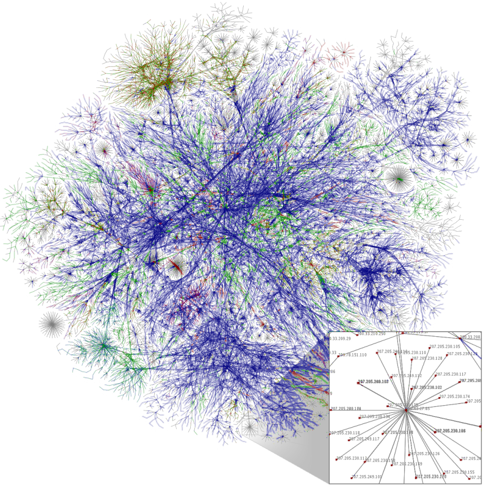
When are scale-free graphs ultra-small?

Júlia Komjáthy

joint with Remco van der Hofstad
Eindhoven University of Technology

Probability Seminar in Bristol,
Nov 4, 2016

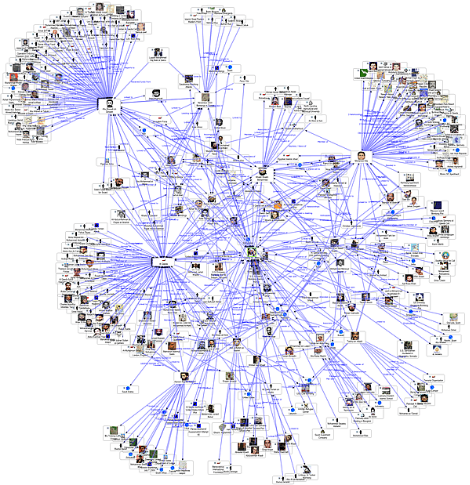
Complex networks 1.



IP level internet network, 2003

from the OPTE project, `opte.org`

Complex networks 2.



A Tweet-network

from Sentinel Visualiser, `fmsasg.com/SocialNetworkAnalysis/`

Degree plots

Empirical degree distributions are fitted to:

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(Pure) power laws

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$$\mathbf{P}(D = x) = \frac{C}{x^\tau}$$

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Truncation parameter ξ_n might depend on the size of the network.
For $x \ll \xi_n$: a power law,
for $x \approx \xi_n$: exponential decay.

Pure power laws

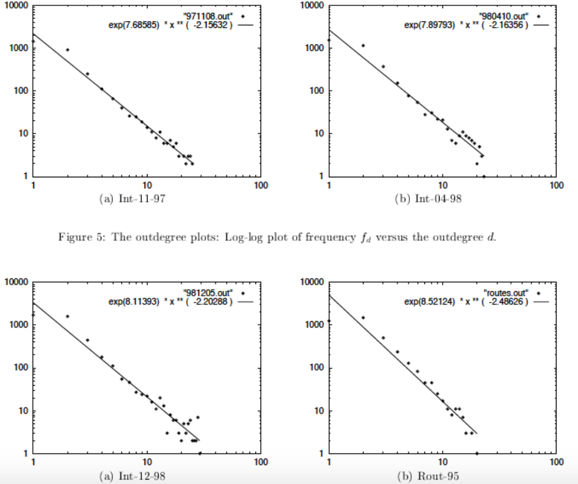

Figure : Growing IP level internet network: a pure power law

from Faloutsos *et al*, 1999

Pure and truncated power laws

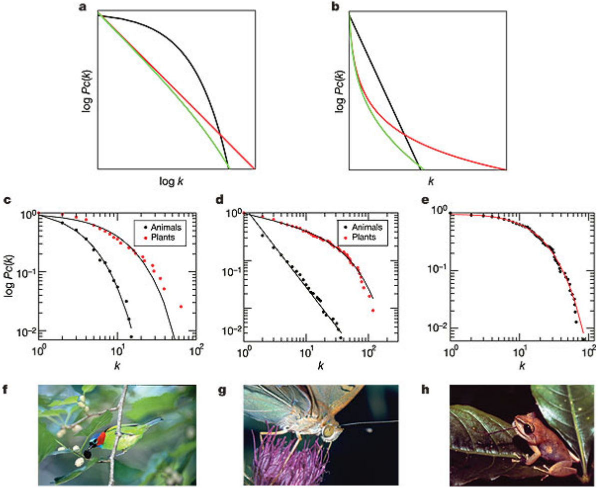

Figure : Ecological networks: pure and truncated power laws, exponential decay

from Montoya, Pimm, Solé, Nature 2006

Examples

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Pure power laws

- internet backbone network,
- metabolic reaction networks,
- telephone call graphs,
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Truncated power laws

- movie actor network,
- air transportation networks,
- co-authorship networks,
- brain functional networks,
- ecological networks.

Scale free vs ultra small

Def: scale free

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Def: ultrasmall world

A network is called an *ultrasmall world* when

$$\mathrm{d}_G(u, v) = O(\log \log n).$$

Scale free $\stackrel{?}{=}$ ultra small

Typical distances vs τ

How does the exponent relate to the (ultra)small world property?

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For pure power laws, $\tau > 3$ implies *small world*.

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Truncated scale free $\stackrel{?}{=}$ ultrasmall world

Goal of this talk

How does the truncation point ξ_n affect the ultrasmall world property?

Building a network: the configuration model

[Uniform matching simulator by Robert Fitzner]
[Configuration model simulator by Robert Fitzner]

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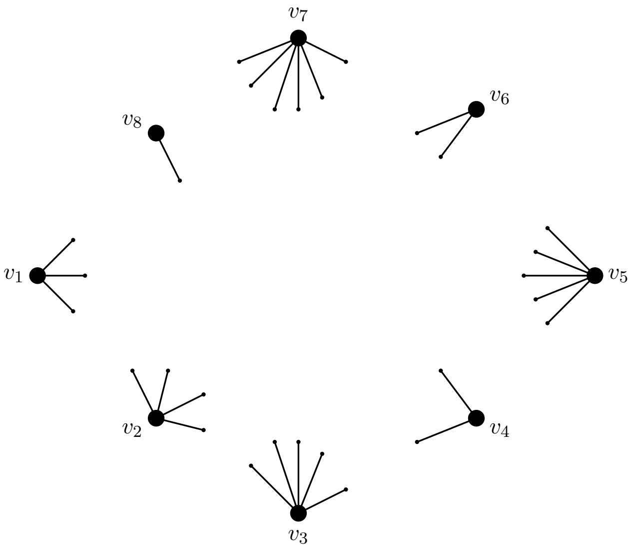

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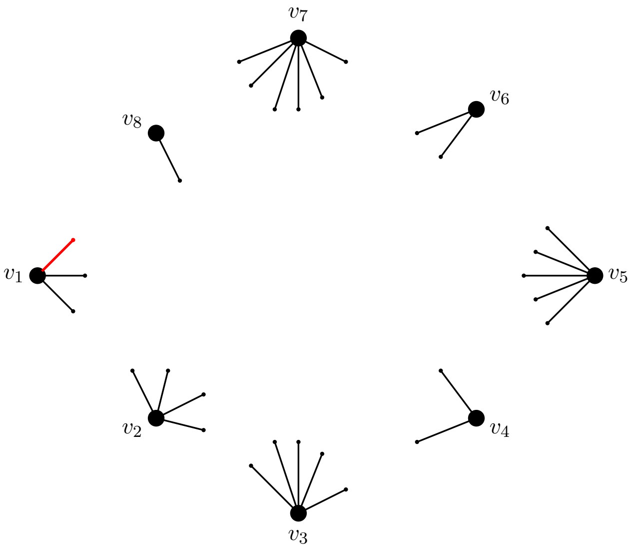

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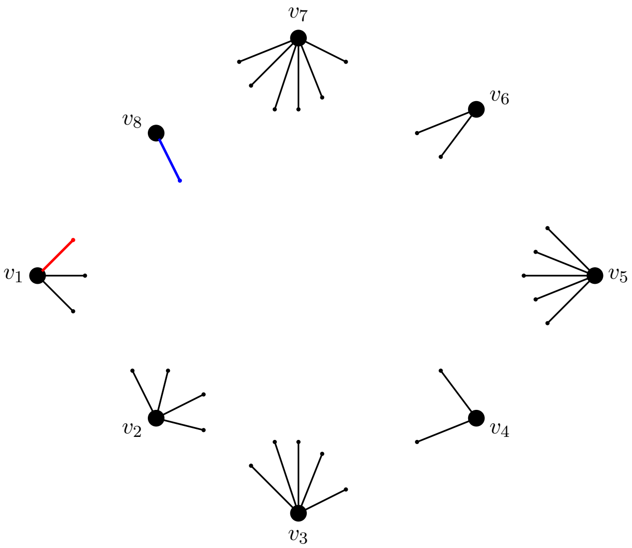

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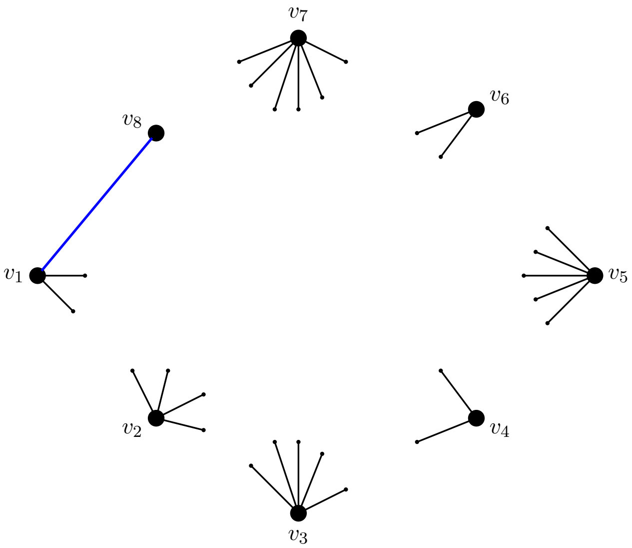

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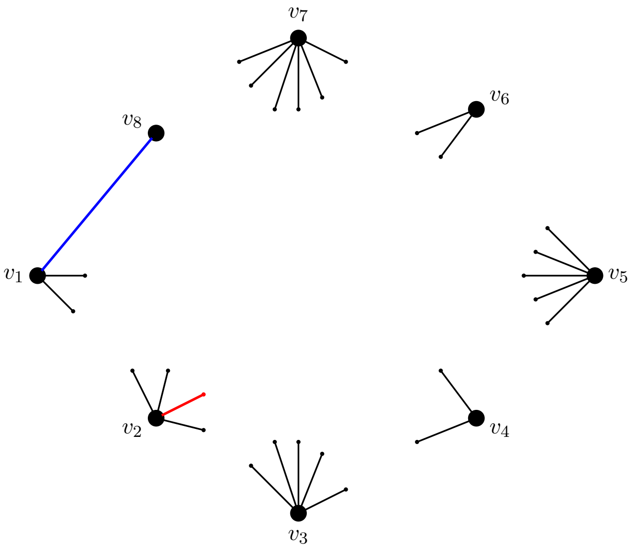

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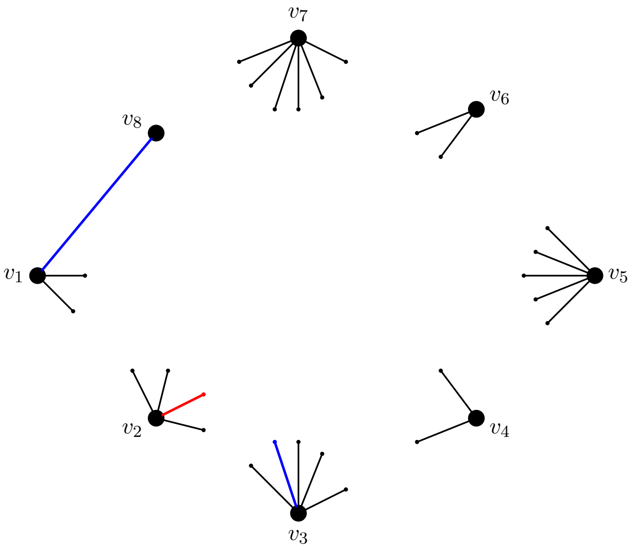

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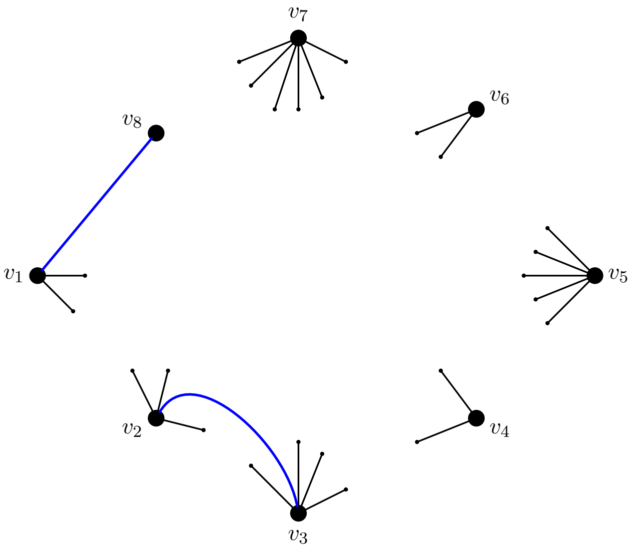

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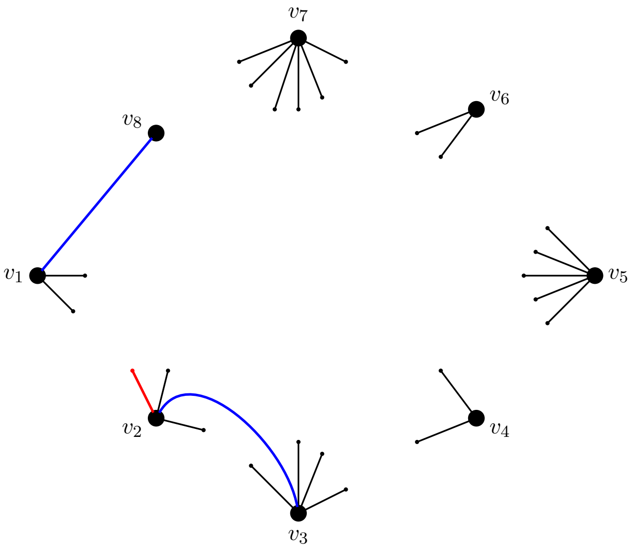

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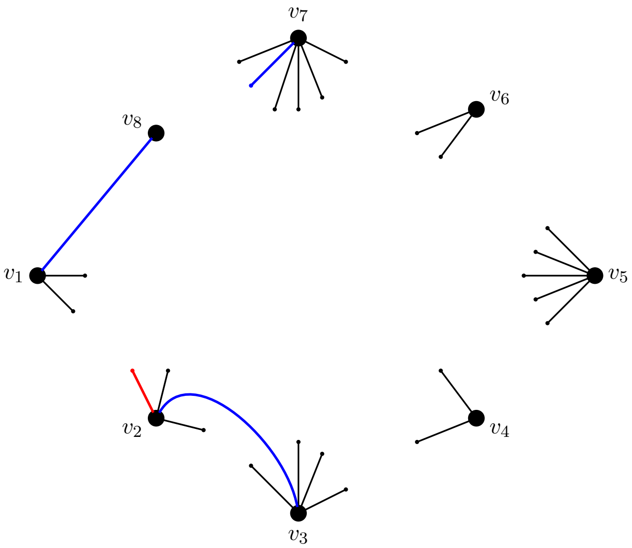

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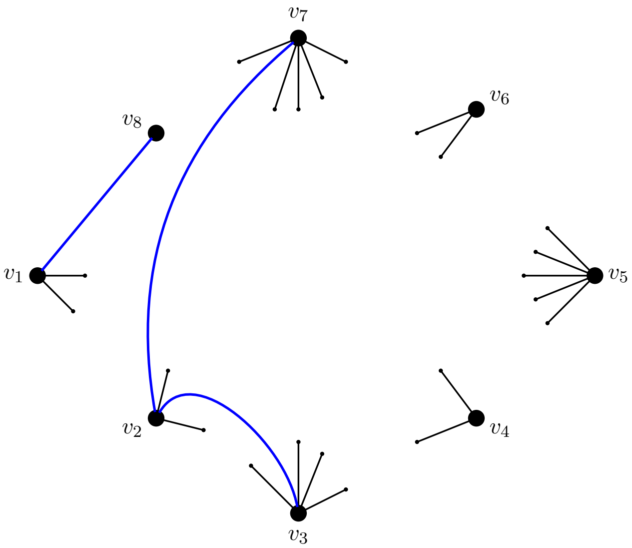

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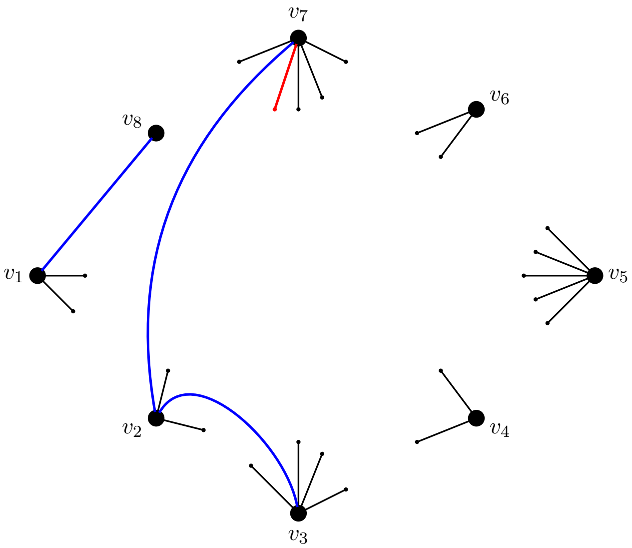

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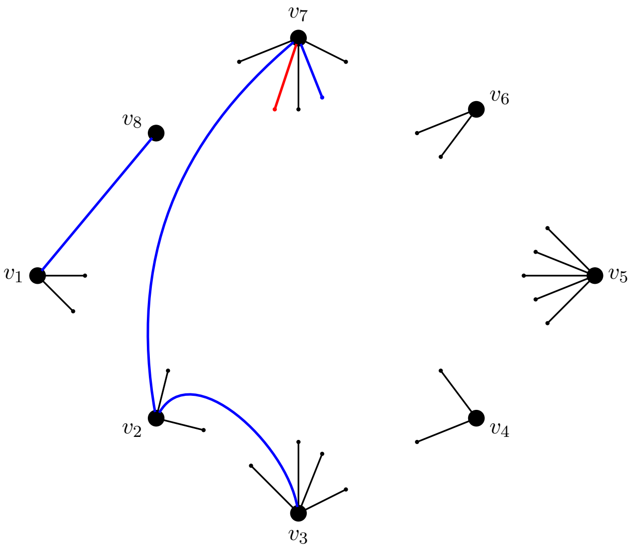

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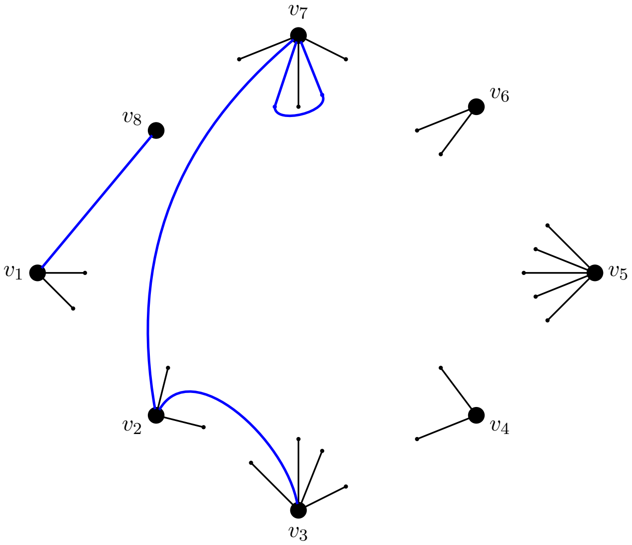

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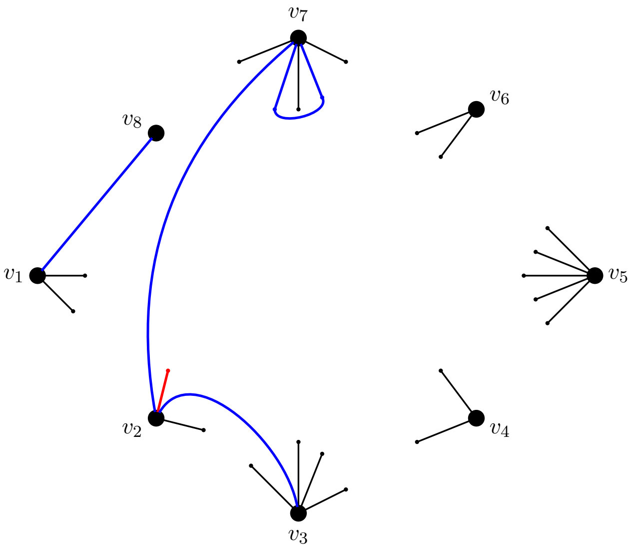

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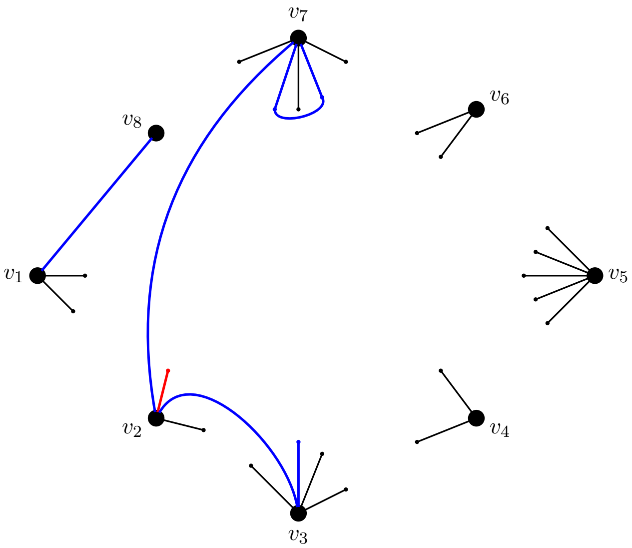

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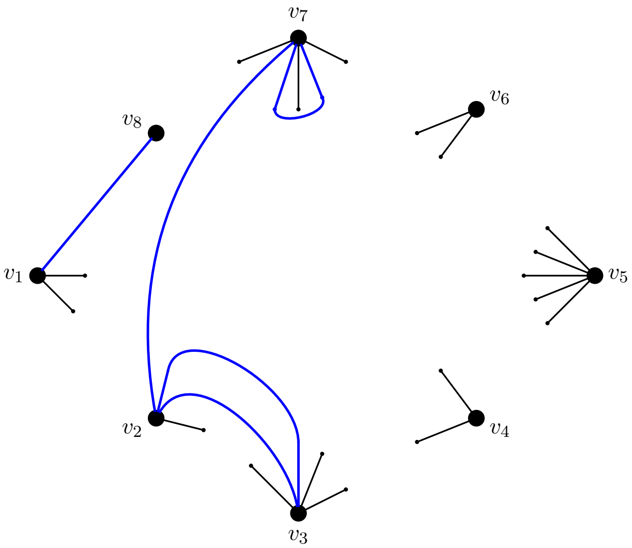

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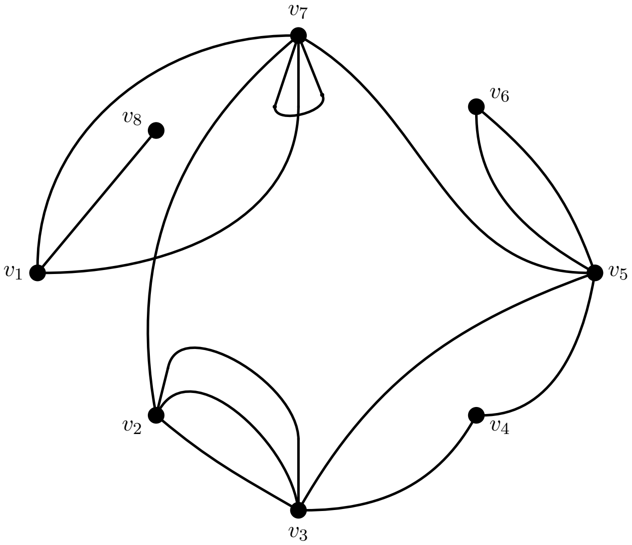

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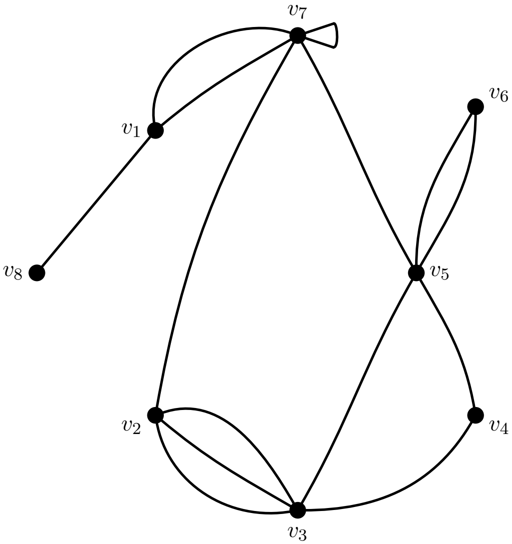

Degree assumptions

Empirical degree distribution:

$$F_n(x) := \frac{1}{n} \sum_{v=1}^{n} \mathbb{1}_{\{d_v \le x\}}.$$

We want to capture all possible degree distributions 'under one hat':

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For $\tau \in (2,3)$, and some $\beta_n > 0$,

$$1 - F_n(x) = \frac{L_n(x)}{x^\tau}, \qquad (TrPL)$$

holds for all $x \le n^{\beta_n(1-\varepsilon)}$ for all $\varepsilon > 0$. $L_n(x)$ is a slowly varying function.

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$1 - F_n(n^{\beta_n(1+\varepsilon)}) = 0$ for all $\varepsilon > 0$.

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The answer: truncated scale free $\neq$ ultrasmall

Heuristic theorem (v/d Hofstad, K)

Consider the configuration model with empirical degree distribution satisfying (*TrPL*) with $\beta_n \gg \frac{1}{(\log n)^{1-\delta}}$ for some $\delta \in (0,1)$. Then

$$\mathrm{d}_G(u,v) - \frac{2\log\log(n^{\beta_n})}{|\log(\tau-2)|} - \frac{1}{\beta_n(3-\tau)}$$

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The tight random variable shows *log-log periodicity.*

We also determine its limit along subsequences.

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- When $\beta_n = 1/(\log n)^{1-\delta}$, then $\mathrm{d}_G(u,v) = O((\log n)^{1-\delta})$,
  Truncation allows to *interpolate* between small and ultrasmall.

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Since Newman, Strogatz, Watts '00, it was believed that
(at least for $\tau > 3$)

$$\mathrm{d_G}(u, v) = \frac{\log n}{\log \nu_n} + \text{tight}$$

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  $\beta_n \equiv \beta$ yields bounded distances
- Dorogovtsev, Mendes, Samukhin '03: no,
  there is also a term $\frac{2 \log \log(\xi_n)}{|\log(\tau-2)|}$, with ξ_n the point of truncation.

Proof idea

Distance between hubs

Distance between hubs

Let v_1, v_2 be two vertices with degrees $n^{x_1 \beta_n}, n^{x_2 \beta_n}$, for $x_1, x_2 > \tau - 2$.

Let's count the expected paths of length z between them!

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The probability of matching z pairs of half-edges:

$$\frac{1}{H_n - 1} \cdot \frac{1}{H_n - 3} \cdot \dots \cdot \frac{1}{H_n - 2z - 1}$$

Distance between hubs

Distance between hubs

Let v_1, v_2 be two vertices with degrees $n^{x_1\beta_n}, n^{x_2\beta_n}$, for $x_1, x_2 > \tau - 2$.

Let's count the expected paths of length z between them!
The probability of matching z pairs of half-edges:

$$\frac{1}{H_n - 1} \cdot \frac{1}{H_n - 3} \cdot \dots \cdot \frac{1}{H_n - 2z - 1} = (1 + o(1)) \frac{1}{(\mathbf{E}[D_n]n)^z}$$

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The number of ways to choose these half-edges via fixed vertices

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$v_1 = \pi_0,\ \pi_1, \dots, \pi_{z-1},\ \ \pi_z = v_2$

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$v_1 = \pi_0,\ \pi_1, \dots, \pi_{z-1},\ \ \pi_z = v_2$

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$$d_{v_1} \cdot \qquad d_{\pi_1}(d_{\pi_1} - 1) \cdot \dots \cdot \qquad d_{\pi_{z-1}}(d_{\pi_{z-1}} - 1) \cdot d_{v_2}$$

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$$d_{v_1} \cdot \sum_{\pi_1=1}^{n} d_{\pi_1}(d_{\pi_1} - 1) \cdot \dots \cdot \sum_{\pi_{z-1}=1}^{n} d_{\pi_{z-1}}(d_{\pi_{z-1}} - 1) \cdot d_{v_2}$$

Distance between hubs

$$\mathbf{E}[\#\mathrm{Path}_{v_1,v_2}(z)] =$$

Distance between hubs

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$$= C \cdot$$

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$$= C \cdot \frac{1}{n} \cdot$$

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$$= C \cdot \frac{1}{n} \cdot d_{v_1} \cdot d_{v_2} \cdot n^{(z-1)(3-\tau)\beta_n}$$

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\begin{aligned}
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&= C \cdot \frac{1}{n} \cdot d_{v_1} \cdot d_{v_2} \cdot n^{(z-1)(3-\tau)\beta_n} \\
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What is the smallest z so that this does not tend to 0?

Distance between hubs

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\end{aligned}
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What is the smallest z so that this does not tend to 0?

$$x_1\beta_n + x_2\beta_n + (z-1)(3-\tau)\beta_n > 1$$

Distance between hubs

$$\mathbf{E}[\#\mathrm{Path}_{v_1,v_2}(z)] = (1+o(1))\frac{1}{\mathbf{E}[D_n]n} \cdot d_{v_1} \cdot \left(\sum_{v=1}^{n} \frac{d_v(d_v-1)}{\mathbf{E}[D_n]n}\right)^{z-1} d_{v_2}$$
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What is the smallest z so that this does not tend to 0?

$$x_1 + x_2 + (z-1)(3-\tau) > 1/\beta_n$$

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$$(z-1)(3-\tau) > 1/\beta_n - x_1 - x_2$$

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What is the smallest z so that this does not tend to 0?

$$z-1>\frac{1/\beta_n-x_1-x_2}{3-\tau}.$$

$$z_{\min} := \left\lceil\frac{1/\beta_n-x_1-x_2}{3-\tau}\right\rceil+1.$$

Distance between hubs

Distance between hubs

Let v_1, v_2 be two vertices with degrees $n^{x_1\beta_n}, n^{x_2\beta_n}$, for $x_1, x_2 > \tau - 2$. Then whp

$$\mathrm{d}_G(v_1, v_2) = \left\lceil \frac{1/\beta_n - x_1 - x_2}{3 - \tau} \right\rceil + 1 = z_{\min},$$

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and

$$\mathbf{E}[\#\mathrm{Path}_{v_1,v_2}(z_{\min})] = n^{f^{up}(1+o_{\mathbf{P}}(1))},$$

where $f^{up} = \left\lceil \frac{1/\beta_n - x_1 - x_2}{3-\tau} \right\rceil - \frac{1/\beta_n - x_1 - x_2}{3-\tau}$ is an 'upper fractional part'.

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Proof

$$\mathbf{P}(\exists \text{ a path shorter than } z_{\min})$$

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Proof

$$\mathbf{P}(\exists \text{ a path shorter than } z_{\min}) \le \mathbf{E}[\#\mathrm{Path}_{v_1,v_2}(z_{\min}-1)] \to 0.$$

The other direction

$$\mathrm{Var}[\#\mathrm{Path}_{v_1,v_2}(z)] = \mathbf{E}[\#\mathrm{Path}_{v_1,v_2}(z)]^2 \cdot$$

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This tends to zero if and only if $\min\{x_1, x_2\} > \tau - 2$.

From here, Chebyshev's inequality finishes the proof.

Comment

Distance between hubs

Let v_1, v_2 be two vertices with degrees $n^{x_1\beta_n}, n^{x_2\beta_n}$, for $x_1, x_2 > \tau - 2$. Then whp

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so the formula from physics is valid *only* between hubs!

How to get to the hubs?

When constructing the shortest path, how long does it take to get to the hubs?

Neighborhood growth

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Growth rate heuristic

$\mathrm{Ball}_{k_n}^{(u)}, \mathrm{Ball}_{k_n}^{(v)}$ grow double-exponentially as long as their size is 'reasonably small'.

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$\mathrm{Ball}_{k_n}^{(u)}, \mathrm{Ball}_{k_n}^{(v)}$ grow double-exponentially as long as their size is 'reasonably small'. I.e., $\exists$ random variables $(Y_k^{(u)}, Y_k^{(v)}) \xrightarrow{d} (Y^{(u)}, Y^{(v)})$ s.t.,$q = u, v$

$$\mathrm{Ball}_{k_n}^{(q)} = \exp\left\{ Y_{k_n}^{(q)} \left(\frac{1}{\tau - 2}\right)^{k_n} \right\}.$$

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Growth rate heuristic

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Stopping time

Let $t(n^\varrho) := \sup\{k_n : \max\{\mathrm{Ball}_{k_n}^{(u)}, \mathrm{Ball}_{k_n}^{(v)}\} \le n^\varrho\}$, and for $q = u, v$:

$$Y_n^{(q)} := (\tau - 2)^{t(n^\varrho)} \log \mathrm{Ball}_{t(n^\varrho)}^{(q)},$$

then $(Y_n^{(u)}, Y_n^{(v)}) \xrightarrow{d} (Y^{(u)}, Y^{(v)})$.

$$\exp\left\{ Y_n^{(q)} \left(\frac{1}{\tau-2}\right)^k \right\} = n^{\varrho}$$

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$$Y_n^{(q)} \left(\frac{1}{\tau-2}\right)^k = \log n^{\varrho}$$

$$\left(\frac{1}{\tau-2}\right)^k = (\varrho \log n)/Y_n^{(q)}$$

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$$\left(\frac{1}{\tau-2}\right)^k = (\varrho \log n)/Y_n^{(q)}$$

$$k = \frac{\log\log n - \log(\varrho/Y_n^{(q)})}{|\log(\tau-2)|}$$

$$\exp\left\{Y_n^{(q)}\left(\frac{1}{\tau-2}\right)^k\right\} = n^{\varrho}$$

$$Y_n^{(q)}\left(\frac{1}{\tau-2}\right)^k = \log n^{\varrho}$$

$$\left(\frac{1}{\tau-2}\right)^k = (\varrho\log n)/Y_n^{(q)}$$

$$t(n^{\varrho}) = \left\lfloor \frac{\log\log n - \log(\varrho/Y_n^{(q)})}{|\log(\tau-2)|} \right\rfloor$$

Shell structure

Step 1

One can find a vertex of degree $\approx \mathrm{Ball}^{(q)}_{t(n^{\varrho})}$ in the balls.

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Structure the high-degree part of the graph in layers of roughly equal degree (on a $\log\log$ scale).

Shell structure

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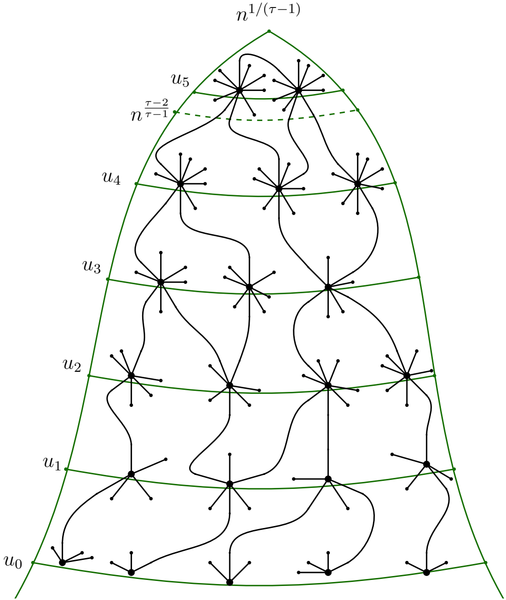
Structure the high-degree part of the graph in layers of roughly equal degree (on a $\log\log$ scale).

Shell i:

$$\Gamma_i = \{v : d_v \geq n^{\varrho(\tau-2)^{-i}}(1+o(1))\}$$

Like shells of an onion, to get to the core of the graph.

The nested shells

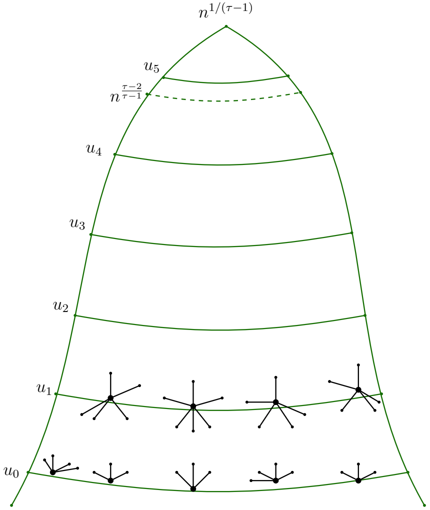


The nested shells

$N(A)$:=neighbors of A

Layer connecting lemma

$$\Gamma_i \subset N(\Gamma_{i+1}) \quad \text{whp}$$



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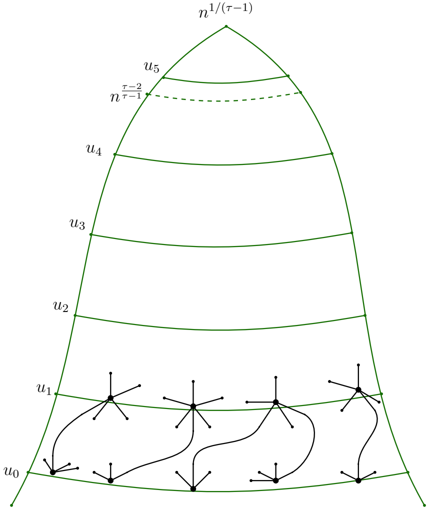

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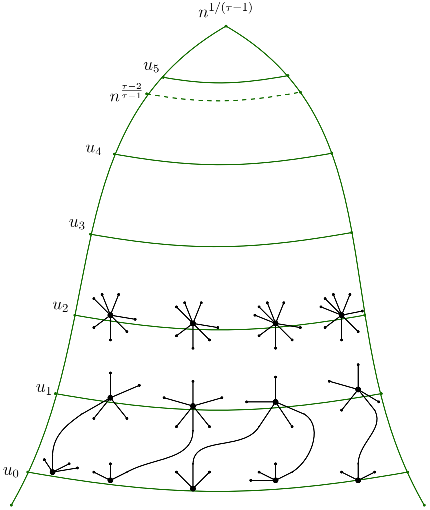

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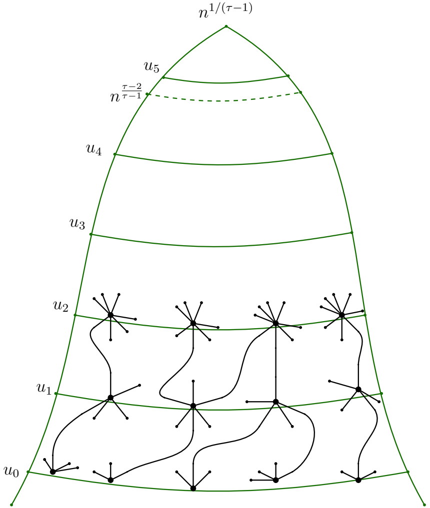

The nested shells

$N(A)$:=neighbors of A

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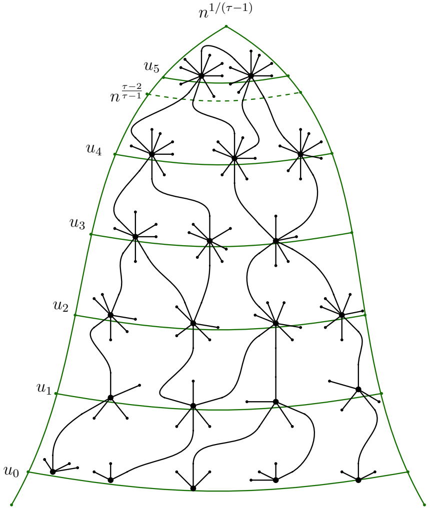
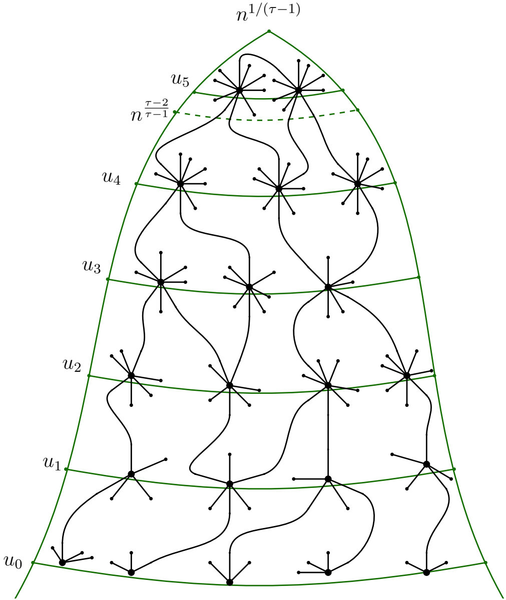


2nd step: establishing the path to the hubs



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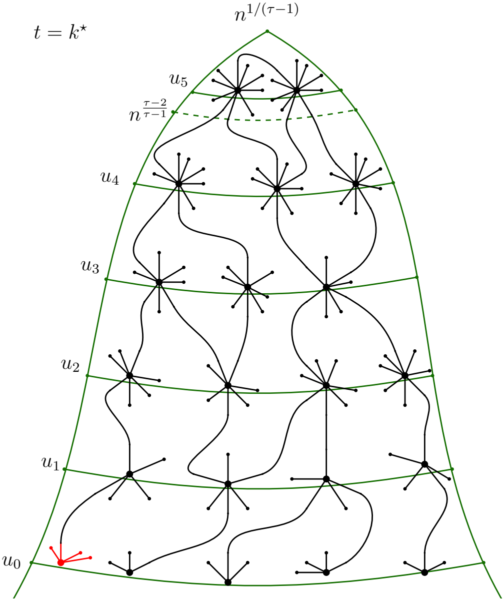

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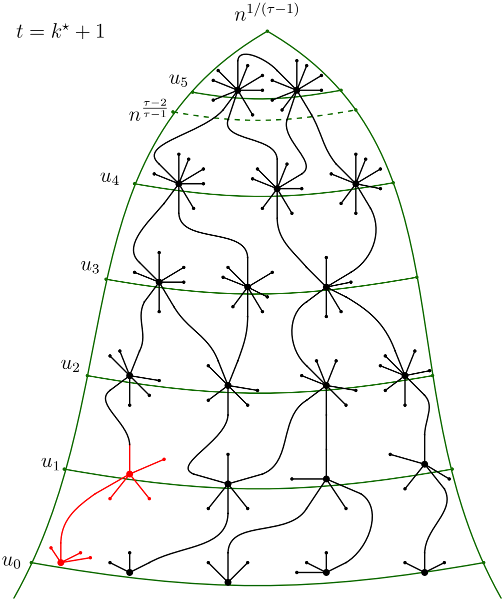

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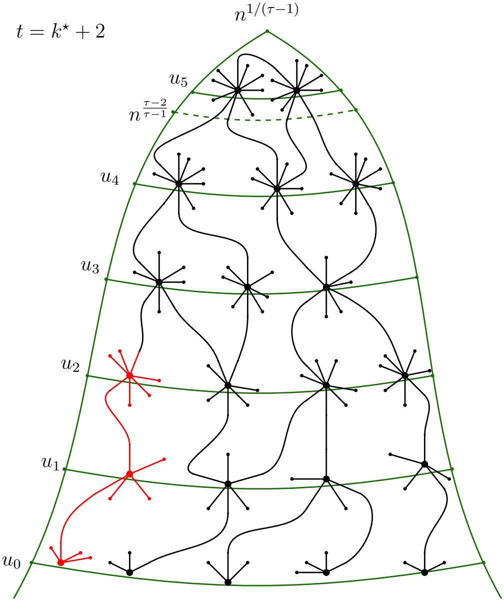

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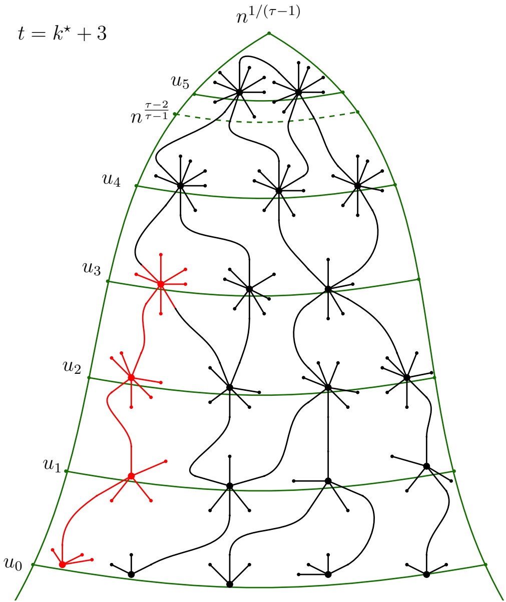

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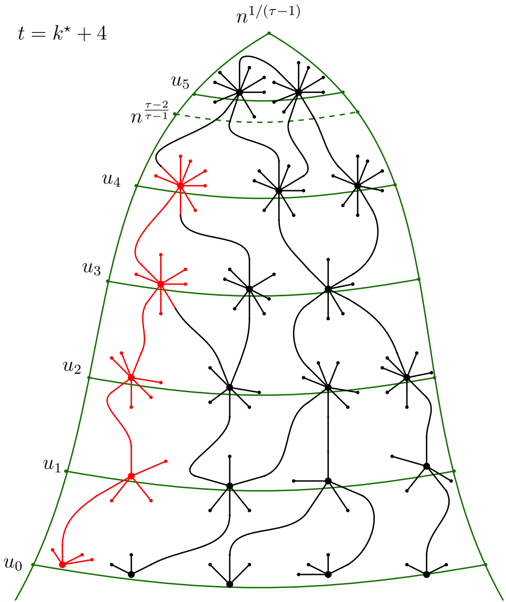

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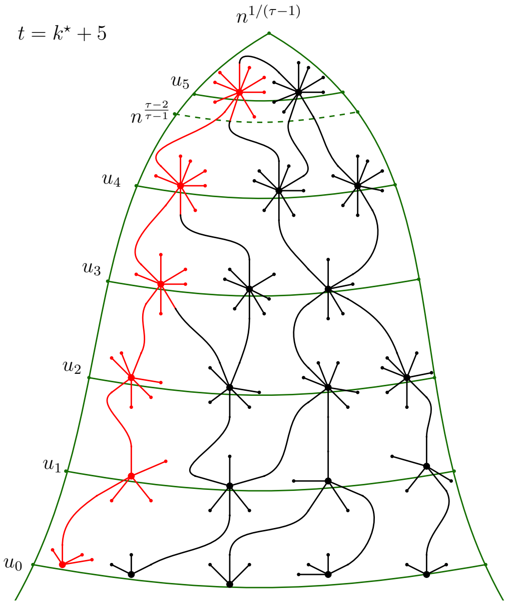

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- Maximal degree in the graph: $M = n^{\beta_n}$

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$$1/(\tau-2)^{i+1} > \beta_n/\varrho$$

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with $e^{(q)}_n \in (-2,0)$.

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Observation

$T^{(q)}_{hub}$ does not depend on ρ! ☺

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with $e_n^{hub} \in \left(\frac{-2}{3-\tau} - 1, \frac{-2(\tau-2)}{3-\tau}\right)$.

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☺☺☺

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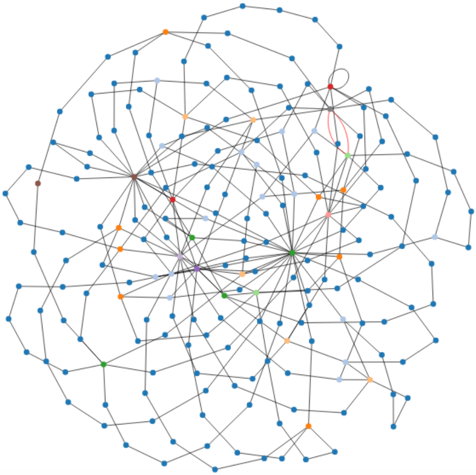

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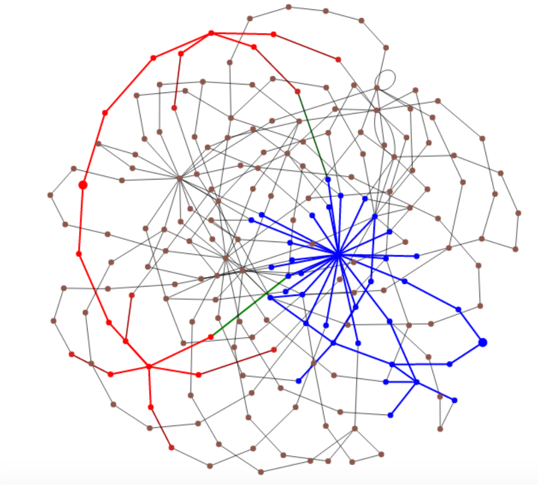

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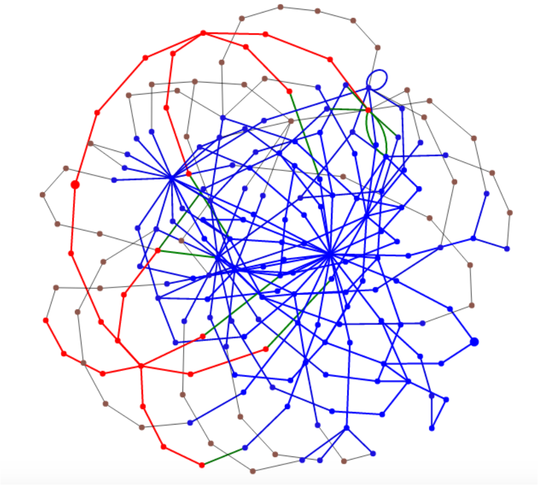

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