and recalling $r^{-D} = n$,

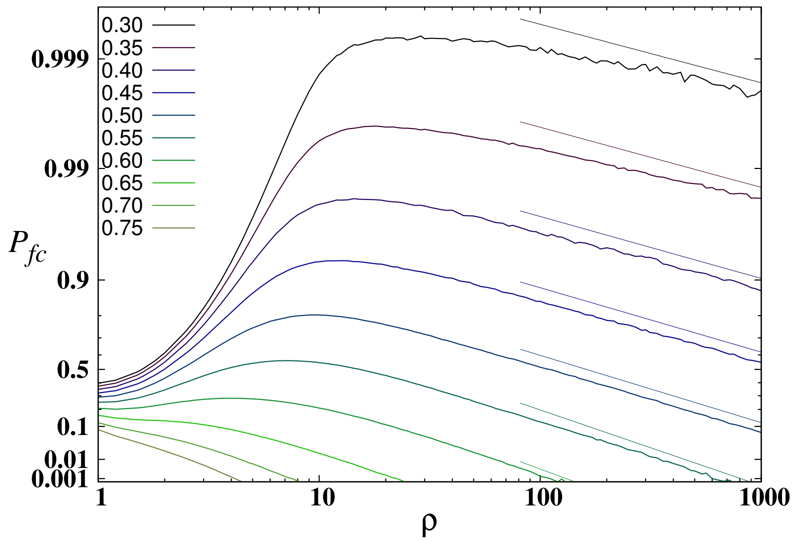
$$a(r^{-d}\rho) = a(\rho)$$

So, log periodic; likely approximately constant:

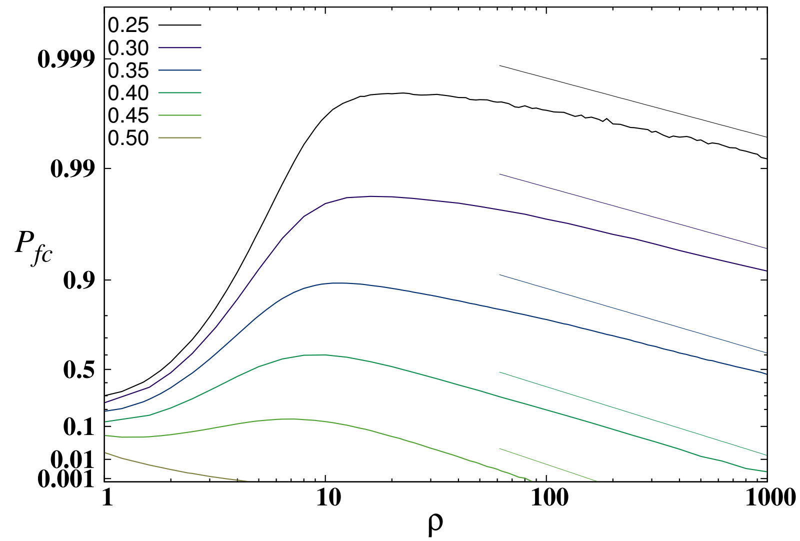
$$P_{fc} \approx \text{exp}\left(-a\rho^{D/d}\right)$$

Numerical confirmation

Note agreement for both small and large P_{fc}



F_2



F_3

Nonunifom measures (work in progress)

Now, nodes are distributed with respect to a more general intensity measure.

Motivation:

- Non-uniform densities arise naturally from mobility
- Many natural and built environments can be described as fractal

It is likely that strong inhomogeneities will affect the RGG properties of

1. Poisson distribution of isolated nodes
2. Relation between isolated nodes and connectivity

Penrose (arxiv 2015) on inhomogeneous random graphs gives Poisson results assuming ϵ-homogeneous and $H(r) < C < 1$, both of which are violated here.

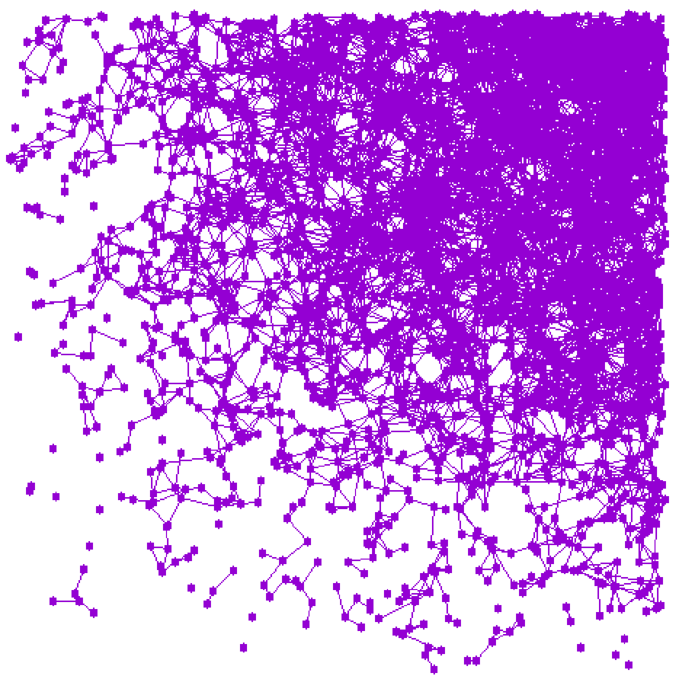
Models

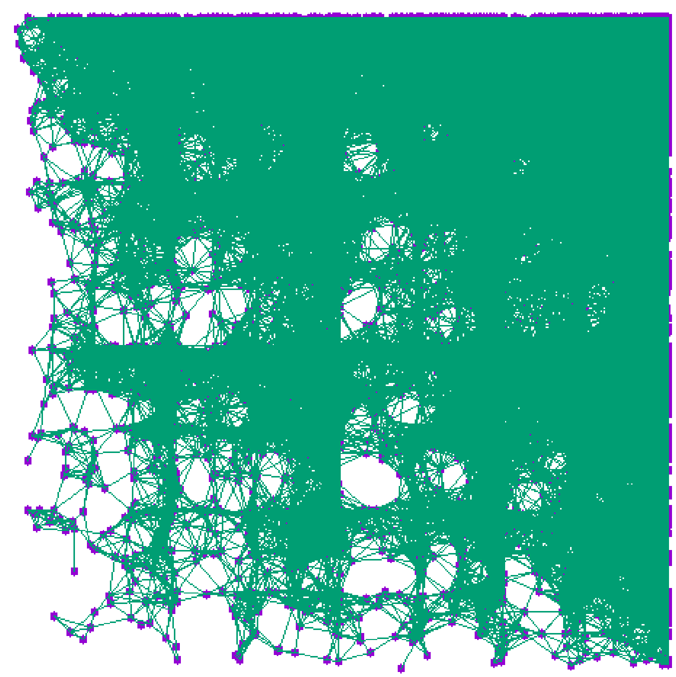
We consider two models in detail:

- A unit square with density $4xy$ (left)

- A self-similar measure splitting the square into four equal quadrants, with measures p^2, $p(1-p)$, $p(1-p)$, $(1-p)^2$ for some $0 < p \leq 0.5$. $p = 0.5$ is the uniform measure. $p = 0.3$ is shown on the right.

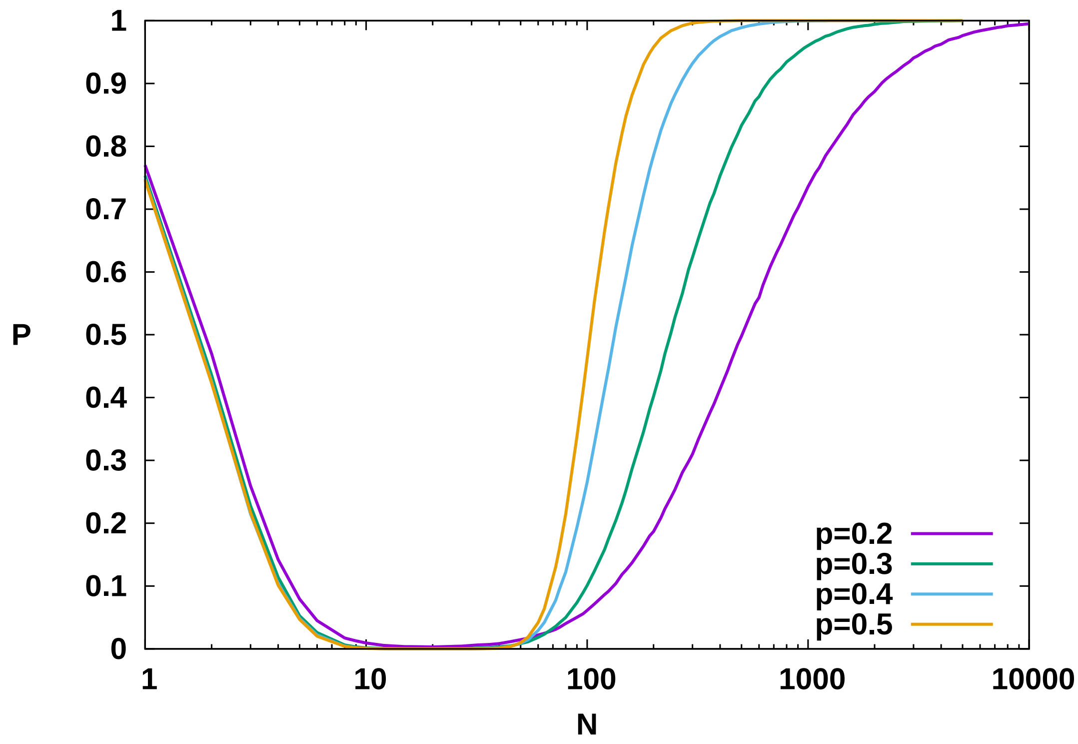
Others under consideration include uniform measures on fractals and on cusps.





Effect on connectivity

Nonuniformity makes connectivity more “random,” ie, the transition is spread over a much greater range of densities.



Quantifying non-Poissonness

Given a discrete distribution P_j for $j = 0, 1, 2, \ldots$ we can quantify non-Poissonness using factorial cumulants:

$$q_n = \frac{d^n}{dt^n} \ln \mathbb{E}(t^X)\bigg|_{t=0}$$

of which the first few are

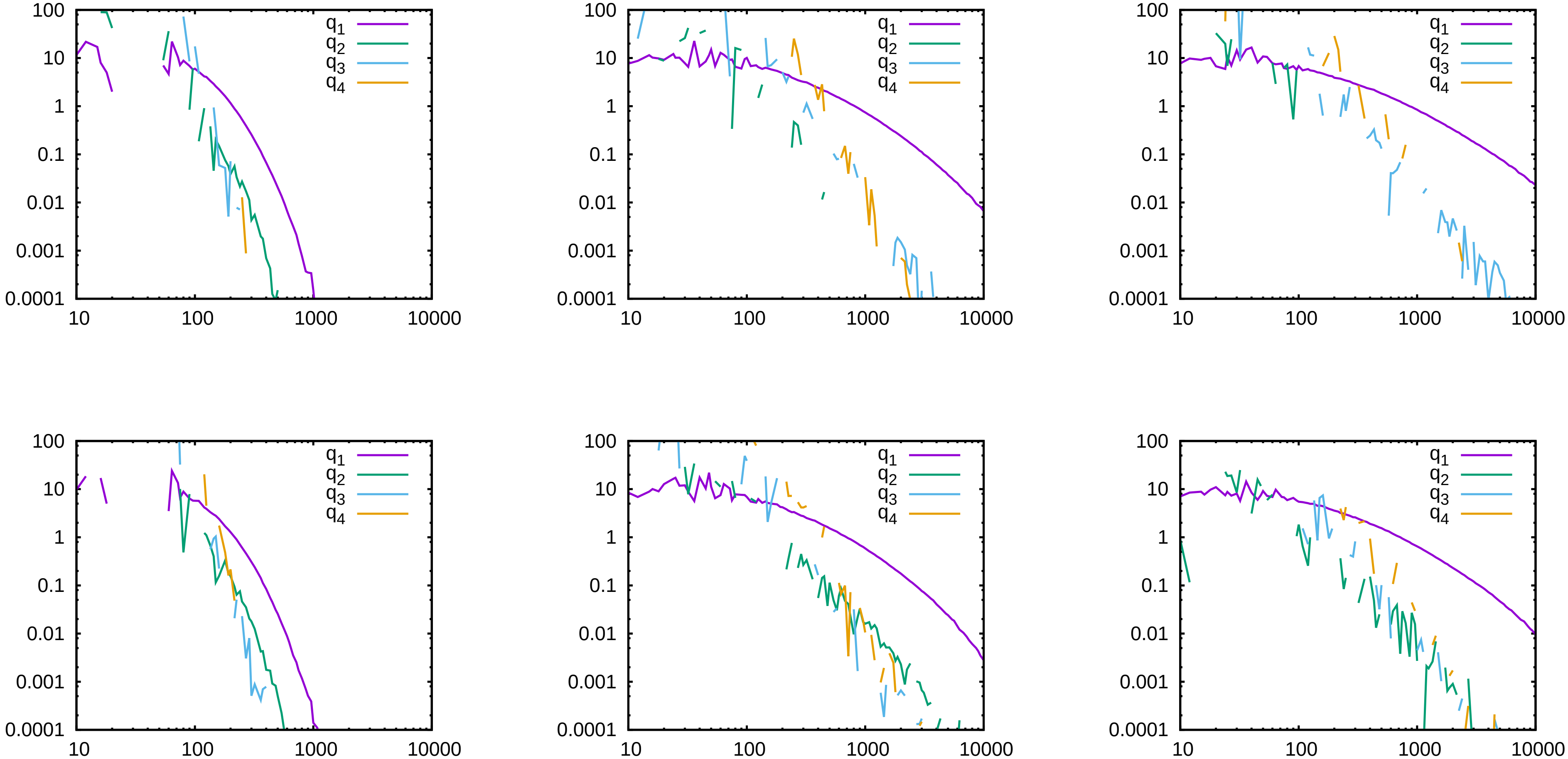
$$\begin{aligned}
q_1 &= \tilde{P}_1 \\
q_2 &= 2\tilde{P}_2 - \tilde{P}_1^2 \\
q_3 &= 6\tilde{P}_3 - 6\tilde{P}_2\tilde{P}_1 + 2\tilde{P}_1^3 \\
q_4 &= 24\tilde{P}_4 - 24\tilde{P}_3\tilde{P}_1 - 12\tilde{P}_2^2 + 24\tilde{P}_2\tilde{P}_1^2 - 6\tilde{P}_1^4
\end{aligned}$$

where $\tilde{P}_j = P_j/P_0$.

For Poisson, all $q_n = 0$ for $n > 1$.

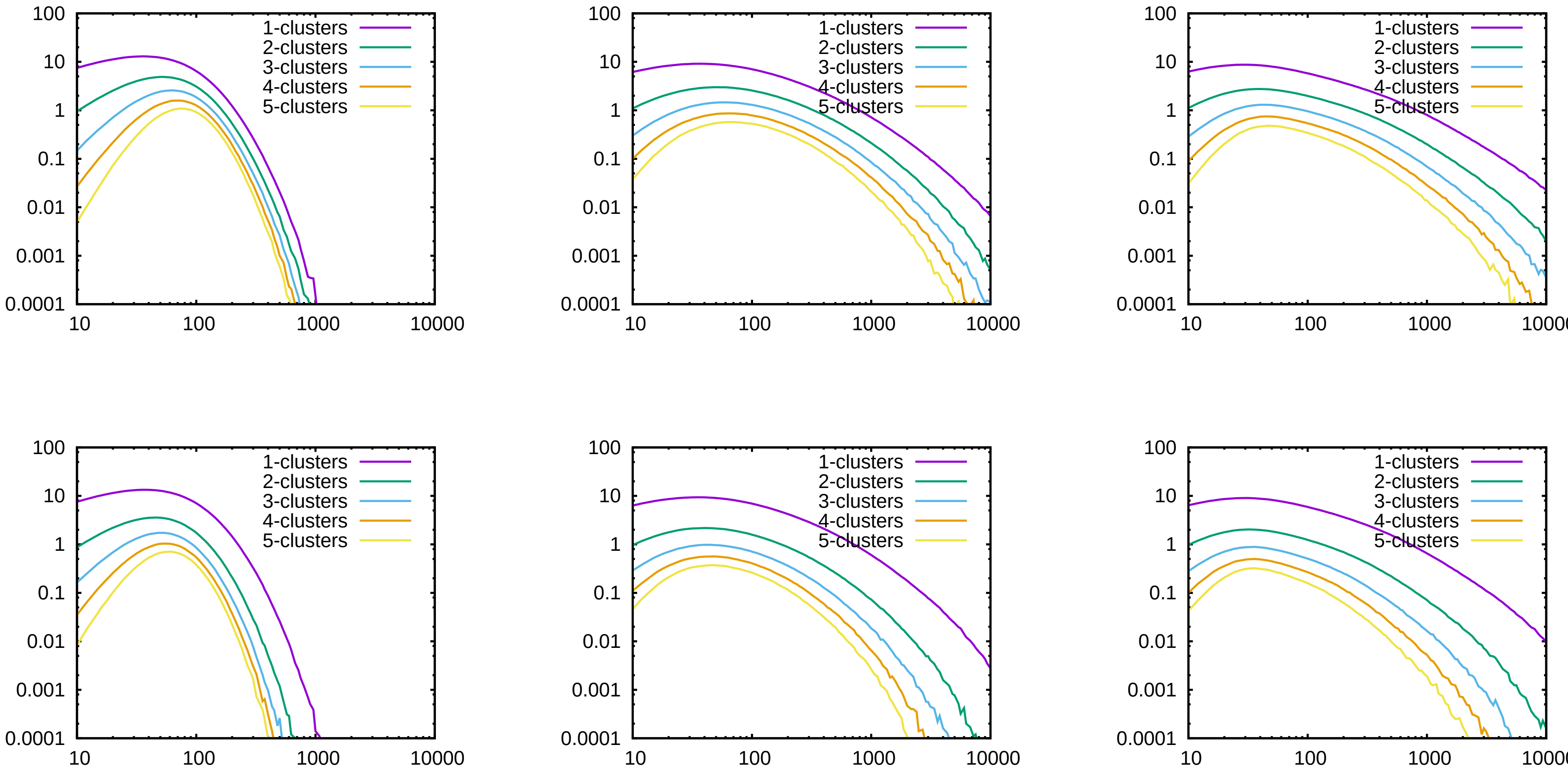
q_n numerics

$r_0 = 0.1$. Top $\eta = \infty$, bottom $\eta = 2$. Uniform left, $p = 0.3$ middle, $4xy$ right.



Occurrence of small clusters

$r_0 = 0.1$. Top $\eta = \infty$, bottom $\eta = 2$. Uniform left, $p = 0.3$ middle, $4xy$ right.



Small clusters - first steps

Let's have a RGG in 1D with density $\rho(x) = cx^\alpha$ for $x > 0$ and study isolated nodes. $H(r)$ is the connection probability, assumed unit disk. $\bar{H} = 1 - H$.

The expected number of isolated nodes is

$$\int_0^\infty \rho(x) \exp\left[-\int_0^\infty H(|x-y|)\rho(y)dy\right] dx = c^{-\alpha} \exp\left[-\frac{c}{\alpha+1}\right] \Gamma(\alpha+1)$$

assuming high density and hence ignoring contributions for $x > 1$.

The expected number of 2-clusters is

$$\int_0^\infty \int_0^\infty \rho(x)\rho(y)H(|x-y|) \exp\left[\int (\bar{H}(|x-z|)\bar{H}(|y-z|) - 1)\rho(z)dz\right] dxdy$$

(modifying a formula of G. Last). Again assuming high density this becomes

$$\frac{c^{-2\alpha}}{\alpha+1} \exp\left[-\frac{c}{\alpha+1}\right] \Gamma(2\alpha+2)$$

Thus the ratio is proportional to $c^{-\alpha}$ so 2-clusters become rarer more quickly at higher α (more inhomogeneous).

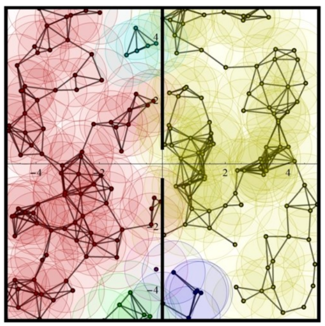
Outlook

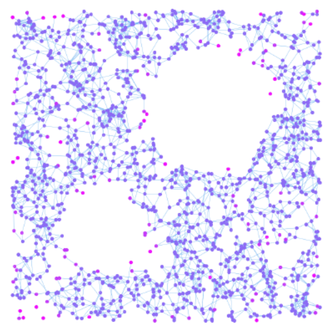
Random connection models are more realistic and well-behaved.

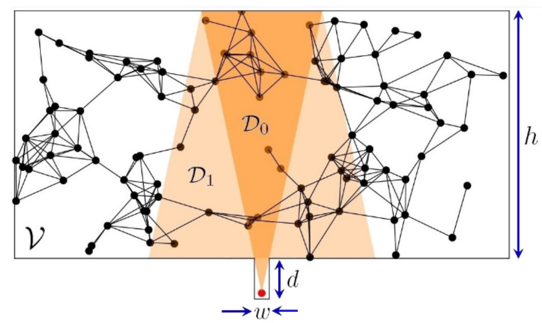
Nonuniform measures are more realistic and well-behaved.

Analysis is more challenging; these models are far from understood yet.

Other results/in progress: Non-convex domains, betweenness, interference, mobility, spectrum . . .







Mitigate lack of connectivity? Other network properties?

Postdoc available!

SPATIALLY EMBEDDED NETWORKS

EPSRC

Engineering and Physical Sciences Research Council