The Matrix Dyson Equation in random matrix theory

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INTRODUCTION

Basic question [Wigner]: What can be said about the statistical properties of the eigenvalues of a large random matrix? Do some universal patterns emerge?

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \quad \Longrightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

$N =$ size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

Wigner Ensemble: i.i.d. entries

$H = (h_{jk})$ real symmetric or complex hermitian $N \times N$ matrix

Entries are i.i.d. up to $h_{jk} = \bar{h}_{kj}$ (for $j < k$), with normalization

$$\mathbb{E} h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}.$$

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are of order one: (on average)

$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 = \mathbb{E}\frac{1}{N}\mathrm{Tr} H^2 = \frac{1}{N}\sum_{ij}\mathbb{E}|h_{ij}|^2 = 1$$

If h_{ij} is Gaussian, then GUE, GOE.

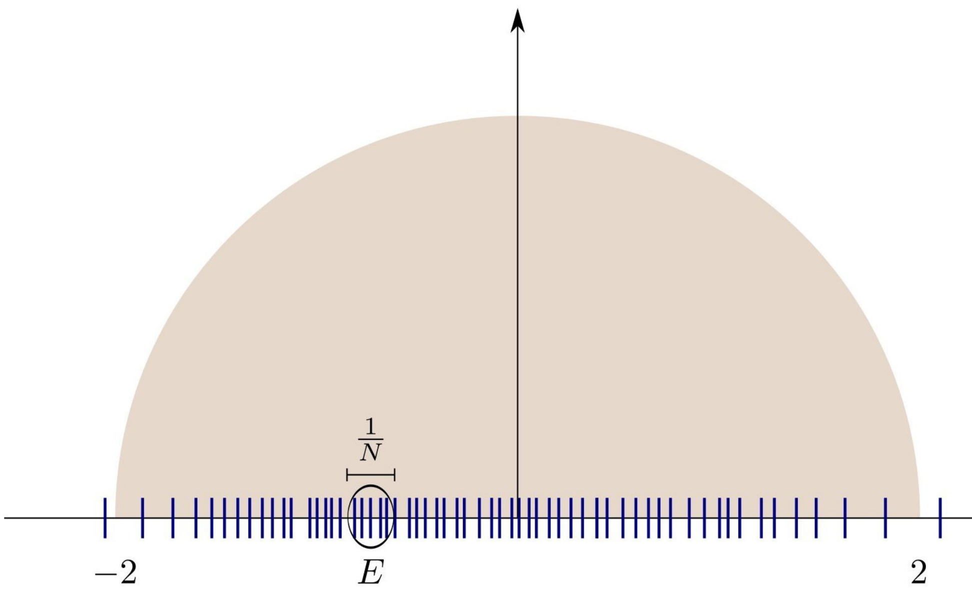
Global vs. local law

Global density: Semicircle Law

Typical ev. gap $\approx \frac{1}{N}$ (bulk)

- Does semicircle law hold just above this scale? ($\Longrightarrow$ local semicircle law)

- How do eigenvalues behave exactly on this scale? ($\Longrightarrow$ WDM universality)



Wigner's revolutionary observation: The global density may be model dependent, but the gap statistics is very robust, it depends only on the symmetry class (hermitian or symmetric).

In particular, it can be determined from the Gaussian case (GUE/GOE).

SINE KERNEL FOR CORRELATION FUNCTIONS

Probability density of the eigenvalues: $p(x_1, x_2, \ldots, x_N)$

The k-point correlation function is given by

$$p_N^{(k)}(x_1, x_2, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} p(x_1, \ldots x_k, x_{k+1}, \ldots, x_N) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N$$

$k = 1$ point correlation function: density ϱ

Rescaled correlation functions at energy E (in the bulk, $\varrho(E) > 0$)

$$p_E^{(k)}(\mathbf{x}) := \frac{1}{[\varrho(E)]^k} p_N^{(k)}\Big(E + \frac{x_1}{N\varrho(E)}, E + \frac{x_2}{N\varrho(E)}, \ldots, E + \frac{x_k}{N\varrho(E)}\Big)$$

Rescales the gap $\lambda_{i+1} - \lambda_i$ to O(1).

Local correlation statistics for GUE [Gaudin, Dyson, Mehta]

$$\lim_{N\to\infty} p_E^{(k)}(\mathbf{x}) = \det\Big\{S(x_i - x_j)\Big\}_{i,j=1}^{k}, \qquad S(x) := \frac{\sin \pi x}{\pi x}$$

Wigner-Dyson-Mehta universality: Local statistics is universal in the bulk spectrum for any Wigner matrix; only symmetry type matters.

Solved for any symmetry class by the three step strategy
[Bourgade, E, Schlein, Yau, Yin: 2009-2014]

Related results:

[Johansson, 2000] Hermitian case with large Gaussian components

[Tao-Vu, 2009] Hermitian case via moment matching.

(Similar development for the edge, for β-log gases and for many related models, such as sample covariance matrices, sparse graphs, regular graphs etc).

Three-step strategy

1. Local density law down to scales $\gg 1/N$

   (Needed in entry-wise form, i.e. control also matrix elements G_{ij} the resolvent $G(z) = (H - z)^{-1}$ and not only TrG)

2. Use local equilibration of Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component

3. Use perturbation theory to remove the tiny Gaussian component.

Steps 2 and 3 need Step 1 as an input but are considered standard since very general theorems are available.

Step 1 is model dependent.

Models of increasing complexity

- Wigner matrix: i.i.d. entries, $s_{ij} := \mathbb{E}|h_{ij}|^2$ are constant $(= \frac{1}{N})$.
  (Density $=$ semicircle; $G \approx$ diagonal, $G_{xx} \approx G_{yy}$)

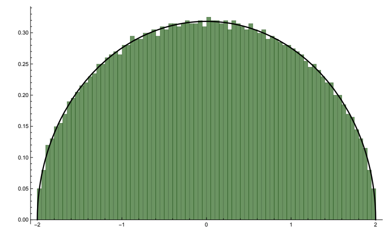
- Generalized Wigner matrix: indep. entries, $\sum_j s_{ij} = 1$ for all i.
  (Density $=$ semicircle; $G \approx$ diagonal, $G_{xx} \approx G_{yy}$)

- Wigner type matrix: indep. entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \approx$ diagonal, $G_{xx} \not\approx G_{yy}$)

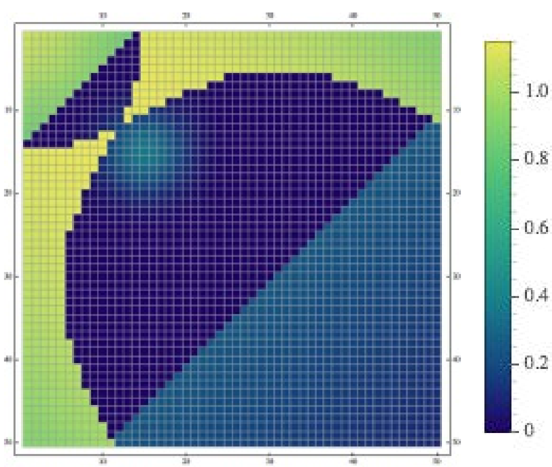
- Correlated Wigner matrix: correlated entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \not\approx$ diagonal)

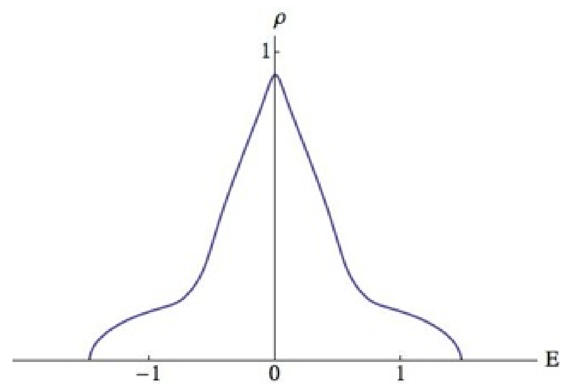
Variance profile and limiting density of states (DOS)

$$\sum_j s_{ij} = 1 \qquad \Longleftrightarrow$$



General variance profile $s_{ij} = \mathbb{E}|h_{ij}|^2$: not the semicircle any more.

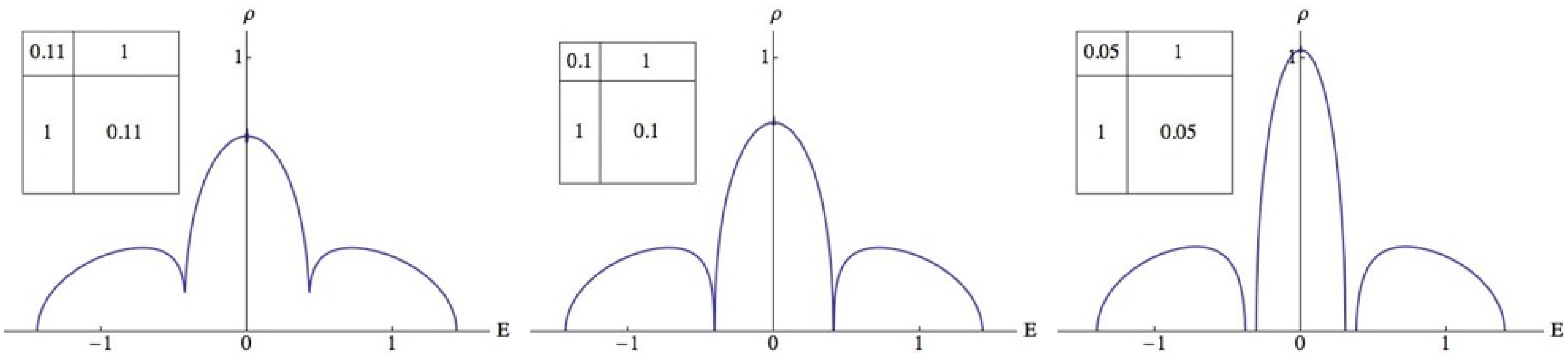




$\sum_j s_{ij} \neq$ const $\qquad \Longrightarrow \qquad$ Density of states

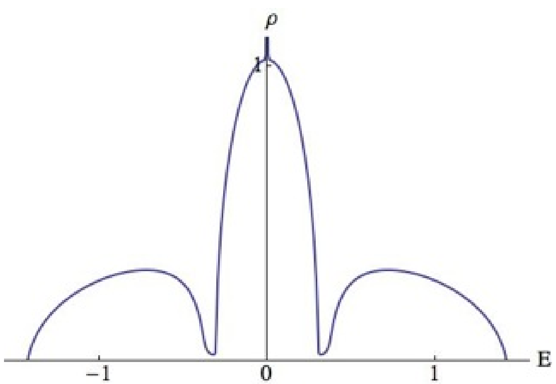
Features of the DOS for Wigner-type matrices

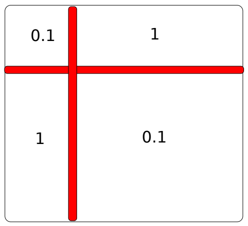
1) Support splits via cusps:



(Matrices in the pictures represent the variance matrix)

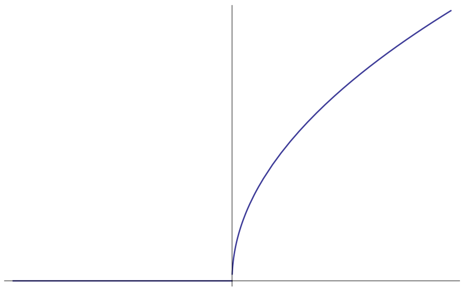
2) Smoothing of the S-profile avoids splitting ($\Rightarrow$ single interval)



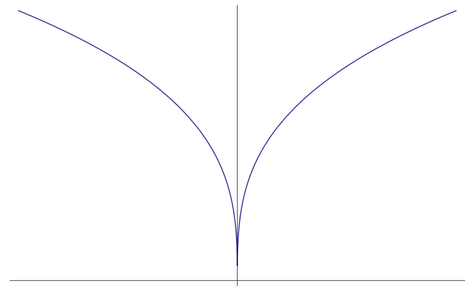


DOS of the same matrix as above but discontinuities in S are regularized

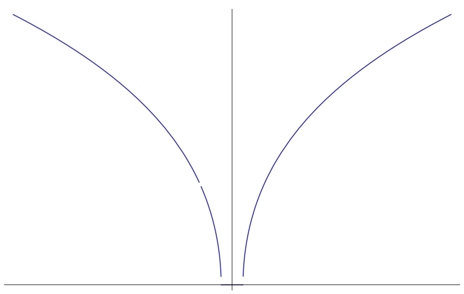
Universality of the DOS singularities for Wigner-type models



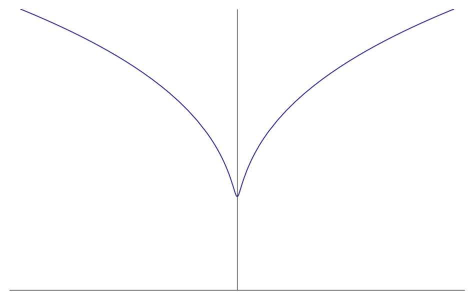
Edge, $\sqrt{E}$ singularity



Cusp, $|E|^{1/3}$ singularity



Small-gap



Smoothed cusp

$$\frac{(2+\tau)\tau}{1+(1+\tau+\sqrt{(2+\tau)\tau})^{2/3}+(1+\tau-\sqrt{(2+\tau)\tau})^{2/3}} \qquad \frac{\sqrt{1+\tau^2}}{(\sqrt{1+\tau^2}+\tau)^{2/3}+(\sqrt{1+\tau^2}-\tau)^{2/3}-1}-1$$

$$\tau := \frac{|E|}{\text{gap}}, \qquad \tau := \frac{|E|}{(\text{minimum })^{1/3}}$$

Main theorems (informally)

Theorem [Ajanki-E-Krüger, 2014] Let $H = H^*$ be a Wigner-type matrix with general variance profile $c/N \le s_{ij} \le C/N$. Then optimal local law (including edge) and bulk universality hold.

Theorem [Ajanki-E-Krüger, 2016] Let $H = H^*$ be correlated

$$H = A + \frac{1}{\sqrt{N}} W$$

where A is deterministic, decaying away from the diagonal; W is random with $\mathbb{E}W = 0$ and fast decaying correlation:

$$\mathrm{Cov}\Big(\phi(W_A), \psi(W_B)\Big) \le \frac{C_K \|\nabla\phi\|_\infty \|\nabla\psi\|_\infty}{[1 + \mathrm{dist}(A, B)]^K}$$

for all K and for any subsets A, B of the index set. Assume

$$\mathbb{E}|\mathbf{u}^* W \mathbf{v}|^2 \ge c\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \qquad \forall \mathbf{u}, \mathbf{v}.$$

Then optimal local law and bulk universality hold.

(Special translation invariant corr. structure: independently by [Che, 2016])

Matrix Dyson Equation

For any $z \in \mathbb{C}_+$, consider the equation (we set $A = \mathbb{E}H = 0$)

$$-\frac{1}{M} = z + \mathcal{S}[M], \qquad M = M(z) \in \mathbb{C}^{N\times N} \tag{1}$$

with the "super-operator"

$$\mathcal{S}[R] := \mathbb{E}\Big[HRH\Big], \qquad \mathcal{S} : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$$

Fact: [Girko, Pastur, Wegner, Helton-Far-Speicher] The MDE has a unique solution with $\mathrm{Im}\, M \geq 0$ and it is a Stieltjes transform of a matrix-valued measure

$$M(z) = \frac{1}{\pi}\int \frac{V(\omega)\mathrm{d}\omega}{\omega - z}, \qquad z \in \mathbb{C}_+$$

Define the density of states

$$\varrho(\omega) := \frac{1}{\pi N}\mathrm{Tr} V(\omega), \qquad \omega \in \mathbb{R}$$

Theorem [AEK] (i) ϱ is compactly supported, Hölder continuous.
(ii) $V(\omega) \gtrsim \varrho(\omega)$
(iii) M_{xy} has a fast offdiag decay away from the spectral edge.

Local law for the correlated case

Theorem [AEK] In the bulk spectrum, $\varrho(\Re z) \geq c$, we have

$$|G_{xy}(z) - M_{xy}(z)| \lesssim \frac{1}{\sqrt{N \,\text{Im}\, z}}, \qquad \left| \frac{1}{N}\text{Tr}G(z) - \frac{1}{N}\text{Tr}M(z) \right| \lesssim \frac{1}{N \,\text{Im}\, z}$$

with very high probability.

M is typically not diagonal, so G has nontriv off-diagonal component.

We also have the "usual" Corollaries:

- Complete delocalization of corresponding eigenvectors
- Rigidity of bulk eigenvalues (ev's are almost in the $1/N$-vicinity of the quantiles of the DOS).
- Wigner-Dyson-Mehta universality in the bulk

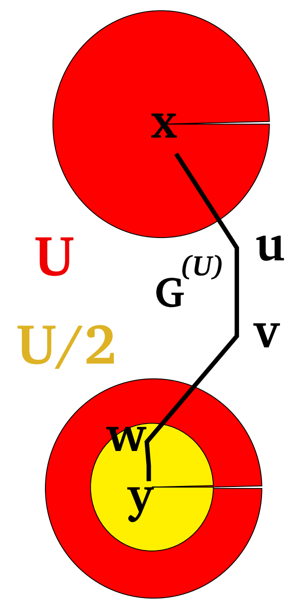
Derivation of the Matrix Dyson Equation

$$G(z) := (H - z)^{-1} \qquad \delta_{xy} + zG_{xy} = \sum_u h_{xu} G_{uy}$$

Let U be a (large) neighborhood of $\{x, y\}$. Let $H^{(U)}$ be the removal of U rows/columns from H and $G^{(U)}$ is its resolvent. Using

$$G = G^{(U)} - G^{(U)}\big[H - H^{(U)}\big]G,$$

$$G_{uy} = -\sum_{v \notin U} \sum_{w \in U} G^{(U)}_{uv} h_{vw} G_{wy}, \qquad \text{for } u \notin U.$$



Thus

$$\delta_{xy} + zG_{xy} = \sum_{u \in U} h_{xu} G_{uy} - \sum_{u,v \notin U} \sum_{w \in U} h_{xu} G^{(U)}_{uv} h_{vw} G_{wy}$$

Here $G^{(U)}_{uv}$ is (almost) indep of h_{xu} and h_{vw} for $w \in \frac{1}{2}U$ (for $w \in U \setminus \frac{1}{2}U$ we use the decay of G_{wy})

First sum is neglected, the uv sum in the second is close to its expectation.

The uv sum is close to its expectation

$$\sum_{u,v\notin U} h_{xu} G^{(U)}_{uv} h_{vw} \approx \sum_{u,v\notin U} \mathbb{E}\Big[h_{xu}h_{vw}\Big] G^{(U)}_{uv} \approx \Big(\mathcal{S}[G^{(U)}]\Big)_{xw}$$

Undoing the removal of U, we get

$$\delta_{xy} + zG_{xy} \approx -\sum_{w} \Big(\mathcal{S}[G]\Big)_{xw} G_{wy}$$

i.e.

$$I + zG \approx -\mathcal{S}[G]G$$

Thus G approximately solves the matrix Dyson equation (MDE)

$$-\frac{1}{M} = z + \mathcal{S}[M], \qquad \text{or} \qquad I + zM = -\mathcal{S}[M]M.$$

Key question: Stability of MDE under small perturbation.

Then we could conclude that

$$G \approx M$$

Dyson equations and their stability operators

Name	Dyson Eqn	For	Stab. op	Feature
Wigner $\mathbb{E}\lvert h_{ij}\rvert^2 = s_{ij} = \frac{1}{N}$	$-\frac{1}{m} = z + m$	$m \approx \frac{1}{N}\mathrm{Tr}G$	$\frac{1}{1-m^2\lvert e\rangle\langle e\rvert}$	$m = m_{sc}$ is explicit
Gen. Wigner $\sum_j s_{ij} = 1$	$-\frac{1}{m} = z + m$	$m \approx \frac{1}{N}\mathrm{Tr}G$	$\frac{1}{1-m^2 S}$	Split S as $S^\perp + \lvert e\rangle\langle e\rvert$
Wigner-type s_{ij} arbitrary	$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}$	$m_x \approx G_{xx}$	$\frac{1}{1-\mathbf{m}^2 S}$	$\mathbf{m}$ to be determined
Corr. Wigner $\mathbb{E}h_{xy}h_{uw} \not\asymp \delta_{xw}\delta_{yu}$	$-\frac{1}{M} = z + \mathcal{S}[M]$	$M_{xy} \approx G_{xy}$	$\frac{1}{1-M\mathcal{S}[\cdot]M}$	Matrix eq. Super-op

- Gen. Wigner could be studied via a scalar equation only (in practice a vector eq. is also considered for G_{xx})
- Wigner-type needs vector equation even for the density
- Corr. Wigner needs matrix equation.

Mechanism for stability I. Generalized Wigner

For gen. Wigner, m is the Stieltjes tr. of the semicircle:

$$|m(z)| \leq 1 - c\eta, \qquad \operatorname{Im} m(z) \approx \varrho(E), \qquad z = E + i\eta$$

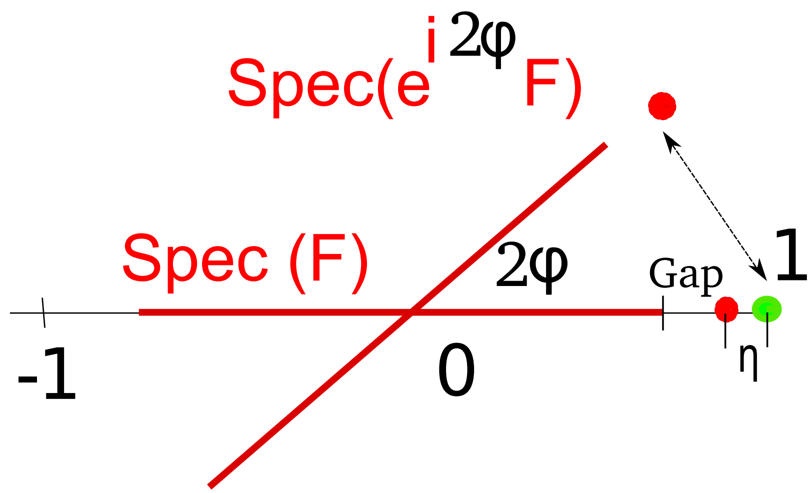
The variance matrix $\|S\| \leq 1$, with $Se = e$ and a gap in $Spec(S)$.

$$1 - m^2 S = 1 - e^{2i\varphi} F, \qquad m = |m|\, e^{i\varphi}, \quad F := |m|^2 S$$

F is symmetric, Spec$(F) \subset (-1, 1)$

In the bulk $\varphi \sim \operatorname{Im} m \neq 0$

$$\Big\| \frac{1}{1 - m^2 S} \Big\| = \Big\| \frac{1}{e^{-2i\varphi} - F} \Big\| \leq \frac{C}{\varphi}$$



At the edge use the gap, the isolated eigenspace $Fe = |m|^2 e$ is treated separately.

Mechanism for stability II. Wigner-type

$$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}, \qquad S = s_{ij}, \qquad \mathbf{m} = (m_i)$$

Why is $(1 - \mathbf{m}^2 S)$ invertible at all? [here $(\mathbf{m}^2 S)_{ij} := m_i^2 S_{ij}$]

Take Im-part and symmetrize

$$\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|} = \eta|\mathbf{m}| + |\mathbf{m}|S|\mathbf{m}|\,\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|}$$

Since $\mathrm{Im}\,\mathbf{m} \geq 0$, by Perron-Frobenius, $F := |\mathbf{m}|S|\mathbf{m}| \leq 1 - c\eta$

Lemma. If F is self-adjoint with $Ff = \|F\|f$ and a gap, then

$$\left\|\frac{1}{U - F}\right\| \leq \frac{C}{\mathrm{Gap}(F)\,\Big|1 - \|F\|\langle f, Uf\rangle\Big|}, \qquad \text{for any } U \text{ unitary}$$

Thus, we have stability (albeit weaker)

$$\left\|\frac{1}{1 - \mathbf{m}^2 S}\right\| = \left\|\frac{1}{e^{-2i\varphi} - F}\right\| \leq \frac{C}{(\min \varphi_j)^2}$$

Mechanism for stability III. Matrix Dyson Equation

Theorem [AEK] Let $\mathcal{S} : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ be *flat*, i.e.

$$\frac{c}{N}\mathrm{Tr}R \le \mathcal{S}[R] \le \frac{C}{N}\mathrm{Tr}R, \qquad \forall R \ge 0$$

and decay

$$\Big|\mathcal{S}[R]_{xy}\Big| \le \frac{C_K \|R\|_{\max}}{(1+|x-y|)^K}, \qquad \|R\|_{\max} := \max_{ab} |R_{ab}|$$

For small D, $\exists$ a unique solution $G = G(D)$ of the perturbed MDE

$$-1 = (z+\mathcal{S}[M]\Big)M, \qquad -1 = (z+\mathcal{S}[G]\Big)G + D,$$

that is linearly stable in strong sense

$$\|G(D_1) - G(D_2)\|_{\max} \le C\|D_1 - D_2\|_{\max}$$

Matrix stability operator

Define the *sandwiching* operator on matrices: $\mathcal{C}_R[T] := RTR$

Lemma: $M = M(z)$ be the solution to MDE, then

$$\Big\| \frac{1}{1 - M\mathcal{S}[\cdot]M} \Big\| = \Big\| \frac{1}{1 - \mathcal{C}_M \mathcal{S}} \Big\| \le \frac{C}{[\varrho(z) + \mathrm{dist}(z, \mathrm{supp}(\varrho)]^{100}}$$

with C depending on M in a controlled way.

Key: find the "right" symmetrization $\mathcal{F}$ despite the noncommutative matrix structure.

Need the analogue of

$$\mathbf{m} = e^{i\boldsymbol{\varphi}}|\mathbf{m}|, \qquad F = |\mathbf{m}|S|\mathbf{m}|, \qquad |1 - \mathbf{m}^2 S| = |e^{-2i\boldsymbol{\varphi}} - F|$$

Answer: "Polar decompose" M into a commuting "quarter" magnitude $W > 0$ and a phase U (unitary)

$$M = \mathcal{C}_{\sqrt{\mathrm{Im} M}} \mathcal{C}_W [U^*] = \sqrt{\mathrm{Im}\, M}\, W U^* W \sqrt{\mathrm{Im}\, M}$$

$$W := \left[1 + \left(\frac{1}{\sqrt{\mathrm{Im} M}} \mathrm{Re}\, M \frac{1}{\sqrt{\mathrm{Im} M}} \right)^2 \right]^{\frac{1}{4}}, \qquad U := \frac{\frac{1}{\sqrt{\mathrm{Im} M}} \mathrm{Re}\, M \frac{1}{\sqrt{\mathrm{Im} M}} - i}{W^2}$$

Define

$$\mathcal{F} := \mathcal{C}_W \mathcal{C}_{\sqrt{\mathrm{Im} M}} \mathcal{S} \mathcal{C}_{\sqrt{\mathrm{Im} M}} \mathcal{C}_W$$

Then $\mathcal{F}$ is selfadjoint (wrt. HS scalar product), has a unique normalized eigenmatrix F with e.v. $\|\mathcal{F}\| \le 1$ and a spectral gap:

$$\left\| \frac{1}{1 - \mathcal{C}_M \mathcal{S}} \right\| \lesssim \left\| \frac{1}{\mathcal{U} - \mathcal{F}} \right\| \lesssim \frac{1}{\mathrm{Gap}(\mathcal{F}) \ \left| 1 - \|\mathcal{F}\| \langle F, UFU \rangle \right|}$$

Then we prove

$|1 - \|\mathcal{F}\| \langle F, UFU \rangle| \ge c, \qquad \mathrm{Gap}(\mathcal{F}) \ge c \qquad$ with some $c = c(\varrho)$. $\square$

Summary

- We gave a quantitative analysis of the solution of the Matrix Dyson Equation and its stability.

- For correlated random matrices with short range correlation in both symmetry classes we proved

  - Optimal local law in the bulk

  - Wigner-Dyson-Mehta bulk universality

Outlook

- Add arbitrary external field ($A = \mathbb{E}H$) – work in progress

- Cusp analysis for Wigner type – work in progress

- Edge analysis for MDE – work in progress

- No. of intervals in suppϱ in terms of block structure of S or $\mathcal{S}$?